Beyond Base 10: Non-decimal Based Number Systems

- What is the decimal based number system?
- How do other number systems work (binary, octal and hex)
- How to convert to and from non-decimal number systems to decimal
- Binary math

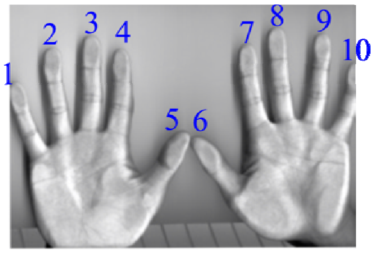
What Is Decimal?

Base 10

- 10 unique symbols are used to represent values

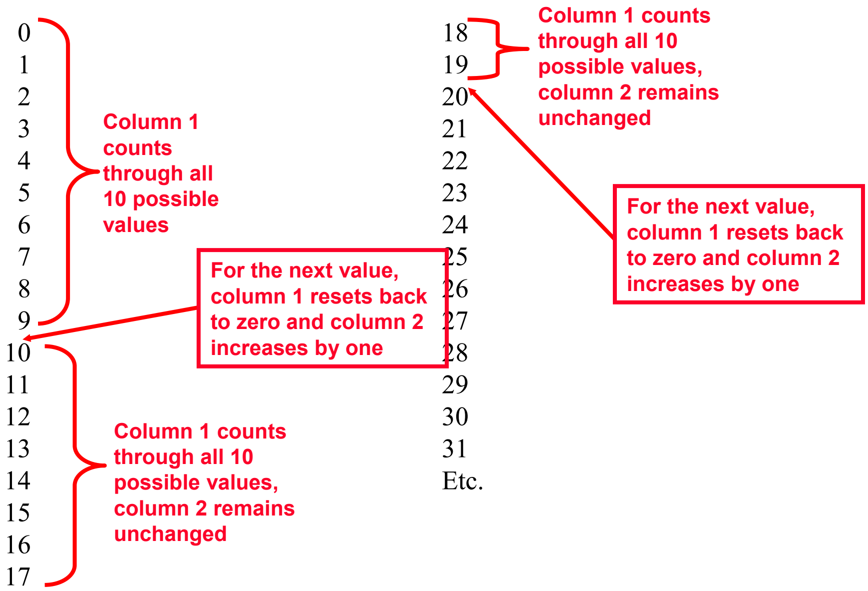
0
1
2
3
4
5
6
7
8
9
10
:

The number of digits is based on…the number of digits



The largest decimal value that can be represented by a single decimal digit is 9 = base(10) - 1

How Does Decimal Work?



0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

18
19
20
21
22
23
24
25
26
27
28
29
30
31
Etc.

Column 1 counts through all 10 possible values

For the next value, column 1 resets back to zero and column 2 increases by one

Column 1 counts through all 10 possible values, column 2 remains unchanged

Column 1 counts through all 10 possible values, column 2 remains unchanged

For the next value, column 1 resets back to zero and column 2 increases by one

Decimal

Base ten

Employs ten unique symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

Each digit can only take on the value from 0 – 9

- Once a column has traversed all ten values then that column resets back to zero (as does it’s right hand neighbours) and the column to it’s immediate left increases by one.

Recall: Computers Don’t Do Decimal!

Most parts of the computer work in a discrete state:

- On/off
- True/false
- Yes/No

These two states can be modeled with the binary number system

Binary

Base two

Employs two unique symbols (0 and 1)

Each digit can only take on the value 0 or the value 1

- Once a column has traversed both values then that column resets back to zero (as does it’s right hand neighbours) and the column to it’s immediate left increases by one.

Counting In Binary

Decimal value	Binary value	Decimal value	Binary value
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111

Why Bother With Binary?

1. Representing information
   - ASCII (American Standard Code for Information Interchange)
   - Unicode
2. It's the language of the computer

1. Representing Information: ASCII

Uses 7 bits to represent characters

Max number of possibilities = 2^7 = 128 characters that can be represented

e.g., 'A' is 65 in decimal or 01000001in binary. In memory it looks like this:

0	1	0	0	0	0	0	1

1. Representing Information: ASCII (2)

ASCII	Decimal	Binary
Invisible (control characters)	0 – 31	00000000 – 00011111
Punctuation, mathematical operations	32 – 47	00100000 – 00101111
Characters 0 - 9	48 - 57	00110000 – 00111001
Comparators and other miscellaneous characters : ; ? @	58 – 64	00111010 – 01000000
Alphabetic (upper case A - Z)	65 - 90	01000001 – 01011010
More miscellaneous characters [\ ] ^ _ '	91 – 96	01011011 – 01100000
Alphabetic (lower case a - z)	97 – 122	01100001 – 01111010
More miscellaneous characters { | } ~ DEL	123 – 127	01111011 - 01111111

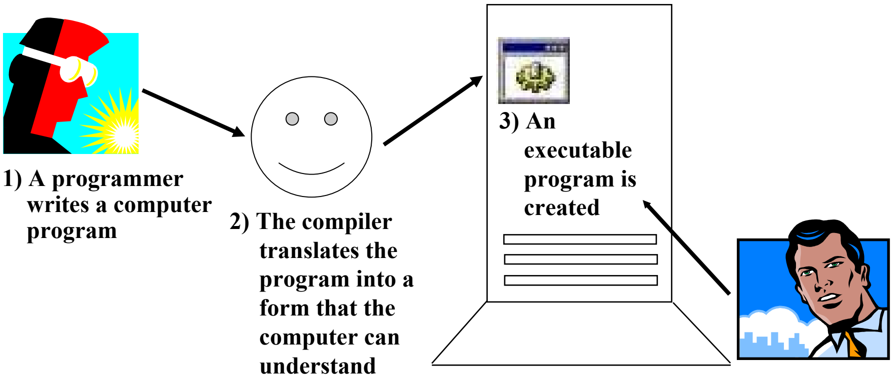
1. Representing Information: Unicode

Uses 16 bits (or more) to represent information

Max number of possibilities = 2^{16} = 65536 characters that can be represented (more if more bits are used)

2. Computer Programs

Binary is the language of the computer



1) A programmer writes a computer program

2) The compiler translates the program into a form that the computer can understand

3) An executable program is created

4) Anybody who has this executable installed on their computer can then run (use) it.

A Problem With Binary

1001 0100 1100 1100?

1001 0100 1100 0100?

1001 0100 1100 0011?

Binary is not intuitive for human beings and one string of binary values can be easily mistaken for another

A Shorthand For Binary: Octal

Machine language	**Octal value**
1010111000000	012700
1001010000101	011205

Octal

Base eight

Employs eight unique symbols (0 - 7)

Largest decimal value that can be represented by 1 octal digit = 7 = base(8) - 1

Table Of Octal Values

Decimal value	Octal value	Decimal value	Octal value
0	0	8	10
1	1	9	11
2	2	10	12
3	3	11	13
4	4	12	14
5	5	13	15
6	6	14	16
7	7	15	17

Problems With Binary: Got Worse As Computers Got More Powerful

1001 0100 1000 0000 1100 0100 0110 1010?

Or

1001 0100 1000 0000 1100 0100 0110 1011?

Hexadecimal: An Even More Compact Way Of Representing Binary Instructions

Machine	**Hexadecimal**
language	**value**
1010011000001	14C1
110000011100000	60E0

Example from 68000 Family Assembly Language by Clements A.

Hexadecimal (Hex)

Base sixteen

Employs sixteen unique symbols (0 – 9, followed by A - F)

Largest decimal value that can be represented by 1 hex digit = 15

Table of Hexadecimal Values

Decimal value	Hexadecimal value	Decimal value	Hexadecimal value
0	0	9	9
1	1	10	A
2	2	11	B
3	3	12	C
4	4	13	D
5	5	14	E
6	6	15	F
7	7	16	10
8	8	17	11

Summary (Decimal, Binary, Octal, Hex)

Decimal	Binary	Octal	Hex	Decimal	Binary	Octal	Hex
0	0000	0	0	8	1000	10	8
1	0001	1	1	9	1001	11	9
2	0010	2	2	10	1010	12	A
3	0011	3	3	11	1011	13	B
4	0100	4	4	12	1100	14	C
5	0101	5	5	13	1101	15	D
6	0110	6	6	14	1110	16	E
7	0111	7	7	15	1111	17	F

Arbitrary Number Bases

Base N

Employs N unique symbols

Largest decimal value that can be represented by 1 digit = Base (N) - 1

Converting Between Different Number Systems

Binary to/from octal

Binary to/from hexadecimal

Octal to/from hexadecimal

Decimal to any base

Any base to decimal

Converting Between Bases: Notation

$$\underbrace{DDDDD_{b1}}_{\text{Number 'DDDDD' in base 'b1'}} \longrightarrow \underbrace{?????_{b2}}_{\text{Convert to some number in another base 'b2'}}$$

Binary To Octal

Machine language	Octal value
1010111000000	012700
1001010000101	011205

Binary To Octal

3 binary digits equals one octal digit (remember $2^3=8$)

Form groups of three starting at the decimal

- For the integer portion start grouping at the decimal and go left
- For the fractional portion start grouping at the decimal and go right

e.g. $101\ 100_2 = ???_8$

$5\ \ 4_8$

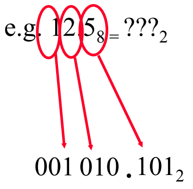
Octal To Binary

1 octal digit equals = 3 binary digits

Split into groups of three starting at the decimal

•For the integer portion start splitting at the decimal and go left

•For the fractional portion start splitting at the decimal and go right



e.g. $12.5_{8} = ???_{2}$

$001\ 010\ .\ 101_{2}$

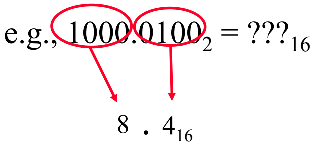
Binary To Hexadecimal

4 binary digits equals one hexadecimal digit (remember 2^4=16)

Form groups of four at the decimal

•For the integer portion start grouping at the decimal and go left

•For the fractional portion start grouping at the decimal and go right



e.g., $1000.0100_{2} = ???_{16}$

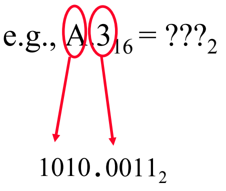
$8\ .\ 4_{16}$

Hexadecimal To Binary

1 hex digit equals = 4 binary digits

Split into groups of four starting at the decimal

- For the integer portion start splitting at the decimal and go left
- For the fractional portion start splitting at the decimal and go right

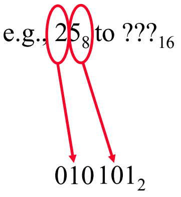


e.g., $A.3_{16} = ???_2$

1010.0011_2

Octal To Hexadecimal

Convert to binary first!

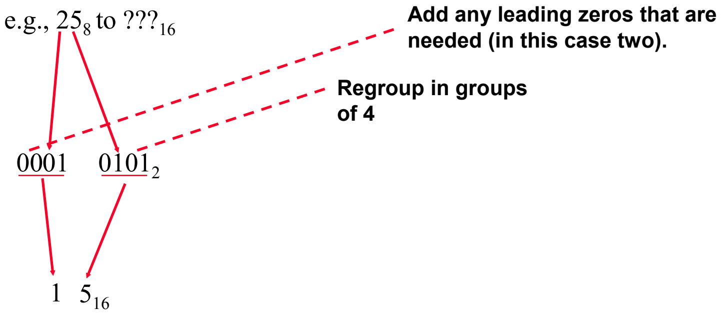


e.g., 25_8 to $???_{16}$

010101_2

Octal To Hexadecimal

Convert to binary first!



e.g., 25_8 to $???_{16}$

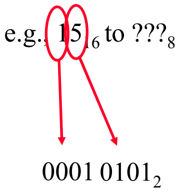
Add any leading zeros that are needed (in this case two).

Regroup in groups of 4

$0001 \quad 0101_2$

$1 \quad 5_{16}$

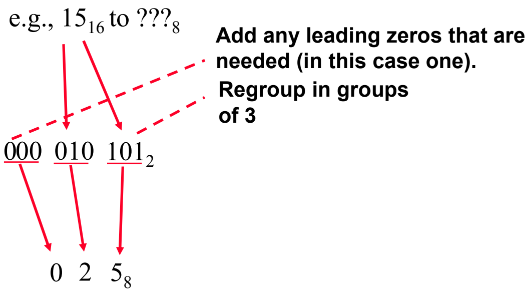
Hexadecimal To Octal



e.g., 15_{16} to $???_8$

$0001\ 0101_2$

Hexadecimal To Octal



e.g., 15_{16} to $???_8$

Add any leading zeros that are needed (in this case one).

Regroup in groups of 3

000 010 101_2

0 2 5_8

Decimal To Any Base

Split up the integer and the fractional portions

1) For the integer portion:

a. Divide the integer portion of the decimal number by the target base.
b. The remainder becomes the first integer digit of the number (immediately left of the decimal) in the target base.
c. The quotient becomes the new integer value.
d. Divide the new integer value by the target base.
e. The new remainder becomes the second integer digit of the converted number (second digit to the left of the decimal).
f. Continue dividing until the quotient is less than the target base and this quotient becomes the last integer digit of the converted number.

Decimal To Any Base (2)

2) For the fractional portion:

a. Multiply by the target base.

b. The integer portion (if any) of the product becomes the first rational digit of the converted number (first digit to the right of the decimal).

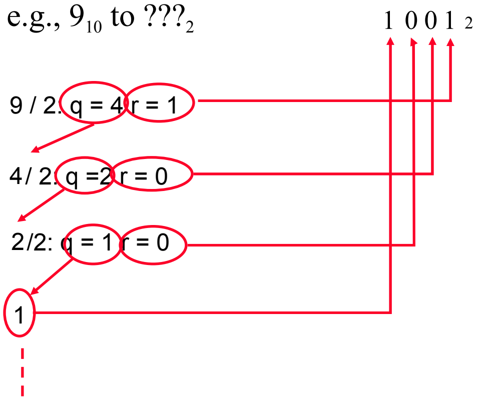
c. The non-rational portion of the product is then multiplied by the target base.

d. The integer portion (if any) of the new product becomes the second rational digit of the converted number (second digit to the right of the decimal).

e. Keep multiplying by the target base until either the resulting fractional part of the product equals zero or you have the desired number of places of precision.

Decimal To Any Base (2)

e.g., 9_{10} to $???_2$



$1\ 0\ 0\ 1_2$

9 / 2: q = 4 r = 1

4/ 2: q =2 r = 0

2/2: q = 1 r = 0

1

Stop dividing! (quotient less than target base)

Converting From A Number In Any Base To Decimal

Evaluate the expression: the base raised to some exponent[1], multiply the resulting expression by the corresponding digit and sum the resulting products.

Example:

1 0 -1 ← Position of digits (superscript)

$1\ 1.\ 0_2$ ← Number to be converted

Value in decimal = $(1 \times 2^{1}) + (1 \times 2^{0}) + (0 \times 2^{-1}) = (1 \times 2)+(1 \times 1)+0 = 3$

General formula:

3 2 1 0 -1 -2 -3 ← Position of digits

d7 d6 d5 d4. d3 d2 $d1_b$ ← Number to be converted

Value in decimal = $(digit7*b^{3}) + (digit6*b^{2}) + (digit5*b^{1}) + (digit4*b^{0}) + (digit3*b^{-1}) + (digit2*b^{-2}) + (digit1*b^{-3})$

[1] The value of this exponent will be determined by the position of the digit (superscript)

Any Base To Decimal (2)

e.g., 12_8 to $???_{10}$

Recall the generic formula:

1 0 ← Position of digits (superscript)

Decimal value (D.V.) = $d2\ d1_8$: ← Number to be converted

$= (d2*8^{1}) + (d1*8^{0})$

Any Base To Decimal (2)

e.g., 12_8 to $???_{10}$

1 0 ← Position of the digits

1 2 ← Number to be converted

Base = 8

$$\text{Value in decimal} = (1*8^1) + (2*8^0)$$
$$= (1*8) + (2*1)$$
$$= 8 + 2$$
$$= 10_{10}$$

Addition In Binary: Five Cases

Case 1: sum = 0, no carry out

$$\begin{array}{r} 0 \\ +\ \underline{0} \\ 0 \end{array}$$

Case 2: sum = 1, no carry out

$$\begin{array}{r} 0 \\ +\ \underline{1} \\ 1 \end{array}$$

Case 3: sum = 1, no carry out

$$\begin{array}{r} 1 \\ +\ \underline{0} \\ 1 \end{array}$$

Case 4: sum 0, carry out = 1

$$\begin{array}{rr} {\scriptstyle 1} & \\ & 1 \\ + & \underline{1} \\ 1 & 0 \end{array}$$

1 + 1 = 2 (in decimal)
= 10 (in binary)

Addition In Binary: Five Cases (2)

Case 5: Sum = 1, Carry out = 1

$$\begin{array}{rr} & {}^{1}\\ & 1\\ & 1\\ + & \underline{1}\\ 1 & 1 \end{array}$$

1 + 1 + 1 = 3 (in decimal)
= 11 (in binary)

Subtraction In Binary *Using Borrows* (4 cases)

Case 1:

$$\begin{array}{rr} & 0\\ - & \underline{0}\\ & 0 \end{array}$$

Case 2:

$$\begin{array}{rr} & 1\\ - & \underline{1}\\ & 0 \end{array}$$

Case 3:

$$\begin{array}{rr} & 1\\ - & \underline{0}\\ & 1 \end{array}$$

Case 4:

$$\begin{array}{rr} & \overset{0}{\cancel{1}}\ \overset{2}{\cancel{0}}\\ - & \underline{1}\\ & 1 \end{array}$$

The amount that you borrow equals the base

- Decimal: Borrow 10
- Binary: Borrow 2

Overflow: A Real World Example

You can only represent a finite number of values

Overflow: Binary

Occurs when you don't have enough bits to represent a value (“wraps around” to zero)

Binary (1 bit)	Value
0	0
1	1

0	0
1	1
:	:

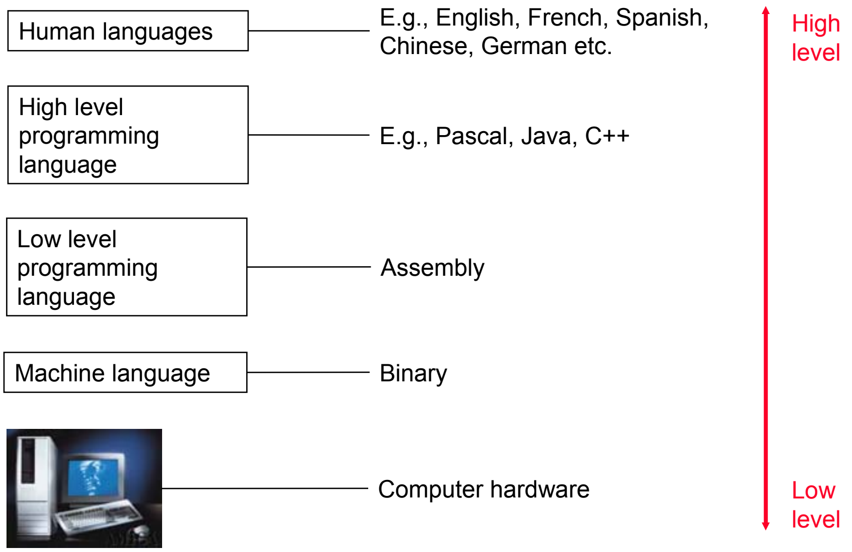
Binary (2 bits)	Value
00	0
01	1
10	2
11	3

00	0
01	1
10	2
11	3
:	:

Binary (3 bits)	Value
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

000	0
001	1
:	:

Terminology: High Vs. Low Level



Human languages — E.g., English, French, Spanish, Chinese, German etc.

High level programming language — E.g., Pascal, Java, C++

Low level programming language — Assembly

Machine language — Binary

Computer hardware

High level

Low level

You Should Now Know

- What is meant by a number base.
- How binary, octal and hex based number systems work and what role they play in the computer.
- How to/from convert between non-decimal based number systems and decimal.
- How to perform simple binary math (addition and subtraction).
- What is overflow, why does it occur and when does it occur.
- What is the difference between a high and low level programming language.