Abstract Computability[1]

Robin Cockett

Department of Computer Science
University of Calgary
Alberta, Canada

robin@cpsc.ucalgary.ca

FMCS June 2007

[1] Joint work with Pieter Hofstra

Restriction categories

Turing categories

Reducibility

Partial combinatory algebras

Forever undecided

Defining restriction categories

A **restriction category** is a category with a restriction operator

$$\frac{A \xrightarrow{f} B}{A \xrightarrow[\overline{f}]{} A}$$

satisfying the following four equations:

$$\begin{array}{llll} \textbf{[R.1]} & \overline{f} f = f & \textbf{[R.2]} & \overline{f}\,\overline{g} = \overline{g}\,\overline{f} \\ \textbf{[R.3]} & \overline{f}\,\overline{g} = \overline{\overline{f} g} & \textbf{[R.4]} & f\overline{g} = \overline{fg} f \end{array}$$

Restriction categories are abstract partial map categories.

Basic results

In any restriction category $\mathbb{X}$:

- $e : A \longrightarrow A$ with $\overline{e} = e$ is called a **restriction idempotent**. The restriction idempotents on A form a semilattice $\mathcal{O}(A)$. Think off these as the "open" sets of the object.
- A map : $A \longrightarrow B$ is **total** in case $\overline{f} = 1$. All monics are total maps and total maps compos: the total maps form a subcategory $\mathrm{Tot}(\mathbb{X})$.
- The hom-sets are partially ordered $f \leq g \Leftrightarrow \overline{f}g = f$.
- Two parallel arrows are compatible $f \smile g$ in case $\overline{f}g = \overline{g}f$ (are the same where they are both defined).

Basic results (cont.)

- A **restriction isomorphism** or **partial isomorphism** is an $f : A \longrightarrow B$ which has a (restriction) inverse $f^{(-1)}$, which is necessarily unique, such that $f^{(-1)} f = \overline{f^{(-1)}}$ and $f f^{(-1)} = \overline{f}$.
  A **restriction monic** is a monic restriction isomorphism: such splits its unique retraction.
- For any class, E, of idempotents, $\mathrm{Split}_E(\mathbb{X})$ is a restriction category with

$$\frac{e_1 \xrightarrow{f} e_2}{e_1 \xrightarrow[e_1 \overline{f}]{} e_1}$$

Examples

- Any category is "trivially" a restriction category by setting $\overline{f} = 1$.
- Sets and partial maps is a restriction category. More generally every partial map category $\mathsf{Par}(\mathbb{X}, \mathcal{M})$ is a restriction category.

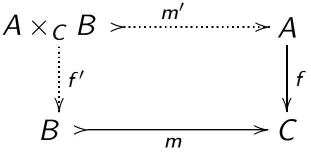
$$\frac{(m, f) : A \longrightarrow B}{(m, m) : A \longrightarrow A}$$

- An inverse monoid is a one object restriction category with all maps partial isomorphisms.
- Topological spaces with partial continuous maps, whose domains of definition are open subsets: $\mathsf{Par}(\mathsf{Top}, \mathsf{open})$.
- The opposite of the category of semilattices with stable maps (binary meet preserving maps) is a restriction category.
- Partial recursive maps on products of the natural numbers.

Completeness of restriction categories

An $\mathcal{M}$-stable system of monics satisfies:

- Each $m \in \mathcal{M}$ is monic
- Composites of maps in $\mathcal{M}$ are themselves in $\mathcal{M}$
- All isomorphisms are in $\mathcal{M}$
- Pullbacks along of an $\mathcal{M}$-map along any map always exists and is an $\mathcal{M}$-map.



Given such an $\mathcal{M}$ one can form the partial map category $\mathrm{Par}(\mathbb{X}, \mathcal{M})$ ...

Completeness of restriction categories

Theorem

(Cockett-Lack) Every restriction category has a full, structure preserving, embedding into the $\mathcal{M}$-partial map category of a category with a stable system of monics $\mathcal{M}$.

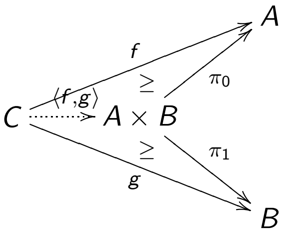
There is also a Yoneda representation (Mulry):

$$\mathcal{Y} : \mathbb{X} \longrightarrow \mathsf{Par}(\mathsf{Set}^{\mathsf{Tot}(\mathsf{Split}(\mathbb{X}))^{\mathrm{op}}}, \widehat{\mathcal{M}})$$

Cartesian restriction categories

A **cartesian restriction category** is a restriction category with partial products:

- It has a restriction final object 1:
  - Each A has a total map $! : A \longrightarrow 1$
  - If $A \xrightarrow{f} 1$ then $f = \overline{f}!$.
- It has restriction products in case for every A and B there is a cone $(A \times B, \pi_0, \pi_1)$ such that given any other cone there is a unique comparison map



such that $\overline{g}f = \langle f, g\rangle\pi_0$ and $\overline{f}g = \langle f, g\rangle\pi_1$.

Cartesian restriction categories

Partial products are examples of restriction limits ...

The following equations hold in any cartesian restriction category:

- Letting $\Delta = \langle 1, 1 \rangle$ then Δ is total, $\Delta \pi_i = 1$
- $\overline{h}\langle f, g \rangle = \langle \overline{h} f, g \rangle = \langle f, \overline{h} g \rangle$
- $\overline{\langle f, g \rangle} = \overline{f}\,\overline{g}$

In the total category the partial products become ordinary products.

Theorem

If restriction idempotents split then $\mathbb{X}$ is a cartesian restriction category if and only if $\mathrm{Tot}(\mathbb{X})$ is a cartesian category.

Meets in restriction categories

A restriction category has **meets** if

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B}{A \xrightarrow[f \cap g]{} B}$$

where $f \cap g \leq f$, $f \cap g \leq g$, $f \cap f = f$, $h(f \cap g) = hf \cap hg$. This makes $f \cap g$ the meet of f and g in the hom-set lattice.

Theorem (Jackson and Stokes)

A restriction category has meets if and only if it is a full subcategory of a partial map category $\mathrm{Par}(\mathbb{X}, \mathcal{M})$ *where* $\mathbb{X}$ *has equalizers and all regular monics are contained in* $\mathcal{M}$*.*

Discrete restriction categories

An object X in a *cartesian* restriction category is **discrete** in case its diagonal map

$$\Delta : X \longrightarrow X \times X$$

is a partial isomorphism. A cartesian restriction category is **discrete** in case very object is discrete.

In $\mathrm{Par}(\mathrm{Top}, \mathrm{Open})$ the discrete objects are precisely discrete topological spaces.

Discrete restriction categories

Theorem

A cartesian restriction category is discrete if and only if it has meets.

PROOF: Note $\Delta(\pi_0 \cap \pi_1) = \Delta\pi_0 \cap \Delta\pi_1 = 1 \cap 1$ while

$$\begin{aligned} \overline{\pi_0 \cap \pi_1} &= \overline{\pi_0 \cap \pi_1}\langle \pi_0, \pi_1 \rangle = \langle \overline{\pi_0 \cap \pi_1}\pi_0, \overline{\pi_0 \cap \pi_1}\pi_1 \rangle \\ &= \langle \pi_0 \cap \pi_1, \pi_0 \cap \pi_1 \rangle = (\pi_0 \cap \pi_1)\Delta \end{aligned}$$

Conversely set $f \cap g = \langle f, g \rangle \Delta^{(-1)}$. □

Joins and disjoins

A restriction category has a **restriction zero** in case there is a zero map between every pair of objects $A \xrightarrow{0} B$ (with $f0 = 0$ and $0g = 0$) such that $\overline{0_{A,B}} = 0_{A,A}$.

A restriction category has **joins** if it has a restriction zero and if whenever $f \smile g$ there is a join $f, g \leq f \vee g$ such that whenever $f, g \leq h$ then $f \vee g \leq h$ which is stable that is $h(f \vee g) = hf \vee hg$ (this implies $(f \vee g)h = fh \vee gh$).

Joins and disjoins

A restriction category has **disjoins** if it has a restriction zero and whenever $f \perp_0 g$, that is $\overline{f}g = 0 = \overline{g}f$, then there is a stable join $f, g \leq f \sqcup g$ such that $f, g \leq h$ then $f \sqcup g \leq h$ and $h(f \sqcup g) = hf \sqcup hg$.

Having joins implies having disjoins (but not conversely).

A cartesian restriction category has **joins** (resp, **disjoins**) in case $0 \times h = 0$ and $(f \vee g) \times h = (f \times h) \vee (g \times h)$ (resp. $(f \sqcup g) \times h = (f \times h) \sqcup (g \times h)$).

Coproducts ...

A restriction category has coproducts if it has coproducts (and initial object) in the ordinary sense such that the coprojections (and the initial maps) are total.

The coproduct functor necessarily preserves the restriction: $\overline{f+g} = \overline{f} + \overline{g}$.

A cartesian restriction category is a **distributive restriction category** in case it has a restriction zero and the products distribute over the coproducts.

Remark: Distributive restriction categories have disjoins with respect to the relation $f \perp g$ if $\langle \overline{f} | \overline{g} \rangle : A + A \longrightarrow A$ is a partial isomorphism.

Disjoins and coproducts

Theorem

In a split restriction category with disjoins whenever e_1 and e_2 are disjoint restriction idempotents then the split of $e_1 \sqcup e_2$ is a coproduct of the splitting of e_1 and the splitting of e_2.

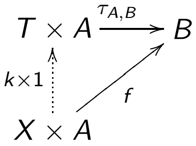
(i.e. certain coproducts are forced by the presence of disjoins)

PROOF: Note that if $f : e_1 \longrightarrow X$ and $g : e_2 \longrightarrow X$ then $f \sqcup g : e_1 \sqcup e_2 \longrightarrow X$ is the unique map required to establish the copairing. □

TURING CATGEORIES

$\mathbb{T}$ is a **Turing category** if

- it is a cartesian restriction category
- it has a **Turing object**, T:



this an object T with for each A and B a **Turing morphism**, $\tau_{A,B}$, such that for each f there is a total k, called a **code** for f, making the diagram above commute.

Note: none of this structure is canonical!

Turing categories

Theorem

In a Turing category, with a Turing object T, every object A is a retract of T.

PROOF: Consider

$$\begin{array}{ccc} T \times 1 & \xrightarrow{\tau_{1,A}} & A \\ {\scriptstyle m_A \times 1}\uparrow & \nearrow{\scriptstyle \pi_1} & \\ A \times 1 & & \end{array}$$

Then we have $A \lhd^{r_A}_{m_A} T$ where $r_A = \langle 1, ! \rangle \tau_{1,A}$. □

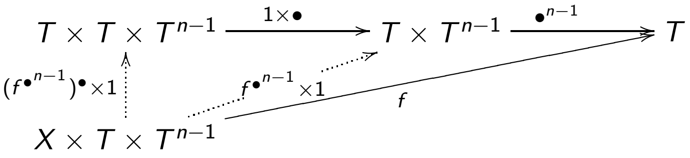
In particular $1 \lhd T$ and $T \times T \lhd T$.

Normalizing Turing structure

Theorem

A cartesian restriction category is a Turing category if and only if there is an object T, of which every object is a retract, which has a Turing morphism $T \times T \xrightarrow{\tau_{T,T}} T$.

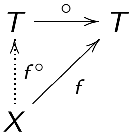
PROOF: The difficulty is to prove that if every object is a retract of T then having a Turing morphism $\bullet = \bullet^1 = \tau_{T,T}$ suffices. For $n > 1$ assume we have $\bullet^{n-1}$ defined inductively then:



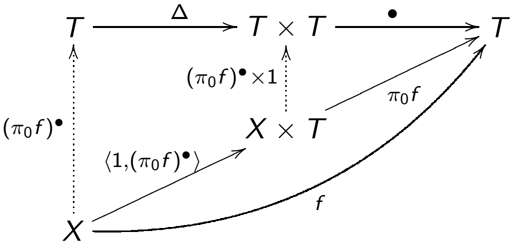
provides $f^{\bullet^n} = (f^{\bullet^{n-1}})^{\bullet}$.

Normalizing Turing structure

But what about $\circ = \bullet^0$?



Set this to $\circ = T \xrightarrow[\Delta]{} T \times T \xrightarrow[\bullet]{} T$. Now we have



Normalizing Turing structure

Finally for an arbitrary object A by assumption we have $A \triangleleft_{r_A}^{m_A} T$ so we may define:

$$\begin{aligned} & T \times A_1 \times \ldots \times A_n \xrightarrow{\tau_{A_1 \times . \times A_n, B}} B \\ = \quad & T \times A_1 \times \ldots \times A_n \xrightarrow{1 \times m_{A_1} \times \ldots \times m_{A_n}} T \times T^n \xrightarrow{\bullet^n} B \xrightarrow{r_b} B \end{aligned}$$

Clearly this is a Turing morphism. □

Thus a Turing category is determined once one has a retractive generator with a (self-)Turing morphism.

Reducibility

In any restriction category say that a restriction idempotent $e' : X$ (many-one) **reduces** to $e : Y$, write $e' \leq_m e$, if there is a total map $f : X \longrightarrow Y$ so that $\overline{fe} = e'$.

Say that e' 1-reduces to e, $e' \leq_1 e$ if there is a monic f with $\overline{fe} = e'$.

Say that $e : X$ is **m-complete** in case every e' m-reduces to e, that is $e' \leq_m e$. Similarly e is **1-complete** is every e' 1-reduces to e.

Reducibility (cont.)

Recall $K = \overline{\circ} = \overline{\Delta \bullet}$ are those computations which terminate on their own codes.

Theorem

In any Turing category $K = \overline{\circ}$ is m-complete.

PROOF: Suppose $e : X$ then

$$\begin{array}{ccc} T & \xrightarrow{\circ} & T \\ {\scriptstyle (em_X)^\circ} \uparrow & & \uparrow {\scriptstyle m_X} \\ X & \xrightarrow[e]{} & X \end{array}$$

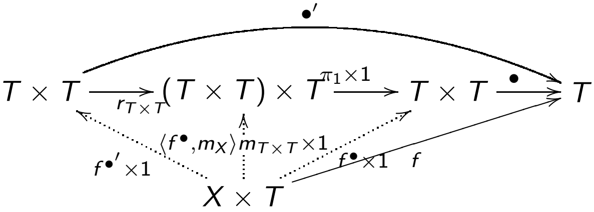
and

$$\begin{aligned} \overline{(em_X)^\circ K} &= \overline{(em_X)^\circ \overline{\circ}} \\ &= \overline{(em_X)^\circ \circ} \\ &= \overline{em_X} = \overline{e} = e \end{aligned}$$

□

1-reducibility

There is no guarantee that f° is monic but if it was K would be 1-complete.
We will use "padding" to obtain an alternative Turing morphism which has this property. Modify the Turing morphism



define $f^{\bullet'} = \langle f^\bullet, m_X\rangle m_{T\times T}$ note that

$$f^{\bullet'} r_{T\times T}\pi_1 = m_X$$

so, in fact, this is a section so certainly $f^{\circ'}$ is monic!

1-reducibility

Theorem

In any Turing category $K' = \overline{\circ'}$, as defined above, is 1-complete.

Note: this is *stronger* than 1-complete as the morphism along which the reduction is being obtained is a section.

This also illustrates the non-canonical nature of the Turing morphisms (doing this again gives an infinite family of Turing morphisms).

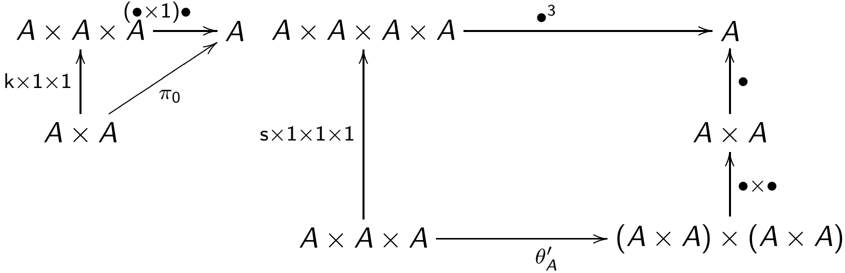
Note: special properties of $\bullet$ may not be preserved by moving to $\bullet'$. For example if $(T, \bullet)$ is a λ-algebra $(T, \bullet')$ will only be a λ-algebra when the Turing category is trivial!

Partial combinatory algebras

In a cartesian restriction category a **partial combinator algebras** is:

- an object A
- a partial map $\bullet : A \times A \longrightarrow A$
- two (total) points $1 \xrightarrow{\mathsf{k}} A$ and $1 \xrightarrow{\mathsf{s}} A$.

satisfying:



and $A \times A \xrightarrow{\mathsf{s} \times 1 \times 1} A \times A \times A \xrightarrow{\bullet^2} A$ is total.

Partial combinatory algebras

Equationally we have:

$$(\mathsf{k} \bullet x) \bullet y = x \qquad ((\mathsf{s} \bullet x) \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z) \qquad x|_{(\mathsf{s} \bullet v) \bullet w} = x$$

These are the usual equations from (total) combinatory algebra with the added requirement (expressed in the last equations) that sxy is total.

Theorem

If $(T, \bullet)$ is a Turing object for a cartesian restriction category it is a partial combinatory algebra.

PROOF: Use the above commuting requirements to define k and s! □

This begs the question: what is the connection between PCAs and Turing categories?

Partial combinatory algebras

Given any cartesian restriction category there is a cartesian restriction functor

$$\Gamma : \mathbb{X} \longrightarrow \mathrm{Par} : A \mapsto \mathrm{points}(A) = \mathrm{Tot}(\mathbb{X})(1, A)$$

Note: this carries a PCA in $\mathbb{X}$ to an "ordinary" PCA in Sets.

Let $\mathbb{X}$ be any cartesian restriction category and suppose $\mathbb{A} = (A, \bullet)$ is an applicative system (i.e. $\bullet : A \times A \longrightarrow .A$ is a partial operation) then $\Gamma(\mathbb{A})$ is an applicative system in Sets. A **set of indices** for $\mathbb{A}$ is a $\mathcal{V} \subseteq \Gamma(\mathbb{A}) = \mathrm{Tot}(\mathbb{X})(1, A)$ which is a sub-applicative system (i.e closed to the application).

A map $A \times ... \times A \xrightarrow{h} A$ in $\mathbb{X}$ is $(\mathbb{A}, \mathcal{V})$-**computable** if there is an index $v \in \mathcal{V}$ with $(v \times 1 \times ... \times 1)\bullet^n = h$. Similarly, the maps $A^n \longrightarrow A^m$ $(m > 0)$ is computable in case each projection $A^n \longrightarrow A$ is computable. $h : A^n \longrightarrow 1$ is computable provided $\overline{h} : A \longrightarrow A$ is computable.

Combinatory completeness

We shall say that an applicative system is **combinatory complete** relative to a set of indices $\mathcal{V}$ in case the $(\mathbb{A}, \mathcal{V})$-computable maps form a cartesian restriction subcategory.

Theorem

An applicative system $\mathbb{A}$*, with respect to a set of indices* $\mathcal{V}$*, is combinatory complete if and only if* $\mathcal{V}$ *contains indices* s *and* k *making* $\mathbb{A}$ *a partial combinatory algebra.*

This is most easily proven using the term logic ... while this is beyond the scope of this talk I shall use it!

PCAs to Turing Categories

This gives an very important method of generating Turing categories:

Theorem ($(\mathbb{A},\mathcal{V})$-computability)

The $(\mathbb{A},\mathcal{V})$-computable maps of any combinatory complete applicative system over any cartesian restriction category form a Turing category $\mathcal{C}(\mathbb{A},\mathcal{V})$ with

$$C_A : \mathcal{C}(\mathbb{A},\mathcal{V}) \longrightarrow \mathbb{X}$$

a faithful cartesian restriction functor.

Given a combinatory algebra in any cartesian restriction category an obvious set of indices to choose is the set of *all* points of the PCA. Conversely one can choose the smallest set generated by a choice of s and k ...

Turing subcategories

Given any cartesian restriction functor from a Turing category $F : \mathbb{T} \longrightarrow \mathbb{X}$ we may factorize it as

$$\mathbb{T} \xrightarrow{E(F)} \mathbb{T}/\cong \xrightarrow{M(F)} \mathbb{X}$$

where $E(F)$ forms the quotient of the category by $f \cong g \Leftrightarrow F(f) = F(g)$ and $M(F)$ is the residual faithful embedding.

$\mathbb{T}/\cong$ is a Turing category, thus, $M(F)$ is a faithful embedding of a Turing category into $\mathbb{X}$:

$$\mathbb{T}/\cong \xrightarrow{M(F)} \mathbb{X}$$

Turing subcategories

Any Turing object $T \in \mathbb{T}$ determines a PCA in $\mathbb{X}$ and a set of indices $\mathcal{V}_F = \{F(p) | p \in \text{points}(T)\}$.
Thus, F induces a faithful functor:

$$C_{F(T)} : \mathcal{C}(F(T), \mathcal{V}_F) \longrightarrow \mathbb{X}$$

Theorem

There is a factorization of any F with domain a Turing category as

$$\mathbb{T} \xrightarrow{F'} \mathrm{Split}(\mathcal{C}(F(T), \mathcal{V}_F)) \longrightarrow \mathrm{Split}(\mathbb{X})$$

Thus, up to splitting, faithful Turing subcategories of $\mathbb{X}$ are determined by combinatory complete applicative systems in $\mathbb{X}$ relative to a set of indices.

FOREVER UNDECIDED

We shall now examine undecidability results in Turing categories. To get off the ground one needs a good notion of complement. Disjoins provide this ...

THEREFORE we shall work from now on in Turing categories with disjoins.

By a **point** of an object A in a caretsian restriction category is meant a map $p : 1 \longrightarrow A$ which is total.

Undecidability of K

Let $\mathbb{T}$ be a Turing category with disjoins.

- A restriction idempotent e is **complemented** (or **recursive**) in case there is a restriction e' with $ee' = 0$ and $e \sqcup e' = 1$.
- A restriction idempotent e is **intuitionistically complemented** in case there is a restriction idempotent $e^{\perp}$ with $ee^{\perp} = 0$ such that whenever $e'e = 0$ $e' \leq e^{\perp}$.

Recall that in a disjoin restriction category if $e : A$ has a complement $e' : A$ then A is the coproduct of the splittings of e and e'.

Undecidability of K

Theorem

In a disjoin Turing category, $\mathbb{T}$, K has a complement if and only if $\mathbb{T}$ is trivial (i.e. exactly one map between each pair of objects).

PROOF: Let K' be an idempotent with $K'K = 0$. Set $v = K'^{\bullet}$ be an index of K' (i.e. $(v \times 1)\bullet = K'$) so that
$vK = v\Delta\bullet = \langle v, v\rangle\bullet = vK'$ but then
$vK' = vK'K' = vKK' = 0 = vK'K = vKK' = vK$ so that if
$K \sqcup K' = 1$ then $0 = \overline{0} = \overline{(vK) \sqcup (vK')} = \overline{v(K \sqcup K')} = \overline{v} = 1$ But this collapses the final object and make the whole category trivial.
□

Note that we have shown that K is "creative" (i.e. given $e = \overline{e}$ with $Ke = 0$ there is a point p_e with $pK = 0 = pe$). Clearly a creative idempotent in disjoin caretsian restriction category has a complement only when the category is trivial.

Below we refine these results ...

Point discreteness ...

A Turing category is **point discrete** if given a point p and a restriction idempotent e with $pe = 0$ then there is a restriction idempotent e' with $e'e = 0$ and $pe' = p$.

A point discrete topological space is *always* a discrete topological space. However, only the following implication holds for Turing (and restriction categories):

Lemma

Every discrete Turing category is point discrete.

PROOF: In any discrete cartesian restriction category all points are open. $\square$

Point discreteness ...

We now show that, for point discreteness Turing categories, K cannot even have an intuitionistic complement:

Theorem

A point discrete Turing category with disjoins which has an intuitionistic complement of K must be trivial.

PROOF: The point v produced earlier is disjoint from the join of K with its intuitionist complement $K^\perp = K \Rightarrow 0$. Thus there is an e containing v disjoint from both these. But $eK = 0$ implies $e \leq (K \Rightarrow 0)$ and so as $e(K \Rightarrow 0) = 0$ this implies $e = 0$ which collapses the category (by collapsing the total point v). □

Point discreteness ...

A Turing category which has *arbitrary* joins (this means each $\mathcal{O}(T)$ is a locale) always has intuitionistic complements defined by: $e_\perp = e \Rightarrow 0 = \bigvee\{e' | e'e = 0\}$.

Corollary

An (arbitrary) join Turing category cannot be point discrete or discrete unless it is trivial.

This means non-trivial Topological Turing categories generated from PCAs in $\mathrm{Par}(\mathrm{Top}, \mathrm{open})$ cannot be discrete ...

Separability

A pair of total map f_0 and f_1 with common codomain are **separable** if there are a pair of restriction idempotents e_0 and e_1 such that $f_i e_i = f_i$ and $e_0 e_1 = 0$.

If the total maps are points, following topological terminology, we that the points satisfy the **Hausdorff** condition.

Theorem

It $\mathbb{T}$ is a split disjoin Turing category then the following are equivalent:

(i) $\mathbb{T}$ *is a distributive Turing category;*

(ii) $\mathbb{T}$ *has a Turing object T which has two separable total endomorphisms $f, g : T \longrightarrow T$;*

(iii) $\mathbb{T}$ *has a Turing object T which has two points satisfying the Hausdorff condition.*

Note that this means a split Turing category with a pair of points which can be separated necessarily has coproducts ...

Separability

PROOF:

(i) $\Rightarrow$ *(ii)* If $\mathbb{T}$ has coproducts then $T \xrightarrow{\sigma_i} T + T \xrightarrow{m_{T+T}} T$ has the pair $\sigma_0 m_{T+T}$ and $\sigma_0 m_{T+T}$ separable by $\overline{r_{T+T}(1+0)}$ and $\overline{r_{T+T}(0+1)}$.

(ii) $\Rightarrow$ *(iii)* If f and g are separated by e_0 and e_1 then kf and kg are separated by e_0 and e_1.

(ii) $\Rightarrow$ *(iii)* Let v_0 and v_1 be elements separated by e_0 and e_1 then $e_0 + e_1 = e_0 \sqcup e_1$ (in the splitting) but $e_i!t_i$ is an idempotent of e_1 whose splitting is the terminal object so $1 + 1$ is present. Finally $(1 + 1) \times T$ is isomorphic to $T + T$. However, as T has copowers it is clear that the copoduct of every pair of objects exists.

$\square$

Separability

Theorem

Every discrete disjoin split Turing category has coproducts (i.e. is distributive).

PROOF: Every point is an open set ... so all we need do is find a couple of disjoint points. However, s and k *must* be disjoint. (The slice category over their intersection is a Turing category in which s = k so it collapses). □

Inseparability

A pair of disjoint restriction idempotents $e_0, e_1 : X$ are **recursively inseparable** in X if there is no complemented idempotent e such that $e_0 \leq e$ and $e_1 \leq e'$.

Theorem (F. Lengyel)

Every non-trivial distributive Turing category has a disjoint pair of inseparable restriction idempotents.

PROOF: The above theorem assures us that we may find two points $t_0, t_1 : 1 \longrightarrow T$ and idempotents e_0, e_1 such that $e_0 e_1 = 0$ (they are disjoint) and $t_i e_i = t_i$ (that is t_0 and t_1 are separable). Set $k_i = \overline{\circ e_i} = \overline{\Delta \bullet e_i}$ now suppose that $k_i \leq u_i$ and $u_0 u_1 = 0$. We wish to show that assuming that $u_0 \sqcup u_1 = 1_T$ forces the category to be trivial.

Inseparability

Consider the map $q = u_0!t_1 \sqcup u_1!t_0$ then q° is a total map with $q^{\circ\circ} = q$. Observe that

$$\begin{aligned} q^\circ k_0 &= \overline{q^\circ k_0} q^\circ = \overline{q^\circ \circ e_0} q^\circ = \overline{q e_0} q^\circ \\ &= \overline{(u_0!t_1 \sqcup u_1!t_0) e_0} q^\circ = \overline{u_0!t_1 e_0 \sqcup u_1!t_0 e_0} q^\circ \\ &= \overline{u_1!t_0} q^\circ = u_1 q^\circ \end{aligned}$$

and similarly $q^\circ k_1 = u_0 q^\circ$. This shows:

$$q^\circ u_1 = \overline{q^\circ u_1} q^\circ = \overline{q^\circ u_1 q^\circ} q^\circ u_1 = \overline{q^\circ k_0} q^\circ u_1 = q^\circ k_0 u_1 = 0$$

and similarly that $q^\circ u_0 = 0$. Finally this gives the following calculation:

$$1_T = \overline{q^\circ} = \overline{q^\circ (u_0 \sqcup u_1)} = \overline{q^\circ u_0} \sqcup \overline{q^\circ u_1} = 0$$

which suffices to show the category is trivial! □

Recursion theorems

The recursion theorems hold in any Turing category. Here is the "second recursion theorem":

Theorem

In any Turing category, for any $F : T \times T \longrightarrow T$. where T is a Turing object there is a point $e : 1 \longrightarrow T$ such that $(e \times 1)\bullet = (e \times 1)F$.

PROOF: This is proven most easily using the term logic: e is the element $\lambda^* fx.(\lambda^* z.f(zz)x)(\lambda^* z.f(zz)x)$. $\Box$

This allows us to obtain a version of Rice's theorem for disjoin Turing categories.

Extensionality

- We say that a restriction idempotent e on a Turing object is **extensional** (with respect to a given choice of Turing structure) in case the following implication holds for every f and g (using the term logic):

$$(e(f(x)) \bullet y = g(x) \bullet y \Rightarrow g(x)_{|e(f(x))} = e(gx)_{|e(f(x))}).$$

- Say that a restriction idempotent e on a Turing object is **non-trivial** in case there are two points, p_0 and p_1 with $p_0 e = p_0$ and $p_1 e = 0$.

Think of f and g as an indexes whose behaviors are the same then the extensionality of e requires that g lies in e *in so far* as f lies in e and is defined.

Extensionality

Lemma

If e is extensional and has a complement e′ then e′ is extensional.

PROOF: Suppose $e'(f(x)) \bullet y = h(x) \bullet y$ then we always have

$$h(x)_{|e'(f(x))} = e(h(x)_{|e'(f(x))}) \sqcup e'(h(x)_{|e'(f(x))})$$

so it suffices to show that $e(h(x)_{|e'(f(x))}) = e(h(x))_{|e'(f(x))} = 0$. Consider

$$\begin{aligned} e(h(x)_{|e'(f(x))}) \bullet y &= h(x) \bullet y_{|e(h(x)),e'(f(x))} \\ &= e'(f(x)) \bullet y_{|e(h(x)),e'(f(x))} \\ &= e'(f(x))_{|e(h(x))} \bullet y \end{aligned}$$

Now using extensionality of e:

$$e'(f(x))_{|e(h(x))} = e(e'(f(x))_{|e(h(x))}) = 0$$

□

Rice's theorem

Theorem (Rice's theorem)

In a non-trivial disjoin Turing category no non-trivial extensional idempotent is complemented.

PROOF: Suppose e with complement e' is extensional (so both are) and non-trivial (so both are). Thus, there are points p_0 and p_1 with $p_0 e = p_0$ and $p_1 e' = p_1$. Using the second recursion theorem define a point h by:

$$h \bullet x = p_1 \bullet x_{|e(h)} \sqcup p_0 \bullet x_{|e'(h)}$$

then

$$\begin{aligned} e(h) \bullet x &= h \bullet x_{|e(h)} = (p_1 \bullet x_{|e(h)} \sqcup p_0 \bullet x_{|e'(h)})_{|e(h)} \\ &= p_1 \bullet x_{|e(h)} = (p_1)_{|e(h)} \bullet x \end{aligned}$$

so using extensionality we have:

$$(p_1)_{|e(h)} = e((p_1)_{|e(h)}) = e(p_1)_{|e(h)} = 0$$

which implies $e(h) = 0$ but by symmetry $e'(h) = 0$ giving $h = 0$ showing the category must collapse. □

Conclusion ...

The basic ideas of computability can be expressed quite smoothly in Turing Categories but ...

The BIG Question:

Can Turing categories bring new insights to computability theory?