The Cup-Length of Stiefel and Projective Stiefel Manifolds

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Abstract

This paper discusses some generalities about cup-length of manifolds and then gives an explicit formula for the \( \mathbb{Z}_2 \)-cup-length of the Stiefel manifolds \( V_{n,r} \), as well as strong lower bounds for the \( \mathbb{Z}_2 \)-cup-length of the projective Stiefel manifolds \( X_{n,r} \), for all \( 1 \leq r \leq n-1 \). A simple formula relating the two cases is given.

We also show the consequences for the Lyusternik-Shnirel’man category, as well as a family of interesting number theoretical identities that arise from the \( V_{n,r} \) calculations.


1 Introduction

Let \( R \) be a commutative ring with 1. We recall that the \( R \)-cup-length \( \cup_R(X) \) of a compact path-connected topological space \( X \) is the largest of all integers \( c \) such that there exist reduced cohomology classes \( a_1, \ldots, a_c \in H^c(X; R) \) with their cup product
\[
a_1 \cup \cdots \cup a_c = a_1 \cdots a_c \neq 0.
\]

In this note, we will concentrate on the two cases \( V_{n,r} \) and \( X_{n,r} \), respectively the real and real projective Stiefel manifolds (defined in the following sections). A short preliminary Section 2 gives a couple of results associated to the \( R \)-cup-length, which is simply written \( \cup(V_{n,r}) \) for all \( n, r \), and a few examples are given. Some purely number theoretical (and perhaps remarkable) identities arising from this formula are stated and proved.

In Section 4, a similar discussion is carried out to give a lower bound for \( \cup(X_{n,r}) \), which is related by

\[
1 + \dim(M)/2 \leq \cup(M) + 1 \leq \beta(M),
\]

where \( \beta(M) \) is the minimal number of smoothly embedded balls needed to cover a closed symplectic manifold \( (M, \omega) \).

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2 Preliminary remarks about \( R \)-cup-length

In this section, we first recall some material in Hatcher [6], Chapter 3. In particular, for a closed connected \( n \)-dimensional manifold \( M \), the notions of \( R \)-orientability and fundamental class \( [M] \in H_n(M; R) \) are defined there as well as the bilinear pairing
\[
T : H^k(M; R) \otimes H^{n-k}(M; R) \to R
\]
given by \( T(\alpha \otimes \beta) = \langle \alpha \cup \beta \rangle[M] \), where \( \alpha \in H^k(M; R), \beta \in H^{n-k}(M; R) \). Poincaré duality for the \( R \)-orientable
manifold is then expressed by [6], Proposition 3.38: The cup-product pairing $T$ is non-singular if $R$ is a field, or if $R = \mathbb{Z}$ and torsion is factored out.

For applications to cup-length it will be convenient to take $R$ to be a field, so we shall henceforth denote it $F$. It will also be convenient to use the natural isomorphism $H^n(M; F) \approx \text{hom}(H_n(M; F), F)$ (6, p. 198) induced by the natural surjection (6, p. 191)

$$h : H^n(C; G) \twoheadrightarrow \text{hom}(H_n(C; G)),$$

and define the “top” cohomology class $\xi_M \in H^n(M; F)$, dual to $[M]$, by $h(\xi_M)([M]) = 1$. We next give two corollaries of the above proposition, the first being a variant of Corollary 3.39 in [6] and the second an application to cup-length that will be useful in proving the main theorems of Sections 3, 4.

**Corollary 2.1.** Let $0 \neq \alpha \in H^k(M; F)$. Then there exists $\beta \in H^{n-k}(M; F)$ such that $\alpha \cup \beta = \xi_M$.

*Proof.* Since $H^k(M; F)$ is a vector space over $F$ and $\alpha \neq 0$, there exists a homomorphism $\varphi : H^k(M; F) \to F$ with $\varphi(\alpha) = 1$. Given any such homomorphism $\varphi$, the fact that $T$ is non-singular means by definition that $\varphi(x) = T(x \otimes \beta)$ for some $\beta \in H^{n-k}(M; F)$. Then $\alpha \cup \beta)[M] = T(\alpha \otimes \beta) = \varphi(\alpha) = 1$ implies $\alpha \cup \beta = \xi_M$. □

**Lemma 2.2.** If $\alpha \in H^d(M; F)$ is a class of maximal cup-length, then $d = n$. In particular, if $F = \mathbb{Z}_2$, then $\alpha = \xi_M$.

*Proof.* If $d < n$, then Corollary 2.1 shows that $\alpha$ cannot have maximal cup-length. □

We remark that while Poincaré duality is of course treated in many texts, it is not clear that its simple application to cup-length in Lemma 2.2 is explicitly stated in the literature. It is implicitly assumed in [7]. Proof of Theorem 1.1. For a space $X$ that is not a manifold, Lemma 2.2 does not hold, an elementary counterexample being $S^m \vee \mathbb{R}P^n$ with $m > n$.

### 3 Cup-length of the Stiefel manifolds

Consider the real Stiefel manifold $V_{n,r}$ of orthonormal $r$-frames in $\mathbb{R}^n$, $1 \leq r \leq n - 1$. It is well known to be a smooth path-connected manifold, indeed a homogeneous space of dimension $d = d_{n,r} = nr - \binom{r+1}{2}$.

Its cohomology and the action of the Steenrod squares are well known and go back to Borel, [2], and Steenrod-Epstein, [12]. For our purposes we can summarize the cohomology as the algebra over $\mathbb{Z}_2$ with generators $x_i \in H^i(V_{n,r})$, $n - r \leq i \leq n - 1$ and the only non-trivial cup-products arising from $x_i^2 = x_{2i}$, $2i \leq n - 1$. After a couple of numerical definitions we give an explicit formula for the $\mathbb{Z}_2$-cup-length of $V_{n,r}$, which we shall write $\text{cup}(V_{n,r})$. First let

$$n - 1 = \sum_{j=1}^{a_{n-1}} 2^{a_j}, \quad a_1 > a_2 > \ldots > a_{a_{n-1}} \quad (1)$$

be the binary expansion of $n - 1$. Here $a_{n-1}$ denotes, as usual, the number of 1’s in this binary expansion. Next, for $k \geq 2$, define

$$b_k = \max\{m : 2^m \leq \frac{n-1}{k-1}, \quad k \geq 2\} = \left\lfloor \log_2 \left(\frac{n-1}{k-1}\right) \right\rfloor. \quad (2)$$

Using (1) and (2), we define

$$\ell'(n, r) = n - 1 + \sum_{j=1}^{a_{n-1}} a_j \cdot 2^{a_j - 1} - \sum_{k=2}^{n-r} 2^{b_k}. \quad (3)$$

We now give three examples with $n = 23$. Here $n - 1 = 2^4 + 2^2 + 2^1$, so $a_1 = 4, a_2 = 2, a_3 = 1$, and one readily finds $b_2 = 4, b_3 = 3, b_4 = b_5 = b_6 = 2, b_7 = \ldots = b_{12} = 1, b_{13} = 0$. The computation is given for Example A, the others being similar.

**Example A:** $\ell'(23, 10) = 22 + 4 \cdot 2^3 + 2 \cdot 2^1 + 1 \cdot 2^0 - 2^4 - 2^3 - 2^2 - 2^1 - 1 = 10$.

**Example B:** $\ell'(23, 18) = 27$.

**Example C:** $\ell'(23, 21) = 43$.

We next define $\ell'(n, r)$, starting with the preliminary definitions

$$m = \lfloor \log_2(n-1) \rfloor,$$

$$r_0 = \left\lfloor \frac{n-1}{2} \right\rfloor, \quad r_1 = \left\lfloor \frac{3(n-1)}{4} \right\rfloor, \quad r_2 = \left\lfloor \frac{7(n-1)}{8} \right\rfloor, \ldots. \quad (4)$$

Elementary calculations then show that

$$r_0 < r_1 < r_2 < \ldots < r_m = \left\lfloor \frac{(2^{m+1} - 1)(n-1)}{2^{m+1}} \right\rfloor = n-1. \quad (5)$$

For convenience, we also set $r_{-1} = 0$.

As before, let $n \geq 2$, $1 \leq r \leq n - 1$, the integers $r_i$ be defined as in (4) above and $r_{q-1} < r_i < r_q$ (for a unique $q$). Then we can define

$$\ell'(n, r) = 2^q r - \sum_{i=1}^{q} 2^{i-1} \left\lfloor \frac{(2^i - 1)(n-1)}{2^i} \right\rfloor. \quad (5)$$

We shall also define

$$\ell''(n, r) = n - 1 - (n - 1 - r) \cdot 2^q + \sum_{j=1}^{a_{n-1}} \min\{a_j, q\} \cdot 2^{a_j - 1}, \quad (6)$$

a definition which uses slightly less machinery than its predecessors.
Theorem 3.1. One has
\[ \text{cup}(V_{n,r}) = \ell(n,r) = \ell'(n,r) = \ell''(n,r). \]

Remark 3.2. In the stable range 2r ≤ n, i.e. q = 0, one has \( \text{cup}(V_{n,r}) = r \) (see also [10]).

The proof given later in Section 5 for the equality \( \text{cup}(V_{n,r}) = \ell(n,r) \) of Theorem 3.1 starts from \( r = n - 1 \) and uses downward induction in \( H^r(V_{n,r}) \). It is possible to prove the equality \( \text{cup}(V_{n,r}) = \ell'(n,r) \) starting from \( r = 1 \) and using upward induction in \( H^r(V_{n,r}) \); however the equality \( \ell(n,r) = \ell'(n,r) = \ell''(n,r) \) is purely number theoretical and we therefore give a purely number theoretical proof of this in Section 5. To illustrate how disparate the two sums \( \ell(n,r) \), \( \ell'(n,r) \) seem, we go back to Example C above, of \( V_{23,21} \). In Theorem 3.1, since the binary expansion \( 22 = 2^4 + 2^2 + 1^1 \) determines the first summation, and \( k = 2 \) in the second summation so we use \( b_2 = \lfloor \log_2(22/1) \rfloor = 4 \), whence
\[ \ell(23,21) = 22 + 4 \cdot 2^3 + 2 \cdot 2^1 + 1 \cdot 2^0 - 2^4 = 43. \]

On the other hand, since \( 20 = r_2 < 21 = r_3 \) implies \( q = 3 \), this gives
\[ \ell'(23,21) = 8 \cdot 21 - 3 \sum_{i=1}^{3} 2^{i-1} \cdot \left( \frac{2^i - 1}{2} \right) = 168 - 11 - 34 - 80 = 43. \]

Theorem 3.1 and the Florenz-Eisenhower inequality give the following for the Lyusternik-Shnirel’man category.

Corollary 3.3. One has \( \text{cat}(V_{n,r}) \geq \ell(n,r) = \ell'(n,r) = \ell''(n,r) \).

We observe, that for \( n \geq 2r \), Nishimoto [10] proved that \( \text{cat}(V_{n,r}) = r \).

4 Cup-length of the projective Stiefel manifolds

In this section, we concentrate on the manifold \( X_{n,r} \) (\( r < n \)), the projective Stiefel manifold, which is obtained from the Stiefel manifold \( V_{n,r} \) of orthonormal \( r \)-frames in \( \mathbb{R}^n \) as the quotient space, by identification of any frame \((v_1, \ldots, v_r)\) with the frame \((-v_1, \ldots, -v_r)\) ([5]).

Let \( \xi_{n,r} \) be the real line bundle associated to the obvious double covering \( V_{n,r} \to X_{n,r} \). By [5], for the \( \mathbb{Z}_2 \)-cohomology ring of \( X_{n,r} \), we have
\[ H^r(X_{n,r}) = \mathbb{Z}_2[y]/(y^N) \otimes V(y_{n-r}, \ldots, y_{N-2}, y_N, \ldots, y_{n-1}), \]
where \( y \in H^1(X_{n,r}) \) is the first Stiefel-Whitney class \( w_1(\xi_{n,r}) \), \( y_j \in H^j(X_{n,r}) \),
\[ N = \min\{j; j \geq n - r + 1, \binom{n}{j} \equiv 1 \pmod{2}\} \]
and \( V(y_{n-r}, \ldots, y_{N-2}, y_N, \ldots, y_{n-1}) \) is the \( \mathbb{Z}_2 \)-vector space, which has the monomials \( \prod_{i=n-r}^{n-1} y_i t_i \), with \( i \neq N - 1 \) and \( t_i \in \{0, 1\} \), as \( \mathbb{Z}_2 \)-basis (\( N \) can be easily calculated for any \( X_{n,r} \)). The dimension of \( X_{n,r} \) is also \( d_{n,r} \) (defined in Section 3).

Recalling the definition (2) of \( b_k \), we now have the following theorem.

Theorem 4.1. One has
\[ \text{cup}(X_{n,r}) \geq \mathcal{L}(n,r) := \text{cup}(V_{n,r}) + N - 1 - 2^{bn}. \]

Since \( \text{cup}(V_{n,r}) \) has already been explicitly calculated in Section 3, indeed via ([3], [5], or [6]), Theorem 4.1 gives an explicit lower bound for \( \text{cup}(X_{n,r}) \).

As an immediate corollary of Theorem 4.1 we have

Corollary 4.2. Let \( X_{n,r} \) (\( 1 \leq r < n \)) be the projective Stiefel manifold. Then
\[ \text{cat}(X_{n,r}) \geq \text{cup}(V_{n,r}) + N - 1 - 2^{bn}. \]

It seems very likely that the strong result \( \text{cup}(X_{n,r}) = \mathcal{L}(n,r) \) is true, but to date neither a proof nor a counterexample (with the help of a computer program developed by the authors) has been found. It is hoped to address this question in a forthcoming note. The next proposition gives a few partial results where equality holds.

Proposition 4.3. The result \( \text{cup}(X_{n,r}) = \mathcal{L}(n,r) \) is true
(a) in the stable range (so here \( \text{cup}(X_{n,r}) = r + N - 2 \)),
(b) if \( n = 2^m \) (so here \( \text{cup}(X_{n,r}) = \ell(n,r) + N - 2 = \ell(n,r) + n - 2 \)),
(c) if \( N = 2 \),
(d) \( \text{cup}(X_{2^r-1,2^r-1}) = 2^r - 2 \).

5 Proofs of the main results

First, we give the proof of \( \text{cup}(V_{n,r}) = \ell(n,r) \) in Theorem 3.1. We prove this in four steps, using the notation \( v_2(q) = p \) for the standard 2-valuation of \( q \), i.e. \( q \) is divisible by \( 2^p \) but not by \( 2^{p+1} \). The top cohomology class, denoted \( \xi_M \) (where now \( M = V_{n,r} \)) in Section 2, will here be denoted simply by \( X \). According to Lemma 2.2, the cup-length is realized by the class \( X \), so one has to look at the relations in \( H^r(V_{n,r}) \) to see how they can give a presentation that maximizes the cup-length of \( X \).

\( (A) \) \text{cup}(V_{2m,2m-1}) = m \cdot 2^{m-1}. \) From Section 3, the top cohomology class of \( V_{2m,2m-1} \) equals \( X := x_1 \cdot x_2 \cdots x_{2^{m-1}} \). This product has length \( 2^{m-1} \) but the cup-length is larger, since some of these classes are decomposable, e.g. (again using Section 3) \( x_2 = x_1^2 \), \( x_4 = x_1^4 \), \( x_6 = x_3^3 \), \( x_8 = x_4^2 \). A little careful counting shows that in \( \{x_1, \ldots, x_{2^{m-1}}\} \), after this decomposition,
exactly $2^{m-1}$ have length 1 (i.e. $x_k$ with $\nu_2(k) = 0$),
exactly $2^{m-2}$ have length 2 ($\nu_2(k) = 1$), etc. Also no
further classes are decomposable. Thus the length after
decomposition equals
\[ 1 \cdot 2^{m-1} + 2 \cdot 2^{m-2} + 4 \cdot 2^{m-3} + \ldots + 2^{m-1} \cdot 1 = m \cdot 2^{m-1}. \]

(B) $\cup(V_{2m+1,2m}) = 2^m + m \cdot 2^{m-1}$. This is a
corollary of (A), since the top class $X$ now has one
additional term $x_{2m} = x_{2m}^1$.

(C) Recalling (1), one now finds
\[
\cup(V_{n,n-1}) = n - 1 + \sum_{j=1}^{a(n-1)} a_j \cdot 2^{a_j-1}.
\]

To verify this, one simply writes
\[
X = \langle x_1 \cdot x_2 \cdots x_{2^{n-1}} \cdot (x_{2^{n+1}} + x_{2^{n+2}}) \cdot (x_{2^{n+1}} + x_{2^{n+1}+1} + x_{2^{n+1}+2}) \cdots \rangle.
\]
Since $a_1 > a_2$, one has
\[
\nu_2(k) = \nu_2(k - 2^{a_1}), 2^{a_1} + 1 \leq k \leq 2^{a_1} + 2^{a_2}.
\]
Thus, from (B), the first bracketed term in the above
expression for $X$ has cup-length $2^{a_1} + a_1 \cdot 2^{a_1-1}$, the
second bracketed term has cup-length $2^{a_2} + a_2 \cdot 2^{a_2-1}$, etc.
Adding these gives the assertion.

(D) We now complete the proof of Theorem 3.1 by
downward induction on $r$. For $r = n - 1$, Theorem 3.1 has no $2^b$
terms, so reduces to (C), giving the start for the
induction. Suppose then it holds for $r = n - s$, $s \geq 1$, so we have $n - r = s$ and Theorem 3.1 reads
\[
\cup(V_{n,r}) = n - 1 + \sum_{j=1}^{a(n-1)} a_j \cdot 2^{a_j-1} - \sum_{k=2}^{s} 2^{b_k}.
\]

Passing to $n - r = s + 1$, the top class $X$ loses $x_s$
(length 1) and its cup-length is thereby shortened by
the further changes $x_s^2$ to $x_{2s}$, $x_s^3$ to $x_{2^2}$, \ldots, $x_s^q$
to $x_{2^{q-1}}$, where $t$ is the largest integer with $s \cdot 2^t \leq n - 1$, or equivalently $2^t \leq \frac{n-1}{s}$. Then, by the definition of $b_k$, we have $t = b_{k+1}$. The net loss in cup-length is thus $1 + (1 + 2 + 4 + \ldots + 2^{b_{k+1} - 1}) = 2^{b_{k+1}}$, thereby completing the inductive step.

Second, we give the proof of $\ell(n,r) = \ell'(n,r)$ in
Theorem 3.1. This proof proceeds by induction on $r$. For $r = 1$, the first part of the proof shows that $\ell(n,1) = \cup(V_{n,1}) = \cup(S_{n-1}) = 1$. Since $r_0 \geq 1$ and $r_{-1} = 0$, we see that $q = 0$, so $\ell'(n,1) = 1 = \ell(n,1)$.

For the inductive step, the induction hypothesis gives
\[
\ell(n,r) = \ell(n,r-1) + 2^{b_{m-r+1}}
= \ell'(n,r-1) + 2^{b_{m-r+1}}
= \ell'(n,r) - 2^q + 2^{b_{m-r+1}},
\]
so it suffices to show that $q = b_{n-r+1}$ for $1 \leq r \leq n - 1$.

Since $r_{q-1} < r \leq r_q$, we have
\[
r \geq r_{q-1} + 1 \geq \frac{(2^q - 1)(n-1)}{2^q} + 1 = n - \frac{n-1}{2^q},
\]
\[
r \leq r_q \leq \frac{(2^{q+1} - 1)(n-1)}{2^{q+1}} + 1 = n - \frac{n-1}{2^{q+1}}.
\]
Rearranging terms yields
\[
(n-1) \cdot 2^{-(q+1)} < n - r \leq (n-1) \cdot 2^{-q},
\]
or equivalently
\[
2^q \leq \frac{n-1}{n-r} < 2^{q+1}.
\]
Finally, taking the base 2 logarithms, we obtain
\[
q \leq \log_2 \left( \frac{n-1}{n-r} \right) < q + 1,
\]
and hence $q = \left[ \log_2 \left( \frac{n-1}{n-r} \right) \right] = b_{n-r+1}$.

Third, we prove that $\ell'(n,r) = \ell''(n,r)$, thus
completing the proof of Theorem 3.1. For simplicity of later
notation, we write the binary representation of $n-1$ in
an alternative way as
\[
n-1 = \sum_{j=0}^{m-1} n_j 2^j, m = 1, n_j \in \{0,1\} \text{ for } 0 \leq j \leq m-1.
\]
This representation is related to (1) as follows:
\[
\begin{align*}
\alpha_1 &= m, \\
\alpha_{\ell(n-1)} &= \min \{ i \mid 0 \leq i \leq m, n_i \neq 0 \}, \\
n_j &= \begin{cases} 1 & \text{when } j \in \{a_1, a_2, \ldots, a_{\ell(n-1)}\}, \\ 0 & \text{otherwise}. \end{cases}
\end{align*}
\]
Fix $i \in \{1, 2, \ldots, m+1\}$. Then
\[
r_{i-1} = \left\lfloor \frac{(2^i - 1)(n-1)}{2^i} \right\rfloor = \left\lfloor \frac{n-1 - \frac{n-1}{2^i}}{2^i} \right\rfloor = \left\lfloor \frac{n-1 - \frac{n-1}{2^i}}{2^i} \right\rfloor.
\]

To determine the floor function of $(n-1)/2^i$, write
\[
\frac{n-1}{2^i} = \frac{1}{2^i} \sum_{j=0}^{m-1} n_j 2^j + \sum_{j=i}^{m} n_j 2^{j-i}.
\]
Now
\[
\frac{1}{2^i} \sum_{j=0}^{m-1} n_j 2^j \leq \frac{1}{2^i} \sum_{j=0}^{m-1} 2^j = \frac{2^i - 1}{2^i} < 1,
\]
so \[ \left[ \frac{n - 1}{2^i} \right] = \sum_{j=1}^{m} n_j 2^{j-i} . \]

It follows that
\[
\ell_i(n) = (n - 1 - \sum_{j=i}^{m} n_j 2^{j-i}) 2^{i-1} = (n - 1) 2^{i-1} - \sum_{j=i}^{m} n_j 2^{j-1},
\]
and hence
\[
\ell'(n, r) = r 2^n - \sum_{i=1}^{q} \ell_i(n) = r 2^n - (n - 1) \sum_{i=0}^{q-1} 2^i = 2^n - 1
\]
and
\[
\sum_{i=1}^{q} \sum_{j=i}^{m} n_j 2^{j-1} = \sum_{j=1}^{m} n_j 2^{j-1} + \sum_{j=2}^{m} n_j 2^{j-1} + \ldots + \sum_{j=q}^{m} n_j 2^{j-1} = n_1 2^0 + n_2 2^1 + \ldots + (q-1)n_{q-1} 2^{q-2} + q \sum_{j=q}^{m} n_j 2^{j-1} = \sum_{i=1}^{q-1} q n_j 2^{j-1} + \sum_{j=q}^{m} n_j 2^{j-1} = \sum_{j=1}^{m} \min\{j, q\} n_j 2^{j-1}.
\]

Thus,
\[
\ell'(n, r) = r 2^n - (n - 1)(2^n - 1) + \sum_{j=1}^{m} \min\{j, q\} n_j 2^{j-1} = (n - 1) - (n - 1 - r) 2^n + \sum_{j=1}^{a(n-1)} \min\{a_j, q\} 2^{a_j - 1} = \ell''(n, r).
\]

\[\square\]

**Proof of Theorem 4.1**

For convenience, we write \( H^*(X_{n,r}) = A \otimes V \), where all cohomology and tensor products are over \( \mathbb{Z}_2 \), \( V = \mathbb{V}(y_{n-r}, \ldots, y_{N-2}, y_N, \ldots, y_{n-1}) \) (as in Section 4), and \( A = \mathbb{Z}_2[y]/(y^N) \). We shall also write \( T_1 \) for the ideal in \( H^*(X_{n,r}) \) generated by \( y \), and similarly \( T_2 \) for the ideal generated by \( y^2 \). Formulæ for the Steenrod squaring operations \( Sq^t(y) \) will be needed, these are due to Gitler and Handel [5], Antoniano [1], and later again given (with a few misprints in [1] corrected) in [3]. We state them once again here in the slightly more convenient form \( Sq^t(y) \) (the older versions give \( Sq^t(y_{q-1}) \)):

\[
Sq^t(y) = \sum_{k=0}^{i} A_k y^{k} y_{q+i-k} + \sum_{0 \leq k < j \leq q} B_{k,j} y^{q+k+i-N-j} y_{N+j-k-1} + \epsilon y^{q+i},
\]

where \( \epsilon = \left( \frac{n}{q+1+2^{t-1}-N} \right) \frac{q+1+2^{t-1}-N}{q-1} \) if \( t := \nu_2(N) \geq 3 \) and \( \epsilon = 0 \) if \( t < 3 \),

\[
A_k = A(q, i, k) = \binom{q-k}{n} \binom{n}{k},
\]

and

\[
B_{k,j} = B(q, i, k, j) = \binom{n}{q+1} \binom{N-1-k}{j-k} \binom{q+1-N}{i-j} \binom{n}{k}.
\]

Just like the calculations of cup-length for the Stiefel manifolds had to take account of relations arising from cup-squares \( x_i x_j \), the calculations for the cup-length of the projective Stiefel manifolds must take account of the relations arising from \( y_q^2 \) (or iterations \( y_q^{2i} \)). These are now much more complicated due to the presence of the first Stiefel-Whitney class \( y \). However, they can be handled using \( y_q^2 = Sq^q(y) \). The AGH (Antonio, Gitler, Handel) formulæ become:

\[
Sq^q(y_q) = \sum_{k=0}^{q} A_k y^{k} y_{2q-k} + \sum_{0 \leq k < j \leq q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1} + \epsilon y^{2q},
\]

where \( \epsilon = \left( \frac{n}{q+1+2^{t-1}-N} \right) \frac{q+1+2^{t-1}-N}{q-1} \) if \( t \geq 3 \) and \( \epsilon = 0 \) if \( t < 3 \),

\[
A_k = A(q, q, k) = \binom{q-k}{n} \binom{n}{k},
\]

and

\[
B_{k,j} = B(q, q, k, j) = \binom{n}{q+1} \binom{N-1-k}{j-k} \binom{q+1-N}{i-j} \binom{n}{k}.
\]

We shall carefully treat the presence of \( y \) by looking at three cases. In all cases, as usual, \( n - r \leq q \leq n - 1, q \neq N - 1 \). The first case is when \( 2q = N - 1, \)
the above formula would give $y_q^2 \in I_1$ (since there is no class $y_{N-1}$). The second case is when $2q \geq n$, similarly, no class $y_{2q}$ exists gives $y_q^2 \in I_1$. The third case is when $2q \leq n - 1$. The three cases will be first stated in the following Lemmas 5.1, 5.2, 5.3 and then proved.

**Lemma 5.1.** The case $2q = N - 1$ cannot occur.

**Lemma 5.2.** One has

$$y_q^2 = \begin{cases} y_{2q} \pmod{I_1}, & 2q \leq n - 1 \\ 0 \pmod{I_1}, & 2q \geq n \end{cases} \ (11)$$

**Lemma 5.3.** If $2q \geq n$, then $y_q^2 \in I_2$.

Proof of Lemma 5.1. Recall that

$$N = \min\{ j : j \geq n - r + 1, \binom{n}{j} \equiv 1 \pmod{2} \}.$$ 

If $N = n - r + 1$, then $N - 1 \leq 2q < 2q$ since $1 \leq n - r \leq q$. So suppose that $N > n - r + 1$. Then $N - 1 \geq n - r + 1$, so $\binom{n}{N-1} \not\equiv 1 \pmod{2}$ by the minimality condition on $N$. We have

$$N\binom{n}{N} = \frac{n!}{N!(n-N)!} = \frac{n!}{n!} \frac{(N-1)!(n-N)!}{(N-1)!(n-(N-1))!} = \binom{n}{n-N-1}.$$ 

Since $\binom{n}{N-1}$ is even, $N\binom{n}{N}$ must be even, and since $\binom{n}{N}$ is odd by the definition of $N$, this forces $N$ to be even. It follows that $N - 1 \neq 2q$.

Proof of Lemma 5.2. First note that the term $ey^{2q}$ in (8) equals $0 \pmod{I_1}$. Second, observe that

$$\sum_{k=0}^q A_k y^k y_{2q-k} \equiv A_0 y_{2q} \ (12)$$

where we have used (9) to evaluate $A_0$ and are also using Lemma 5.1 by implicitly assuming that $y_{2q}$ exists. Thus the proof of Lemma 5.2 will be completed by showing that in (8)

$$\sum_{0 \leq k < j \leq q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1} \equiv 0 \pmod{I_1}.$$ 

To prove this claim, first recall that $q \neq N - 1$ when $y_q \in H^*(X_{n,r})$. Second, using (10) for $B_{k,j}$ together with $j - k = 2q + 1 - N$ for any $y^0$ terms in the second sum in (8), we find

$$B_{k,j} = \binom{n}{q+1} \binom{N-1-k}{2q+1-N} \binom{q+1-N}{N-(q+1)-k} \binom{n}{k}.$$ 

Since $q + 1 \neq N$ as noted above, either $q + 1 < N$ or $q + 1 > N$. In the former case, since also $n - r + 1 \leq q + 1$, the definition of $N$ implies that the first binomial coefficient in (13) equals 0. In the latter case the third binomial coefficient equals 0 since $q + 1 - N > 0$ whereas $N - (q + 1) - k < 0$ (recall that for integers $a > 0, b < 0$, one has $\binom{a}{b} = 0$). The claim and thereby also Lemma 5.2 are thus proved.

Proof of Lemma 5.3. Since $2q \geq n \geq N$, we have $y^{2q} = 0$ so the $ey^{2q}$ term in (8) vanishes. Next, for

$$\sum_{k=0}^q A_k y^k y_{2q-k}$$

in (8), the first term $(k = 0)$ vanishes since $2q \geq n$ and there is no $y_{2q}$ in the cohomology. Since $2q - 1 \geq n - 1$ the $y_{2q-1}$ in the second term $(k = 1)$ also vanishes unless $2q - 1 = n - 1$, i.e. $n = 2q$. But then

$$\binom{n}{1} = 0 \pmod{2}$$

and $A_1 = 0$. This proves that

$$\sum_{k=0}^q \binom{n}{k} y^{2q-k} \in I_2.$$ 

Next we claim the terms in $y^{2q} y_{2q}$ in

$$\sum_{0 \leq k < j \leq q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1}$$

vanish. To prove this claim, first recall that $q \neq N - 1$ here. Second, using (10) for $B_{k,j}$ together with $j - k = 2q + 1 - N$ for $y^0$ in (8), we find

$$B_{k,j} = \binom{n}{q+1} \binom{N-1-k}{2q+1-N} \binom{q+1-N}{N-(q+1)-k} \binom{n}{k}.$$ 

Since $q + 1 \neq N$ as noted above, either $q + 1 < N$ or $q + 1 > N$. In the former case, since also $n - r + 1 \leq q + 1$, the definition of $N$ implies that the first binomial coefficient in (11) equals 0. In the latter case the third binomial coefficient equals 0 since $q + 1 - N > 0$ whereas $N - (q + 1) - k < 0$ (recall that for integers $a > 0, b < 0$, one has $\binom{a}{b} = 0$). The claim is thus proved.

Now we turn to the $y^{2q} y_{2q-1}$ term in the $B_{k,j}$ summation and show that it also vanishes. Since we now have $2q + 1 - N + k - j = 1$, then $2q - N = j - k$ and also $q - j = N - q - k$. Substituting gives

$$B_{k,j} = \binom{n}{q+1} \binom{N-1-k}{2q-N} \binom{q+1-N}{N-q-k} \binom{n}{k}.$$
Next note that the absence of $y_{N-1}$ implies that $q + 1 \neq N$, and also (as above, in the $y^0$ case) $n - r + 1 \leq q + 1$. So either $n - r + 1 \leq q + 1 < N$ or $q + 1 > N$. In the former case the definition of $N$ implies $\binom{n}{q+1} = 0$ whence $B_{k,j} = 0$. In the latter case we have $\binom{n}{q+1} - \binom{N - q - k}{N - q - k}$ with $q + 1 - N > 0$. We may therefore suppose $N - q - k \geq 0$ since otherwise this binomial coefficient vanishes. But then

$$0 < (q + 1 - N) + (N - q - k) = 1 - k$$

and $k \geq 0$ gives $k = 0$ as the only possibility, whence $q + 1 - N = 1$, $N - q - k = 0$, i.e. $q = N$. Then, finally, the second binomial coefficient now equals $\binom{N - 1 - k}{2q - N} = \binom{N - 1}{N} = 0$. Thus $B_{k,j} = 0$ and the sum in (8) reduces to $\sum_{k=0}^{q} A_k y^k y_{2q-k} \in \mathcal{I}_2$.

Completing the proof of Theorem 4.1 is now easy. By Lemma 5.2 any cup-product of maximal cup-length must be in the top dimension $d_{n,r}$ and equal to

$$\xi = y^{N-1} \cdot y_{n-r} \cdots y_{n-2} \cdot y_N \cdots y_{n-1}.$$ 

This gives an immediate lower bound

$$\cup(X_{n,r}) \geq N + r - 2.$$ 

However we can now use the AGH relations (8) to improve this lower bound by decomposing the $y_j$, where possible, and thus obtain a representation with greater cup-length for $\xi$. Since $y^{N-1}$ is present in the product, it suffices to compute all cup-squares modulo $\mathcal{I}_2$. Lemma 5.2 then implies that the cup-squares are identical (apart from notation) in $H^*(X_{n,r})$ modulo $\mathcal{I}_2$, and in $H^*(V_{n,r})$. The difference in the cup-lengths therefore arises entirely from the first Stiefel-Whitney class $y \in H^1(X_{n,r})$, and from the class $x_{N-1} \in H^1(V_{n,r})$ which has no counterpart in $H^*(X_{n,r})$. Recall from (2) that $(N-1)2^b_N \leq n - 1$ whereas $(N-1)2^{b_N+1} > n - 1$. Hence the class $x_{N-1}$ and its square, fourth power, ..., contribute $1 + 2 + 4 + \cdots + 2^k$ to the cup-length of $V_{n,r}$. For the cup-length of $X_{n,r}$ there is the additional contribution by $y^{N-1}$ of length $N - 1$, and the smaller contribution by $y_{2(N-1)}$ and its square, fourth power, ..., which will have length $1 + 2 + 4 + \cdots + 2^{b_{N-1}}$. Thus cup($X_{n,r}$) gets an additional contribution of $N - 1$ from $y$ but a lesser contribution of $2^{b_N}$ due to the absence of $y_{N-1}$, this is exactly (7) so Theorem 4.1 is proved.

Remark 5.4. This proof actually shows that if $\eta = y^{N-1} \gamma \in H^{d_{n,r}}(X_{n,r})$ is a cohomology class in the top dimension, and the AGH relations are applied inside $\gamma$, the maximal cup-length attained in this way is $L(n,r)$.

Proof of Proposition 4.3. (a) Combining Remark 3.1 with Theorem 4.1 gives, in the stable range,

$$\cup(X_{n,r}) \geq r + N - 1 - 2^{b_N}.$$ 

By definition $N \geq n - r + 1$, and stability implies $r < \frac{n + 1}{2}$. Thus $N > n - \frac{n + 1}{2} + 1 = \frac{n + 1}{2}$, from which $\frac{n - 1}{N - 1} < 2$ follows. By definition then $b_N = 0$, giving $\cup(X_{n,r}) \geq r + N - 2$, and this cup-length is realized by

$$\xi = y^{N-1} \cdot y_{n-r} \cdots y_{n-2} \cdot y_N \cdots y_{n-1},$$

noting that in the stable range each $y_q$ is indecomposable. To see that any use of the AGH formulae cannot increase the cup-length of $\xi$, first note that due to stability $2q \geq n$, for all $q \geq n - r$. Thus Lemma 5.3 applies and for each $q$ we have, for some $a_j \in \mathbb{Z}_2$,

$$y_q^2 = a_2 y_q^2 y_{2q-2} + a_3 y_q^3 y_{2q-3} + \cdots + a_{N-1} y_q^{N-1} y_{2q-N+1}.$$ (14)

Relations (14) can only be applied by selecting one of the terms in the right hand sum of (14) for which $a_j \neq 0$, suppose for example $a_2 = 1$, and rewriting $\xi$ as

$$\xi = y^{N-3} \cdot y^2 \cdot y_{2q-2} \cdot \eta$$

$$= y^{N-3} (y_q^2 + a_3 y_q^3 y_{2q-3} + \cdots + a_{N-1} y_q^{N-1} y_{2q-N+1}) \cdot \eta.$$ (15)

where $\eta$ is identical to $\xi$ with $y_{2q-2}$ and $y^{N-1}$ removed. Clearly cup($\eta$) = cup($\xi$) - (N - 1) - 1 = cup($\xi$) - N. Thus, expanding (15) into a sum, the first term has cup-length $N - 3 + 2 + \text{cup}(\eta) = \text{cup}(\xi) - 1$, while the following terms all contain $y^N$ and vanish. A similar calculation for any other term with $a_j = 1, j > 2$ shows a decrease in cup-length even greater than 1.

(b) Here $n = 2^m = N$, so $\xi(X_{n,r}) = y^{n-1} \cdot y_{n-r} \cdots y_{n-2}$. The AGH formulae simplify to

$$y_q^2 = \begin{cases} y_{2q} & 2q \leq n - 1, \\ 0 & 2q \geq n. \end{cases}$$

This is because $A_k = \binom{n}{k}, 0 \leq k \leq q$, equals 1 only for $k = 0$, while $\frac{n}{q+1} = 0, n - r \leq q \leq n - 2$, implies $B_{k,j} = 0$.

Now $\xi(V_{n,r}) = x_{n-r} \cdots x_{n-2} \cdot x_{n-1}$ agrees with $\xi(X_{n,r})$ apart from the extra $x_{n-1}$ in the former and extra $y^{N-1}$ in the latter, furthermore the above calculation shows that the cup-squares are the same in both (since $n - 1 = 2^m - 1$ is odd $x_{n-1}$ is indecomposable). It is easy to see that $b_N = 0$ for $V_{n,r}$. This gives the cup-length of $X_{n,r}$ as equal to $\ell(n,r) + (N - 1) - 1 = \ell(n,r) + N - 1 - 2^{b_N} = L(n,r)$. 

5 Proofs of the main results

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(c) With $N = 2$ we immediately have $r = n - 1$ and $n \equiv 2, 3 \pmod{4}$, as well as $\xi = y \cdot y_2 \cdot y_3 \cdots y_{n-1}$, say $\xi = y \cdot \gamma$. Now Lemma 5.2 implies $y^2 = y_2 + \alpha y y_2 - 1$, $2q \leq n - 1$, $\alpha \in \{0, 1\}$, while Lemma 5.3 implies $y^2_2 = 0$, $2q \geq n$, since $L_2 = 0$ here. Since the relation $y^2_2 = y \cdot \alpha$, $\alpha \neq 0$, does not occur, any decompositions that lengthen $\xi$ must take place in $\gamma$. Then, by Remark 5.4, $\text{cup}(X_{n,n-1}) = L(n, n-1)$.

(d) We have $N = 2^{s-1}$. So the non-zero product in the top dimension is

$$\xi = y^{2^{s-1}-1}y_2^{s-1}y_2^{s-1+1} \cdots y_2^{s-1+2^{s-1}-2}.$$  

As a consequence, $\text{cup}(X_{2^{s-1}, 2^{s-1}})$ is at least $2^{s} - 2$. But for each $y_2$ in $\xi$ we have $2q \geq n = 2^{s} - 1$, so the proof that $\text{cup}(\xi)$ cannot be increased from $2^{s} - 2$ can now proceed exactly as in the stable case (a) above. □

References


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