Graduate J. Math. **6** (2021), 27 – 34

The Cup-Length of Stiefel and Projective Stiefel Manifolds

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Abstract

This paper discusses some generalities about cup-length of manifolds and then gives an explicit formula for the $\mathbb{Z}_2$-cup-length of the Stiefel manifolds $V_{n,r}$, as well as strong lower bounds for the $\mathbb{Z}_2$-cup-length of the projective Stiefel manifolds $X_{n,r}$, for all $1 \leq r \leq n-1$. A simple formula relating the two cases is given.

We also show the consequences for the Lyusternik-Shnirel'man category, as well as a family of interesting number theoretical identities that arise from the $V_{n,r}$ calculations.

MSC 2010. Primary: 57R19; Secondary: 55M30, 11B75, 05A19.

1 Introduction

Let R be a commutative ring with 1. We recall that the *R-cup-length* $\operatorname{cup}_R(X)$ of a compact path-connected topological space X is the largest of all integers c such that there exist reduced cohomology classes $a_1, \ldots, a_c \in \widetilde{H}^*(X;R)$ with their cup product

$$a_1 \cup \cdots \cup a_c = a_1 \cdots a_c \neq 0.$$

In this note, we will concentrate on the two cases $V_{n,r}$ and $X_{n,r}$, respectively the real and real projective Stiefel manifolds (defined in the following sections). A short preliminary Section 2 gives a couple of results associated to the R-cup-length, which is simply written $\operatorname{cup}(X)$ when $R = \mathbb{Z}_2$, as will be the case starting from Section 3. In Section 3, an explicit formula is obtained for $\operatorname{cup}(V_{n,r})$ for all n, r, and a few examples are given. Some purely number theoretical (and perhaps remarkable) identities arising from this formula are stated and proved.

In Section 4, a similar discussion is carried out to give a lower bound for $\operatorname{cup}(X_{n,r})$, which is related by a simple formula to $\operatorname{cup}(V_{n,r})$. Proofs of all the results are given in Section 5.

The Froloff-Elsholz inequality (cf. [4]) $\operatorname{cat}(X) \geq \operatorname{cup}(X)$ relates $\operatorname{cup}(X)$ to another important homotopy invariant, the *Lyusternik-Shnirel'man category* $\operatorname{cat}(X)$. The latter is defined to be the least integer k such that X can be covered by $k+1$ open subsets each of which is contractible in X, and was introduced in 1934 [9]. Thus our results have immediate corollaries for $\operatorname{cat}(V_{n,r})$ (Section 3) and for $\operatorname{cat}(X_{n,r})$ (Section 4). These numbers can be applied, for instance, as the lower bound for the number of critical points that a smooth real-valued function on $V_{n,r}$ or $X_{n,r}$ could have ([4]) (a topic that arises e.g. in calculus courses for smooth real-valued functions on $\mathbb{R}^n$). For a full treatment of these topics see the excellent monograph [3].

Applications of cup-length to symplectic embedding problems are given in [11], p. 161. Specifically, one has the inequality

$$1 + \dim(M)/2 \leq \operatorname{cup}(M) + 1 \leq \beta(M),$$

where $\beta(M)$ is the minimal number of smoothly embedded balls needed to cover a closed symplectic manifold (M, ω).

Acknowledgment: The authors wish to thank the referee for very careful reading of several revisions of this paper, as well as suggestions and corrections which led to substantial improvements.

2 Preliminary remarks about R-cup-length

In this section, we first recall some material in Hatcher [6], Chapter 3. In particular, for a closed connected n-dimensional manifold M, the notions of R-orientability and fundamental class $[M] \in H_n(M;R)$ are defined there as well as the bilinear pairing

$$T : H^k(M;R) \otimes H^{n-k}(M;R) \to R$$

given by $T(\alpha \otimes \beta) = (\alpha \cup \beta)[M]$, where $\alpha \in H^k(M;R)$, $\beta \in H^{n-k}(M;R)$. Poincaré duality for the R-orientable

* This work was supported, in case of the first author, by the Slovak Research and Development Agency under the contract No. APVV-16-0053 and by the grant agency VEGA 1/0596/21. He was partially affiliated with the Mathematical Institute, Slovak Academy of Sciences, Bratislava.

† The second author is supported by NSERC of Canada.

manifold is then expressed by [6], Proposition 3.38 : *The cup-product pairing T is non-singular if R is a field, or if $R = \mathbb{Z}$ and torsion is factored out.*

For applications to cup-length it will be convenient to take R to be a field, so we shall henceforth denote it F. It will also be convenient to use the natural isomorphism $H^n(M;F) \approx \hom_F(H_n(M;F),F)$ ([6], p. 198) induced by the natural surjection ([6], p. 191)

$$h : H^n(C;G) \twoheadrightarrow \hom(H_n(C;G)),$$

and define the "top" cohomology class $\xi_M \in H^n(M;F)$, dual to $[M]$, by $h(\xi_M)([M]) = 1$. We next give two corollaries of the above proposition, the first being a variant of Corollary 3.39 in [6] and the second an application to cup-length that will be useful in proving the main theorems of Sections 3, 4.

Corollary 2.1. *Let $0 \neq \alpha \in H^k(M;F)$. Then there exists $\beta \in H^{n-k}(M;F)$ such that $\alpha \cup \beta = \xi_M$.*

Proof. Since $H^k(M;F)$ is a vector space over F and $\alpha \neq 0$, there exists a homomorphism $\varphi : H^k(M;F) \to F$ with $\varphi(\alpha) = 1$. Given any such homomorphism φ, the fact that T is non-singular means by definition that $\varphi(x) = T(x \otimes \beta)$ for some $\beta \in H^{n-k}(M;R)$. Then $(\alpha \cup \beta)[M] = T(\alpha \otimes \beta) = \varphi(\alpha) = 1$ implies $\alpha \cup \beta = \xi_M$. □

Lemma 2.2. *If $\alpha \in H^d(M;F)$ is a class of maximal cup-length, then $d = n$. In particular, if $F = \mathbb{Z}_2$, then $\alpha = \xi_M$.*

Proof. If $d < n$, then Corollary 2.1 shows that α cannot have maximal cup-length. □

We remark that while Poincaré duality is of course treated in many texts, it is not clear that its simple application to cup-length in Lemma 2.2 is explicitly stated in the literature. It is implicitly assumed in [7], Proof of Theorem 1.1. For a space X that is not a manifold, Lemma 2.2 does not hold, an elementary counterexample being $S^m \vee \mathbb{R}P^n$ with $m > n$.

3 Cup-length of the Stiefel manifolds

Consider the real Stiefel manifold $V_{n,r}$ of orthonormal r-frames in $\mathbb{R}^n$, $1 \leq r \leq n-1$. It is well known to be a smooth path-connected manifold, indeed a homogeneous space of dimension $d = d_{n,r} = nr - \binom{r+1}{2}$. Its cohomology and the action of the Steenrod squares are well known and go back to Borel, [2], and Steenrod-Epstein, [12]. For our purposes we can summarize the cohomology as the algebra over $\mathbb{Z}_2$ with generators $x_i \in H^i(V_{n,r})$, $n-r \leq i \leq n-1$ and the only non-trivial cup-products arising from $x_i^2 = x_{2i}$, $2i \leq n-1$. After a couple of numerical definitions we give an explicit formula for the $\mathbb{Z}_2$-cup-length of $V_{n,r}$, which we shall write $\mathrm{cup}(V_{n,r})$. First let

$$n - 1 = \sum_{j=1}^{\alpha(n-1)} 2^{a_j}, \quad a_1 > a_2 > \ldots > a_{\alpha(n-1)} \tag{1}$$

be the binary expansion of $n-1$. Here $\alpha(n-1)$ denotes, as usual, the number of 1's in this binary expansion. Next, for $k \geq 2$, define

$$b_k = \max\{m : 2^m \leq \frac{n-1}{k-1},\ k \geq 2\} = \left\lfloor \log_2\left(\frac{n-1}{k-1}\right) \right\rfloor . \tag{2}$$

Using (1) and (2), we define

$$\ell(n,r) = n - 1 + \sum_{j=1}^{\alpha(n-1)} a_j \cdot 2^{a_j - 1} - \sum_{k=2}^{n-r} 2^{b_k}. \tag{3}$$

We now give three examples with $n = 23$. Here $n-1 = 2^4 + 2^2 + 2^1$, so $a_1 = 4, a_2 = 2, a_3 = 1$, and one readily finds $b_2 = 4, b_3 = 3, b_4 = b_5 = b_6 = 2$, $b_7 = \ldots = b_{12} = 1, b_{13} = 0$. The computation is given for Example A, the others being similar.

Example A: $\ell(23,10) = 22 + 4 \cdot 2^3 + 2 \cdot 2^1 + 1 \cdot 2^0 - 2^4 - 2^3 - 3 \cdot 2^2 - 6 \cdot 2^1 - 1 = 10$.

Example B: $\ell(23,18) = 27$.

Example C: $\ell(23,21) = 43$.

We next define $\ell'(n,r)$, starting with the preliminary definitions

$$m = \lfloor \log_2(n-1) \rfloor, \text{ and}$$

$$r_0 = \left\lceil \frac{n-1}{2} \right\rceil, r_1 = \left\lceil \frac{3(n-1)}{4} \right\rceil, r_2 = \left\lceil \frac{7(n-1)}{8} \right\rceil, \ldots . \tag{4}$$

Elementary calculations then show that

$$r_0 < r_1 < r_2 < \ldots < r_m = \left\lceil \frac{(2^{m+1}-1)(n-1)}{2^{m+1}} \right\rceil = n-1.$$

For convenience, we also set $r_{-1} = 0$.

As before, let $n \geq 2$, $1 \leq r \leq n-1$, the integers r_i be defined as in (4) above and $r_{q-1} < r \leq r_q$ (for a unique q). Then we can define

$$\ell'(n,r) = 2^q r - \sum_{i=1}^{q} 2^{i-1} \left\lceil \frac{(2^i - 1)(n-1)}{2^i} \right\rceil . \tag{5}$$

We shall also define

$$\ell''(n,r) = n-1-(n-1-r)\cdot 2^q + \sum_{j=1}^{\alpha(n-1)} \min\{a_j, q\} \cdot 2^{a_j - 1} , \tag{6}$$

a definition which uses slightly less machinery than its predecessors.

Theorem 3.1. *One has*

$$\mathrm{cup}(V_{n,r}) = \ell(n,r) = \ell'(n,r) = \ell''(n,r).$$

Remark 3.2. In the stable range $2r \leq n$, i.e. $q = 0$, one has $\mathrm{cup}(V_{n,r}) = r$ (see also [10]).

The proof given later in Section 5 for the equality $\mathrm{cup}(V_{n,r}) = \ell(n,r)$ of Theorem 3.1 starts from $r = n-1$ and uses downward induction in $H^*(V_{n,r})$. It is possible to prove the equality $\mathrm{cup}(V_{n,r}) = \ell'(n,r)$ starting from $r = 1$ and using upward induction in $H^*(V_{n,r})$, however the equality $\ell(n,r) = \ell'(n,r) = \ell''(n,r)$ is purely number theoretical and we therefore give a purely number theoretical proof of this in Section 5. To illustrate how disparate the two sums $\ell(n,r)$, $\ell'(n,r)$ seem, we go back to Example C above, of $V_{23,21}$. In Theorem 3.1, since the binary expansion $22 = 2^4 + 2^2 + 2^1$ determines the first summation, and $k = 2$ in the second summation so we use $b_2 = \lfloor \log_2(22/1) \rfloor = 4$, whence

$$\ell(23,21) = 22 + 4 \cdot 2^3 + 2 \cdot 2^1 + 1 \cdot 2^0 - 2^4 = 43.$$

On the other hand, since $20 = r_2 < 21 = r_3$ implies $q = 3$, this gives

$$\begin{aligned} \ell'(23,21) &= 8 \cdot 21 - \sum_{i=1}^{3} 2^{i-1} \cdot \left\lceil \frac{(2^i - 1)\cdot(22)}{2^i} \right\rceil \\ &= 168 - 11 - 34 - 80 = 43. \end{aligned}$$

Theorem 3.1 and the Froloff-Elsholz inequality give the following for the Lyusternik-Shnirel'man category.

Corollary 3.3. *One has* $\mathrm{cat}(V_{n,r}) \geq \ell(n,r) = \ell'(n,r) = \ell''(n,r)$.

We observe, that for $n \geq 2r$, Nishimoto [10] proved that $\mathrm{cat}(V_{n,r}) = r$.

4 Cup-length of the projective Stiefel manifolds

In this section, we concentrate on the manifold $X_{n,r}$ ($r < n$), the projective Stiefel manifold, which is obtained from the Stiefel manifold $V_{n,r}$ of orthornormal r-frames in $\mathbb{R}^n$ as the quotient space, by identification of any frame $(v_1, \ldots, v_r)$ with the frame $(-v_1, \ldots, -v_r)$ ([5]).

Let $\xi_{n,r}$ be the real line bundle associated to the obvious double covering $V_{n,r} \to X_{n,r}$. By [5], for the $\mathbb{Z}_2$-cohomology ring of $X_{n,r}$, we have

$$H^*(X_{n,r}) = \mathbb{Z}_2[y]/(y^N) \otimes V(y_{n-r}, ..., y_{N-2}, y_N, ..., y_{n-1}),$$

where $y \in H^1(X_{n,r})$ is the first Stiefel-Whitney class $w_1(\xi_{n,r})$, $y_j \in H^j(X_{n,r})$,

$$N = \min\{j; j \geq n-r+1, \binom{n}{j} \equiv 1 (\mathrm{mod}\ 2)\}$$

and $V(y_{n-r}, ..., y_{N-2}, y_N, ..., y_{n-1})$ is the $\mathbb{Z}_2$-vector space, which has the monomials $\prod_{i=n-r}^{n-1} {y_i}^{t_i}$, with $i \neq N-1$ and $t_i \in \{0,1\}$, as $\mathbb{Z}_2$-basis (N can be easily calculated for any $X_{n,r}$). The dimension of $X_{n,r}$ is also $d_{n,r}$ (defined in Section 3).

Recalling the definition (2) of b_k, we now have the following theorem.

Theorem 4.1. *One has*

$$\mathrm{cup}(X_{n,r}) \geq \mathcal{L}(n,r) := \mathrm{cup}(V_{n,r}) + N - 1 - 2^{b_N}. \quad (7)$$

Since $\mathrm{cup}(V_{n,r})$ has already been explicitly calculated in Section 3, indeed via (3), (5), or (6), Theorem 4.1 gives an explicit lower bound for $\mathrm{cup}(X_{n,r})$.

As an immediate corollary of Theorem 4.1 we have

Corollary 4.2. *Let* $X_{n,r}$ $(1 \leq r < n)$ *be the projective Stiefel manifold. Then*

$$\mathrm{cat}(X_{n,r}) \geq \mathrm{cup}(V_{n,r}) + N - 1 - 2^{b_N}.$$

It seems very likely that the stronger result $\mathrm{cup}(X_{n,r}) = \mathcal{L}(n,r)$ is true, but to date neither a proof nor a counterexample (with the help of a computer program developed by the authors) has been found. It is hoped to address this question in a forthcoming note. The next proposition gives a few partial results where equality holds.

Proposition 4.3. *The result* $\mathrm{cup}(X_{n,r}) = \mathcal{L}(n,r)$ *is true*

(a) in the stable range (so here $\mathrm{cup}(X_{n,r}) = r+N-2$*),*

(b) if $n = 2^m$ *(so here* $\mathrm{cup}(X_{n,r}) = \ell(n,r) + N - 2 = \ell(n,r) + n - 2$*),*

(c) if $N = 2$*,*

(d) $\mathrm{cup}(X_{2^s-1,2^{s-1}}) = 2^s - 2$.

5 Proofs of the main results

First, we give the proof of $\mathrm{cup}(V_{n,r}) = \ell(n,r)$ in Theorem 3.1. We prove this in four steps, using the notation $\nu_2(q) = p$ for the standard 2-valuation of q, i.e. q is divisible by 2^p but not by 2^{p+1}. The top cohomology class, denoted ξ_M (where now $M = V_{n,r}$) in Section 2, will here be denoted simply by X. According to Lemma 2.2, the cup-length is realized by the class X, so one has to look at the relations in $H^*(V_{n,r})$ to see how they can give a presentation that maximizes the cup-length of X.

(A) $\mathrm{cup}(V_{2^m,2^m-1}) = m \cdot 2^{m-1}$. From Section 3, the top cohomology class of $V_{2^m,2^m-1}$ equals $X := x_1 \cdot x_2 \cdots x_{2^m-1}$. This product has length $2^m - 1$ but the cup-length is larger, since some of these classes are decomposable, e.g. (again using Section 3) $x_2 = x_1^2$, $x_4 = x_1^4$, $x_6 = x_3^2$, $x_8 = x_1^8, \ldots$. A little careful counting shows that in $\{x_1, ..., x_{2^m-1}\}$, after this decomposition, exactly 2^{m-1} have length 1 (i.e. x_k with $\nu_2(k) = 0$), exactly 2^{m-2} have length 2 ($\nu_2(k) = 1$), etc. Also no further classes are decomposable. Thus the length after decomposition equals

$$1\cdot 2^{m-1} + 2\cdot 2^{m-2} + 4\cdot 2^{m-3} + \ldots + 2^{m-1}\cdot 1 = m\cdot 2^{m-1} \ .$$

(B) $\mathrm{cup}(V_{2^m+1,2^m}) = 2^m + m\cdot 2^{m-1}$. This is a corollary of (A), since the top class X now has one additional term $x_{2^m} = x_1^{2^m}$.

(C) Recalling (1), one now finds

$$\mathrm{cup}(V_{n,n-1}) = n - 1 + \sum_{j=1}^{\alpha(n-1)} a_j \cdot 2^{a_j - 1}.$$

To verify this, one simply writes

$$X = (x_1\cdot x_2 \cdots x_{2^{a_1}})\cdot(x_{2^{a_1}+1}\cdots x_{2^{a_1}+2^{a_2}})\cdot(x_{2^{a_1}+2^{a_2}+1} \cdots x_{2^{a_1}+2^{a_2}+2^{a_3}})\cdots .$$

Since $a_1 > a_2$, one has

$$\nu_2(k) = \nu_2(k - 2^{a_1}),\ 2^{a_1} + 1 \le k \le 2^{a_1} + 2^{a_2}.$$

Thus, from (B), the first bracketed term in the above expression for X has cup-length $2^{a_1} + a_1\cdot 2^{a_1-1}$, the second bracketed term has cup-length $2^{a_2} + a_2\cdot 2^{a_2-1}$, etc. Adding these gives the assertion.

(D) We now complete the proof of Theorem 3.1 by downward induction on r. For $r = n-1$, Theorem 3.1 has no 2^{b_k} terms, so reduces to (C), giving the start for the induction. Suppose then it holds for $r = n - s,\ s \ge 1$, so we have $n - r = s$ and Theorem 3.1 reads

$$\mathrm{cup}(V_{n,r}) = n - 1 + \sum_{j=1}^{\alpha(n-1)} a_j\cdot 2^{a_j-1} - \sum_{k=2}^{s} 2^{b_k}.$$

Passing to $n - r = s + 1$, the top class X loses x_s (length 1) and its cup-length is thereby shortened by the further changes x_s^2 to x_{2s}, x_s^4 to x_{2s}^2, $\ldots$, $x_s^{2^t}$ to $x_{2s}^{2^{t-1}}$, where t is the largest integer with $s\cdot 2^t \le n-1$, or equivalently $2^t \le \frac{n-1}{s}$. Then, by the definition (2) of b_k, we have $t = b_{s+1}$. The net loss in cup-length is thus $1 + (1 + 2 + 4 + \ldots + 2^{b_{s+1}-1}) = 2^{b_{s+1}}$, thereby completing the inductive step.

Second, we give the proof of $\ell(n,r) = \ell'(n,r)$ in Theorem 3.1. This proof proceeds by induction on r. For $r = 1$, the first part of the proof shows that $\ell(n,1) = \mathrm{cup}(V_{n,1}) = \mathrm{cup}(S^{n-1}) = 1$. Since $r_0 \ge 1$ and $r_{-1} = 0$, we see that $q = 0$, so $\ell'(n,1) = 1 = \ell(n,1)$.

For the inductive step, the induction hypothesis gives

$$\begin{aligned} \ell(n,r) &= \ell(n,r-1) + 2^{b_{n-r+1}} \\ &= \ell'(n,r-1) + 2^{b_{n-r+1}} \\ &= \ell'(n,r) - 2^q + 2^{b_{n-r+1}}, \end{aligned}$$

so it suffices to show that $q = b_{n-r+1}$ for $1 \le r \le n-1$. Since $r_{q-1} < r \le r_q$, we have

$$\begin{aligned} r &\ge r_{q-1} + 1 \ge \frac{(2^q-1)(n-1)}{2^q} + 1 = n - \frac{n-1}{2^q} \ , \\ r &\le r_q < \frac{(2^{q+1}-1)(n-1)}{2^{q+1}} + 1 = n - \frac{n-1}{2^{q+1}} \ . \end{aligned}$$

Rearranging terms yields

$$(n-1)\cdot 2^{-(q+1)} < n - r \le (n-1)\cdot 2^{-q},$$

or equivalently

$$2^q \le \frac{n-1}{n-r} < 2^{q+1} \ .$$

Finally, taking the base 2 logarithms, we obtain

$$q \le \log_2\left(\frac{n-1}{n-r}\right) < q + 1 \ ,$$

and hence $q = \left\lfloor \log_2\left(\frac{n-1}{n-r}\right)\right\rfloor = b_{n-r+1}$. □

Third, we prove that $\ell'(n,r) = \ell''(n,r)$, thus completing the proof of Theorem 3.1. For simplicity of later notation, we write the binary representation of $n-1$ in an alternative way as

$$n - 1 = \sum_{j=0}^{m} n_j 2^j,\ n_m = 1,\ n_j \in \{0,1\} \text{ for } 0 \le j \le m-1.$$

This representation is related to (1) as follows:

$$\begin{aligned} a_1 &= m, \\ a_{\alpha(n-1)} &= \min\{i \mid 0 \le i \le m,\ n_i \ne 0\}, \\ n_j &= \begin{cases} 1 & \text{when } j \in \{a_1, a_2, \ldots, a_{\alpha(n-1)}\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Fix $i \in \{1, 2, \ldots, m+1\}$. Then

$$\begin{aligned} r_{i-1} &= \left\lceil \frac{(2^i-1)(n-1)}{2^i} \right\rceil \\ &= \left\lceil n - 1 - \frac{n-1}{2^i} \right\rceil \\ &= n - 1 - \left\lfloor \frac{n-1}{2^i} \right\rfloor. \end{aligned}$$

To determine the floor function of $(n-1)/2^i$, write

$$\frac{n-1}{2^i} = \frac{1}{2^i}\sum_{j=0}^{i-1} n_j 2^j + \sum_{j=i}^{m} n_j 2^{j-i}.$$

Now

$$\frac{1}{2^i}\sum_{j=0}^{i-1} n_j 2^j \le \frac{1}{2^i}\sum_{j=0}^{i-1} 2^j = \frac{2^i-1}{2^i} < 1 \ ,$$

so

$$\left\lfloor \frac{n-1}{2^i} \right\rfloor = \sum_{j=i}^{m} n_j 2^{j-i} .$$

It follows that

$$\begin{aligned} r_{i-1}2^{i-1} &= (n-1-\sum_{j=i}^{m} n_j 2^{j-i})2^{i-1} \\ &= (n-1)2^{i-1} - \sum_{j=i}^{m} n_j 2^{j-1}, \end{aligned}$$

and hence

$$\begin{aligned} \ell'(n,r) &= r2^q - \sum_{i=1}^{q} r_{i-1}2^{i-1} \\ &= r2^q - (n-1)\sum_{i=1}^{q} 2^{i-1} + \sum_{i=1}^{q}\sum_{j=i}^{m} n_j 2^{j-1}. \end{aligned}$$

Now

$$\sum_{i=1}^{q} 2^{i-1} = \sum_{i=0}^{q-1} 2^i = 2^q - 1$$

and

$$\begin{aligned} &\sum_{i=1}^{q}\sum_{j=i}^{m} n_j 2^{j-1} \\ &= \sum_{j=1}^{m} n_j 2^{j-1} + \sum_{j=2}^{m} n_j 2^{j-1} + \cdots + \sum_{j=q}^{m} n_j 2^{j-1} \\ &= n_1 2^0 + 2n_2 2^1 + \cdots + (q-1)n_{q-1}2^{q-2} + q\sum_{j=q}^{m} n_j 2^{j-1} \\ &= \sum_{j=1}^{q-1} j n_j 2^{j-1} + \sum_{j=q}^{m} q n_j 2^{j-1} \\ &= \sum_{j=1}^{m} \min\{j,q\} n_j 2^{j-1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\ell'(n,r) \\ &= r2^q - (n-1)(2^q-1) + \sum_{j=1}^{m} \min\{j,q\} n_j 2^{j-1} \\ &= (n-1) - (n-1-r)2^q + \sum_{j=1}^{\alpha(n-1)} \min\{a_j,q\} 2^{a_j-1} \\ &= \ell''(n,r). \end{aligned}$$

□

Proof of Theorem 4.1.
For convenience, we write $H^*(X_{n,r}) = A \otimes V$, where all cohomology and tensor products are over $\mathbb{Z}_2$, $V = V(y_{n-r},\ldots,y_{N-2},y_N,\ldots,y_{n-1})$ (as in Section 4), and $A = \mathbb{Z}_2[y]/(y^N)$. We shall also write $\mathcal{I}_1$ for the ideal in $H^*(X_{n,r})$ generated by y, and similarly $\mathcal{I}_2$ for the ideal generated by y^2. Formulae for the Steenrod squaring operations $\mathrm{Sq}^i(y_q)$ will be needed, these are due to Gitler and Handel [5], Antoniano [1], and later again given (with a few misprints in [1] corrected) in [8]. We state them once again here in the slightly more convenient form $\mathrm{Sq}^i(y_q)$ (the older versions give $\mathrm{Sq}^i(y_{q-1})$):

$$\begin{aligned} \mathrm{Sq}^i(y_q) = &\sum_{k=0}^{i} A_k y^k y_{q+i-k} + \\ &\sum_{0\le k<j\le i} B_{k,j} y^{q+1+k+i-N-j} y_{N+j-k-1} + \epsilon y^{q+i}, \end{aligned}$$

where $\epsilon = \binom{n}{q+1+2^{t-1}-N}\binom{q+1+2^{t-1}-N}{i-1}$ if $t := \nu_2(N) \ge 3$ and $\epsilon = 0$ if $t < 3$,

$$A_k = A(q,i,k) = \binom{q-k}{q-i}\binom{n}{k}$$

and

$$\begin{aligned} B_{k,j} &= B(q,i,k,j) \\ &= \binom{n}{q+1}\binom{N-1-k}{j-k}\binom{q+1-N}{i-j}\binom{n}{k}. \end{aligned}$$

Just like the calculations of cup-length for the Stiefel manifolds had to take account of relations arising from cup-squares x_q^2, the calculations for the cup-length of the projective Stiefel manifolds must take account of the relations arising from y_q^2 (or iterations $y_q^{2^j}$). These are now much more complicated due to the presence of the first Stiefel-Whitney class y. However, they can be handled using $y_q^2 = Sq^q(y_q)$. The AGH (Antoniano, Gitler, Handel) formulae become:

$$\begin{aligned} \mathrm{Sq}^q(y_q) = &\sum_{k=0}^{q} A_k y^k y_{2q-k} + \\ &\sum_{0\le k<j\le q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1} + \epsilon y^{2q}, \end{aligned} \tag{8}$$

where $\epsilon = \binom{n}{q+1+2^{t-1}-N}\binom{q+1+2^{t-1}-N}{q-1}$ if $t \ge 3$ and $\epsilon = 0$ if $t < 3$,

$$A_k = A(q,q,k) = \binom{q-k}{q-q}\binom{n}{k} = \binom{n}{k} \tag{9}$$

and

$$\begin{aligned} B_{k,j} &= B(q,q,k,j) \\ &= \binom{n}{q+1}\binom{N-1-k}{j-k}\binom{q+1-N}{q-j}\binom{n}{k}. \end{aligned} \tag{10}$$

We shall carefully treat the presence of y by looking at three cases. In all cases, as usual, $n-r \le q \le n-1$, $q \ne N-1$. The first case is when $2q = N-1$, the above formula would give $y_q^2 \in \mathcal{I}_1$ (since there is no class y_{N-1}). The second case is when $2q \geq n$, similarly, no class y_{2q} exists gives $y_q^2 \in \mathcal{I}_1$. The third case is when $2q \leq n-1$. The three cases will be first stated in the following Lemmas 5.1, 5.2, 5.3 and then proved.

Lemma 5.1. *The case* $2q = N-1$ *cannot occur.*

Lemma 5.2. *One has*

$$y_q^2 \equiv \begin{cases} y_{2q} \pmod{\mathcal{I}_1}, & 2q \leq n-1 \\ 0 \pmod{\mathcal{I}_1}, & 2q \geq n \ . \end{cases} \tag{11}$$

Lemma 5.3. *If* $2q \geq n$, *then* $y_q^2 \in \mathcal{I}_2$.

Proof of Lemma 5.1. Recall that

$$N = \min\{j : j \geq n-r+1, \binom{n}{j} \equiv 1 \pmod 2\}.$$

If $N = n-r+1$, then $N-1 \leq q < 2q$ since $1 \leq n-r \leq q$. So suppose that $N > n-r+1$. Then $N-1 \geq n-r+1$, so $\binom{n}{N-1} \not\equiv 1 \pmod 2$ by the minimality condition on N. We have

$$\begin{aligned} N\binom{n}{N} &= N\frac{n!}{N!(n-N)!} = \frac{n!}{(N-1)!(n-N)!} \\ &= \frac{n!}{(N-1)!}\frac{n-(N-1)}{(n-(N-1))!} \\ &= (n-N+1)\binom{n}{N-1}. \end{aligned}$$

Since $\binom{n}{N-1}$ is even, $N\binom{n}{N}$ must be even, and since $\binom{n}{N}$ is odd by the definition of N, this forces N to be even. It follows that $N-1 \neq 2q$. □

Proof of Lemma 5.2. First note that the term ϵy^{2q} in (8) equals 0 (mod $\mathcal{I}_1$). Second observe that

$$\begin{aligned} \sum_{k=0}^{q} A_k y^k y_{2q-k} &\equiv A_0 y_{2q} \qquad (12) \\ &\equiv y_{2q} \equiv \begin{cases} y_{2q} \pmod{\mathcal{I}_1}, & 2q \leq n-1 \\ 0 \pmod{\mathcal{I}_1}, & 2q \geq n \ , \end{cases} \end{aligned}$$

where we have used (9) to evaluate A_0 and are also using Lemma 5.1 by implicitly assuming that y_{2q} exists. Thus the proof of Lemma 5.2 will be completed by showing that in (8)

$$\sum_{0 \leq k < j \leq q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1} \equiv 0 \pmod{\mathcal{I}_1} \ .$$

To prove this claim, first recall that $q \neq N-1$ when $y_q \in H^*(X_{n,r})$. Second, using (10) for $B_{k,j}$ together with $j-k = 2q+1-N$ for any y^0 terms in the second sum in (8), we find

$$B_{k,j} = \binom{n}{q+1}\binom{N-1-k}{2q+1-N}\binom{q+1-N}{N-(q+1)-k}\binom{n}{k}. \tag{13}$$

Since $q+1 \neq N$ as noted above, either $q+1 < N$ or $q+1 > N$. In the former case, since also $n-r+1 \leq q+1$, the definition of N implies that the first binomial coefficient in (13) equals 0. In the latter case the third binomial coefficient equals 0 since $q+1-N > 0$ whereas $N-(q+1)-k < 0$ (recall that for integers $a > 0, b < 0$, one has $\binom{a}{b} = 0$). The claim and thereby also Lemma 5.2 are thus proved. □

Proof of Lemma 5.3. Since $2q \geq n \geq N$, we have $y^{2q} = 0$ so the ϵy^{2q} term in (8) vanishes. Next, for

$$\sum_{k=0}^{q} A_k y^k y_{2q-k}$$

in (8), the first term ($k=0$) vanishes since $2q \geq n$ and there is no y_{2q} in the cohomology. Since $2q-1 \geq n-1$ the y_{2q-1} in the second term ($k=1$) also vanishes unless $2q-1 = n-1$, i.e. $n = 2q$. But then $\binom{n}{1} = 0$ (all modulo 2) and $A_1 = 0$. This proves that $\sum_{k=0}^{q} A_k y^k y_{2q-k} \in \mathcal{I}_2$.

Next we claim the terms in $y^0 y_{2q}$ in

$$\sum_{0 \leq k < j \leq q} B_{k,j} y^{2q+1+k-N-j} y_{N+j-k-1}$$

vanish. To prove this claim, first recall that $q \neq N-1$ here. Second, using (10) for $B_{k,j}$ together with $j-k = 2q+1-N$ for y^0 in (8), we find

$$B_{k,j} = \binom{n}{q+1}\binom{N-1-k}{2q+1-N}\binom{q+1-N}{N-(q+1)-k}\binom{n}{k}.$$

Since $q+1 \neq N$ as noted above, either $q+1 < N$ or $q+1 > N$. In the former case, since also $n-r+1 \leq q+1$, the definition of N implies that the first binomial coefficient in (11) equals 0. In the latter case the third binomial coefficient equals 0 since $q+1-N > 0$ whereas $N-(q+1)-k < 0$ (recall that for integers $a > 0, b < 0$, one has $\binom{a}{b} = 0$). The claim is thus proved.

Now we turn to the $y^1 y_{2q-1}$ term in the $B_{k,j}$ summation and show that it also vanishes. Since we now have $2q+1-N+k-j = 1$, then $2q-N = j-k$ and also $q-j = N-q-k$. Substituting gives

$$B_{k,j} = \binom{n}{q+1}\binom{N-1-k}{2q-N}\binom{q+1-N}{N-q-k}\binom{n}{k}.$$

Next note that the absence of y_{N-1} implies that $q+1 \neq N$, and also (as above, in the y^0 case) $n-r+1 \leq q+1$. So either $n-r+1 \leq q+1 < N$ or $q+1 > N$. In the former case the definition of N implies $\binom{n}{q+1} = 0$ whence $B_{k,j} = 0$. In the latter case we have $\binom{q+1-N}{N-q-k}$ with $q+1-N > 0$. We may therefore suppose $N-q-k \geq 0$ since otherwise this binomial coefficient vanishes. But then

$$0 < (q+1-N) + (N-q-k) = 1-k$$

and $k \geq 0$ gives $k=0$ as the only possibility, whence $q+1-N = 1, N-q-k = 0$, i.e. $q = N$. Then, finally, the second binomial coefficient now equals $\binom{N-1-k}{2q-N} = \binom{N-1}{N} = 0$. Thus $B_{k,j} = 0$ and the sum in (8) reduces to $\sum_{k=0}^{q} A_k y^k y_{2q-k} \in \mathcal{I}_2$. $\square$

Completing the proof of Theorem 4.1 is now easy. By Lemma 2.2 any cup-product of maximal cup-length must be in the top dimension $d_{n,r}$ and equal to

$$\xi = y^{N-1} \cdot y_{n-r} \cdots y_{N-2} \cdot y_N \cdots y_{n-1} \ .$$

This gives an immediate lower bound

$$\mathrm{cup}(X_{n,r}) \geq N + r - 2.$$

However we can now use the AGH relations (8) to improve this lower bound by decomposing the y_j, where possible, and thus obtain a representation with greater cup-length for ξ. Since y^{N-1} is present in the product, it suffices to compute all cup-squares modulo $\mathcal{I}_1$. Lemma 5.2 then implies that the cup-squares are identical (apart from notation) in $H^*(X_{n,r})$ modulo $\mathcal{I}_1$, and in $H^*(V_{n,r})$. The difference in the cup-lengths therefore arises entirely from the first Stiefel-Whitney class $y \in H^1(X_{n,r})$, and from the class $x_{N-1} \in H^*(V_{n,r})$ which has no counterpart in $H^*(X_{n,r})$. Recall from (2) that $(N-1)2^{b_N} \leq n-1$ whereas $(N-1)2^{b_N+1} > n-1$. Hence the class x_{N-1} and its square, fourth power,..., contribute $1+2+4+\ldots+2^{b_N}$ to the cup-length of $V_{n,r}$. For the cup-length of $X_{n,r}$ there is the additional contribution by y^{N-1} of length $N-1$, and the smaller contribution by $y_{2(N-1)}$ and its square, fourth power, ... , which will have length $1+2+4+\ldots+2^{b_N-1}$. Thus $\mathrm{cup}(X_{n,r})$ gets an additional contribution of $N-1$ from y but a lesser contribution of 2^{b_N} due to the absence of y_{N-1}, this is exactly (7) so Theorem 4.1 is proved. $\square$

Remark 5.4. This proof actually shows that if $\eta = y^{N-1} \cdot \gamma \in H^{d(n,r)}(X_{n,r})$ is a cohomology class in the top dimension, and the AGH relations are applied inside γ, the maximal cup-length attained in this way is $\mathcal{L}(n,r)$.

Proof of Proposition 4.3. (a) Combining Remark 3.1 with Theorem 4.1 gives, in the stable range,

$$\mathrm{cup}(X_{n,r}) \geq r + N - 1 - 2^{b_N}.$$

By definition $N \geq n-r+1$, and stability implies $r < \frac{n+1}{2}$. Thus $N > n - \frac{n+1}{2} + 1 = \frac{n+1}{2}$, from which $\frac{n-1}{N-1} < 2$ follows. By definition then $b_N = 0$, giving $\mathrm{cup}(X_{n,r}) \geq r + N - 2$, and this cup-length is realized by

$$\xi = y^{N-1} \cdot y_{n-r} \cdots y_{N-2} \cdot y_N \cdots y_{n-1},$$

noting that in the stable range each y_q is indecomposable. To see that any use of the AGH formulae cannot increase the cup-length of ξ, first note that due to stability $2q \geq n$, for all $q \geq n-r$. Thus Lemma 5.3 applies and for each q we have, for some $a_j \in \mathbb{Z}_2$,

$$y_q^2 = a_2 y^2 y_{2q-2} + a_3 y^3 y_{2q-3} + \ldots + a_{N-1} y^{N-1} y_{2q-N+1}. \tag{14}$$

Relations (14) can only be applied by selecting one of the terms in the right hand sum of (14) for which $a_j \neq 0$, suppose for example $a_2 = 1$, and rewriting ξ as

$$\begin{aligned} \xi &= y^{N-3} \cdot y^2 \cdot y_{2q-2} \cdot \eta \\ &= y^{N-3}[y_q^2 + a_3 y^3 y_{2q-3} + \ldots + a_{N-1} y^{N-1} y_{2q-N+1}] \cdot \eta, \end{aligned} \tag{15}$$

where η is identical to ξ with y_{2q-2} and y^{N-1} removed. Clearly $\mathrm{cup}(\eta) = \mathrm{cup}(\xi) - (N-1) - 1 = \mathrm{cup}(\xi) - N$. Thus, expanding (15) into a sum, the first term has cup-length $N - 3 + 2 + \mathrm{cup}(\eta) = \mathrm{cup}(\xi) - 1$, while the following terms all contain y^N and vanish. A similar calculation for any other term with $a_j = 1, j > 2$ shows a decrease in cup-length even greater than 1.

(b) Here $n = 2^m = N$, so $\xi(X_{n,r}) = y^{n-1} \cdot y_{n-r} \cdots y_{n-2}$. The AGH formulae simplify to

$$y_q^2 = \begin{cases} y_{2q}, & 2q \leq n-1, \\ 0, & 2q \geq n \ . \end{cases}$$

This is because $A_k = \binom{n}{k}$, $0 \leq k \leq q$, equals 1 only for $k = 0$, while $\binom{n}{q+1} = 0$, $n - r \leq q \leq n-2$, implies $B_{k,j} = 0$.

Now $\xi(V_{n,r}) = x_{n-r} \cdots x_{n-2} \cdot x_{n-1}$ agrees with $\xi(X_{n,r})$ apart from the extra x_{n-1} in the former and extra y^{N-1} in the latter, furthermore the above calculation shows that the cup-squares are the same in both (since $n-1 = 2^m - 1$ is odd x_{n-1} is indecomposable). It is easy to see that $b_N = 0$ for $V_{n,r}$. This gives the cup-length of $X_{n,r}$ as equal to $\ell(n,r) + (N-1) - 1 = \ell(n,r) + N - 1 - 2^{b_N} = \mathcal{L}(n,r)$.

(c) With $N = 2$ we immediately have $r = n - 1$ and $n \equiv 2, 3 \pmod 4$, as well as $\xi = y \cdot y_2 \cdot y_3 \cdots y_{n-1}$, say $\xi = y \cdot \gamma$. Now Lemma 5.2 implies $y_q^2 = y_{2q} + \alpha y y_{2q-1}$, $2q \leq n - 1$, $\alpha \in \{0, 1\}$, while Lemma 5.3 implies $y_q^2 = 0$, $2q \geq n$, since $\mathcal{I}_2 = 0$ here. Since the relation $y_q^2 = y \cdot \alpha$, $\alpha \neq 0$, does not occur, any decompositions that lengthen ξ must take place in γ. Then, by Remark 5.4, $\mathrm{cup}(X_{n,n-1}) = \mathcal{L}(n, n-1)$.

(d) We have $N = 2^{s-1}$. So the non-zero product in the top dimension is

$$\xi = y^{2^{s-1}-1} y_{2^{s-1}} y_{2^{s-1}+1} \cdots y_{2^{s-1}+2^{s-1}-2}.$$

As a consequence, $\mathrm{cup}(X_{2^s-1,2^{s-1}})$ is at least $2^s - 2$. But for each y_q in ξ we have $2q \geq n = 2^s - 1$, so the proof that $\mathrm{cup}(\xi)$ cannot be increased from $2^s - 2$ can now proceed exactly as in the stable case (a) above. □

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