Introduction to Restriction Categories

Robin Cockett

Department of Computer Science
University of Calgary
Alberta, Canada

robin@cpsc.ucalgary.ca

Estonia, March 2010

DEFINITION

A **restriction category** is a category with a restriction operator

$$\frac{A \xrightarrow{f} B}{A \xrightarrow[\overline{f}]{} A}$$

satisfying the following four equations:

$$\begin{array}{ll} \textbf{[R.1]}\ \overline{f}f = f & \textbf{[R.2]}\ \overline{f}\ \overline{g} = \overline{g}\ \overline{f} \\ \textbf{[R.3]}\ \overline{f}\ \overline{g} = \overline{\overline{f}g} & \textbf{[R.4]}\ f\overline{g} = \overline{fg}f \end{array}$$

Restriction categories are abstract partial map categories.

MOTIVATING EXAMPLE

Sets and partial maps, Par:

Objects: Sets ..

Maps: $f : A \longrightarrow B$ is a relations $f \subseteq A \times B$ which is deterministic ($x\ f\ y_1$ and $x\ f\ y_2$ implies $y_1 = y_2$);

Identities: $1_A : A \longrightarrow A$ is the diagonal relation $\Delta_A \subseteq A \times A$;

Composition: Relational composition $fg = \{(a, c) | \exists b.(a, b) \in f \& (b, c) \in g\}$;

Restriction: $\overline{f} = \{(a, a) | \exists b.(a, b) \in f\}$.

The restriction gives the *domain of definition* by an idempotent.

BASIC RESULTS

In any restriction category $\mathbb{X}$:

- $\overline{f}\,\overline{f} = \overline{f}$.
- For any monic $\overline{m} = 1_A$ (in particular $\overline{1_A} = 1_A$).
- $\overline{\overline{f}} = \overline{f}$.
- $\overline{fg} = \overline{f\overline{g}}$.

BASIC RESULTS

In any restriction category $\mathbb{X}$:

- $\overline{f}\ \overline{f} = \overline{f}$ as $\overline{f}\ \overline{f} =_{[\mathbf{R.3}]} \overline{\overline{f} f} =_{[\mathbf{R.1}]} \overline{f}$.
- For any monic $\overline{m} = 1_A$ as $\overline{m} m =_{[\mathbf{R.1}]} m = 1_A m$ (in particular $\overline{1_A} = 1_A$).
- $\overline{\overline{f}} = \overline{f}$ as $\overline{\overline{f}} = \overline{\overline{f} 1_A} =_{[\mathbf{R.3}]} \overline{f}\ \overline{1_A} = \overline{f}$.
- $\overline{fg} = \overline{f\overline{g}}$ as

$$\begin{aligned} \overline{f\overline{g}} &=_{[\mathbf{R.4}]} \overline{\overline{fg} f} =_{[\mathbf{R.3}]} \overline{fg}\ \overline{f} \\ &=_{[\mathbf{R.2}]} \overline{f}\ \overline{fg} =_{[\mathbf{R.3}]} \overline{\overline{f} fg} =_{[\mathbf{R.1}]} \overline{fg} \end{aligned}$$

BASIC RESULTS

In any restriction category $\mathbb{X}$ a map $f : A \longrightarrow B$ is **total** when $\overline{f} = 1_A$:

- All monics are total (in particular identity maps are total).
- Total maps compose as f and g total means $\overline{fg} = \overline{f\overline{g}} = \overline{f1_B} = \overline{f} = 1_A$.

Lemma

The total maps of any restriction category form a subcategory $\mathrm{Total}(\mathbb{X}) \subseteq \mathbb{X}$.

Total(Par) is the category of sets and functions ...

BASIC RESULTS

In any restriction category $\mathbb{X}$ the hom-sets are partially ordered:

$$f \leq g \Leftrightarrow \overline{f} g = f$$

- $f \leq f$ ...
- $f \leq g$ and $g \leq h$ implies $f \leq h$ as
  $f = \overline{f} g = \overline{f}\ \overline{g} h = \overline{\overline{f} g} h = \overline{f} h$.
- $f \leq g$ and $g \leq f$ then $f = \overline{f} g = \overline{f}\ \overline{g} g = \overline{\overline{g} f} g = \overline{g} g = g$.

But more $f \leq g$ implies $hfk \leq hgk$ as

$$\overline{hfk}\, hgk = h\overline{fk}\, gk = h\overline{fk}\ \overline{f} gk = h\overline{fk}\, fk = hfk.$$

This means every restriction category is partial order enriched.
In Par $f \leq g$ if and only if $f \subseteq g$.

BASIC RESULTS

In any restriction category $\mathbb{X}$ the hom-sets have a compatibility structure. f is **compatible** to g, $f \smile g$, if and only if:

$$f \smile g \Leftrightarrow \overline{f}g = \overline{g}f$$

In Par this means where both maps are defined they are equal.

Compatibility is always a symmetric, reflexive relation (not transitive in general).

Lemma

In any restriction category;

(i) $f \smile g$ *if and only it* $\overline{f}g \leq f$ *and* $\overline{g}f \leq g$;

(ii) *If* $f \smile g$ *then* $hfk \smile hgk$.

BASIC RESULTS

So far ... in any restriction category $\mathbb{X}$:

- $e : A \longrightarrow A$ with $\overline{e} = e$ is called a **restriction idempotent**. The restriction idempotents on A form a semilattice $\mathcal{O}(A)$. Think of these as the "open" sets of the object.
- A map : $A \longrightarrow B$ is **total** in case $\overline{f} = 1$. All monics are total maps and total maps compose the total maps form a subcategory Total($\mathbb{X}$).
- The hom-sets are partially ordered $f \leq g \Leftrightarrow \overline{f} g = f$.
- Two parallel arrows are compatible $f \smile g$ in case $\overline{f} g = \overline{g} f$ (are the same where they are both defined).

BASIC RESULTS

A **partial isomorphism** is an $f : A \longrightarrow B$ which has a (partial) inverse $f^{(-1)}$ such that $f^{(-1)}f = \overline{f^{(-1)}}$ and $ff^{(-1)} = \overline{f}$.

Lemma

In any restriction category:

(i) If a map in a restriction category has a partial inverse then that partial inverse is unique;

(ii) Partial isomorphisms include isomorphisms and all restriction idempotents;

(iii) Partial isomorphisms are closed to composition.

In Par a partial isomorphism is just a partial map which is monic on its domain.

BASIC RESULTS

Uniqueness of partial inverses:

Suppose $fg = \overline{f}$, $gf = \overline{g}$ and $fh = \overline{f}$, $hf = \overline{h}$ then

$$\begin{aligned} g &= \overline{g}g = gfg = g\overline{f}fg = gfhfg = \overline{g}\,\overline{h}g \\ &= \overline{h}\,\overline{g}g = \overline{h}g = hfg = h\overline{f} = hfh = \overline{h}h = h \end{aligned}$$

BASIC RESULTS

The partial isomorphisms of any restriction category form a subrestriction category. A restriction category in which *all* maps are partial isomorphisms is called an **inverse category**.

Inverse categories are to restriction categories as groupoids are to categories.

RESTRICTION FUNCTORS

A **restriction functor** $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a functor such that, in addition, preserves the restriction $\overline{F(g)} = F(\overline{g})$.

A **(strict) restriction transformation** $\alpha : F \longrightarrow G$ between restriction functors is a natural transformation for which each α_X is total.

A **lax restriction transformation** $\alpha : F \longrightarrow G$ between restriction functors is a natural transformation for which each α_X is total and the naturality square commutes up to inequality:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & \leq & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$

Lemma

Restriction categories, restriction functors, and restriction transformations (resp. lax transformations) organize themselves into a 2-category Rest *(resp.* Rest_l*).*

RESTRICTION FUNCTORS

Restriction functors preserve:

- Restriction idempotents
- Total maps
- Partial isomorphisms
- Restriction monics (= partial isomorphism which are total).

EXAMPLES

- Any category is "trivially" a restriction category by setting $\overline{f} = 1$. This is a *total* restriction category as all maps are total.
- Sets and partial maps is a restriction category – in fact, a split restriction category.
- A meet semilattice S is a restriction category with one object, composition $xy = x \wedge y$, identity the top, and restriction defined by $\overline{x} = x$.
- An inverse monoid (an inverse semigroup with a unit) is a one object inverse category and thus is a restriction category. An inverse monoid is a monoid with an inverse operation $(\_)^{(-1)}$ which has
  - $(x^{(-1)})^{(-1)} = x$,
  - $(xy)^{(-1)} = y^{(-1)}x^{(-1)}$,
  - $xx^{(-1)}x = x$
  - $xx^{(-1)}yy^{(-1)} = yy^{(-1)}xx^{(-1)}$

EXAMPLES Take a directed graph, G, form a category where

Objects: Nodes of G

Maps: $A \xrightarrow{((A,s,B),S)} B$ where S is a finite prefix-closed set of paths out of A, and $(A,s,B) \in S$ is a path from $A \longrightarrow B$ called the **trunk**. Being prefix closed requires that if (A,rt,C) is a path in S, then (A,r,C') is a path in S.

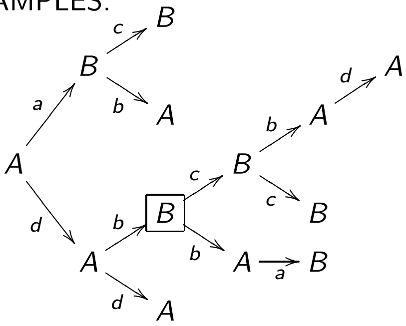
Composition: Given, $A \xrightarrow{((A,s,B),S)} B$ and $B \xrightarrow{((B,t,C),T)} C$ take the composite to be:

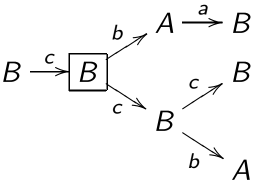
$$((A,s,B),S)((B,t,C),T) : A \longrightarrow C = ((A,st,C), S \cup (A,s,$$

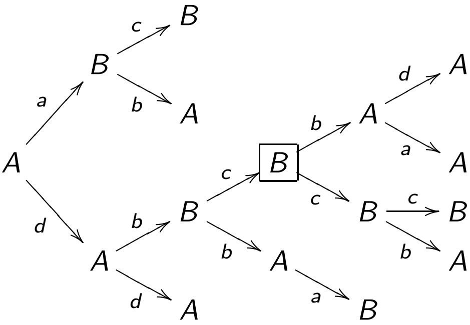
Identities: $((A,[],A),\{(A,[],A)\}) : A \longrightarrow A.$

Restriction: $\overline{((A,s,B),S)} = ((A,[],A),S)$

EXAMPLES:

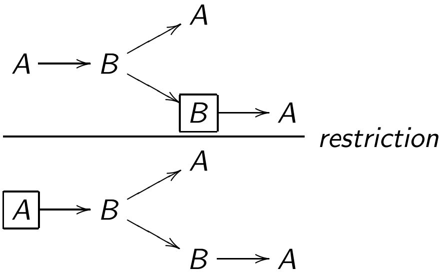






EXAMPLES

The restriction can be displayed as:



Notice that in a restriction category generated by a graph, the only total map are the identity maps $(X, [\,], X)$. Thus the only monics are the identities: this is in contrast to the free category (or path category) in which all maps are monic.

EXAMPLES

The category of meet semilattices with *stable* maps, StabSLat, is a *co*restriction category.

Objects: Meet semilattices $(L, \wedge, \top)$;

Maps: Stable maps $f : L_1 \longrightarrow L_2$ such that $f(x \wedge y) = f(x) \wedge f(y)$ (but $\top$ not necessarily preserved).

Identity: As usual the identity map ...

Composition: As usual ...

Corestriction If $f : L_1 \longrightarrow L_2$ then $\overline{f} : L_2 \longrightarrow L_2; x \mapsto f(\top) \wedge x$.

Lemma

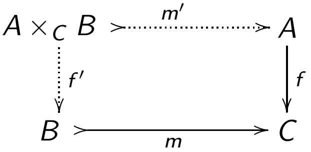
Every restriction category, $\mathbb{X}$*, has a "fundamental restriction functor"*

$$\mathcal{O} : \mathbb{X} \longrightarrow \mathsf{StabSLat}^{\mathrm{op}}$$

$\mathcal{M}$-CATEGORIES

A stable system of monics $\mathcal{M}$ in a category $\mathbb{X}$ is a class of maps satisfying:

- Each $m \in \mathcal{M}$ is monic
- Composites of maps in $\mathcal{M}$ are themselves in $\mathcal{M}$
- All isomorphisms are in $\mathcal{M}$
- Pullbacks along of an $\mathcal{M}$-map along any map always exists and is an $\mathcal{M}$-map.



An $\mathcal{M}$-category $(\mathbb{X}, \mathcal{M})$ is a category $\mathbb{X}$ equipped with a stable system of monics $\mathcal{M}$.

Think the category of sets with all injective maps $(\text{Set}, \text{Monic})$.

$\mathcal{M}$-CATEGORIES

- For any stable system of monics $\mathcal{M}$, if $mn \in \mathcal{M}$ and m is monic, then $n \in \mathcal{M}$.
- Functors between $\mathcal{M}$-categories, called $\mathcal{M}$-functors, must preserve the selected monics *and* pullbacks of these monics.
- Natural transformations are "tight" (Manes) in the sense that they are cartesian over the selected monics.

Lemma

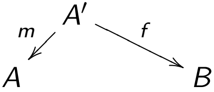
$\mathcal{M}$-categories, $\mathcal{M}$-functors, and tight transformations form a 2-category $\mathcal{M}\mathrm{Cat}$.

PARTIAL MAP CATEGORIES

The partial map category of an $\mathcal{M}$-category, written $\mathrm{Par}(\mathbb{C}, \mathcal{M})$ is a (split) restriction category:

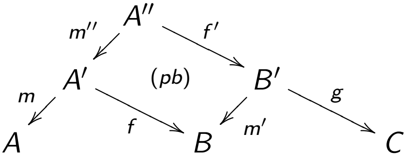
Objects: $A \in \mathbb{C}$;

Maps: $(m, f) : A \longrightarrow B$ (up to equivalence) with $m : A' \longrightarrow A$ is in $\mathcal{M}$ and $f : A' \longrightarrow B$ is a map in $\mathbb{C}$:



Identities: $(1_A, 1_A) : A \longrightarrow A$;

Composition: $(m', g)(m, f) = (mm'', gf')$:



Restriction: $\overline{(m, f)} = (m, m)$.

This gives a (2-)functor

$$\mathrm{Par} : \mathcal{M}\mathrm{Cat} \longrightarrow \mathrm{Rest}$$

PARTIAL MAP CATEGORIES

Examples of partial map categories:

- $\mathrm{Par}(\mathrm{Set}, \mathrm{Monic})$ is sets and partial maps.
- Consider the category of topological spaces with continuous maps Top: a special class of monics is the inclusions of open sets (up to iso.) open this gives $\mathrm{Par}(\mathrm{Top}, \mathrm{open})$ as a partial map category and therefore a restriction category. Now $\mathcal{O}(X)$ is just the lattice of open sets.
- Consider the category of commutative rings CRing: a *localization* is a ring homomorphism induced by freely adding (multiplicative) inverses of maps. Some facts: localizations compose and include isomorphisms, pushouts along localizations are localizations, localizations are epic. So $(\mathrm{CRing}^{\mathrm{op}}, \mathrm{loc})$ is and $\mathcal{M}$-category: $\mathrm{Par}(\mathrm{CRing}^{\mathrm{op}}, \mathrm{loc})$ is essentially the subject matter of algebraic geometry!!!

MORAL: Restriction occur everywhere AND they can be very non-trivial!

SPLITTING IDEMPOTENTS

A **restriction monic** is a monic which is a partial isomorphism:

Lemma

In any restriction category the following are equivalent:

(i) *A monic partial isomorphism;*

(ii) *A total partial isomorphism;*

(iii) *A section which splits a restriction idempotent.*

Corollary

In any restriction category:

(i) *Every restriction monic splits a* unique *restriction idempotent and has a* unique *retraction.*

(ii) *Composites of restriction monics are restriction monic.*

A restriction category is a **split restriction category** if every restriction idempotent is split by a restriction monic.

Partial map categories are split restriction categories.

SPLITTING IDEMPOTENTS For any class, E, of idempotents, $\mathrm{Split}_E(\mathbb{X})$ is a restriction category with:

Objects: Idempotents $e \in E$

Maps: $f : e_1 \longrightarrow e_2$ where $e_1 f e_2 = f$

Identities: $e : e \longrightarrow e$

Composition: As before ...

Restriction: $$\frac{e_1 \xrightarrow{f} e_2}{e_1 \xrightarrow[e_1 \overline{f}]{} e_1}$$

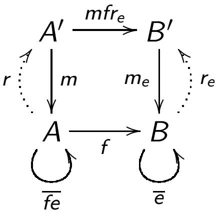
In $\mathrm{Split}_E(\mathbb{X})$ the idempotents in E are split.

We shall be mostly interested in $\mathrm{Split}_r(\mathbb{X})$ where we split the restriction idempotents: this is always a split restriction category.

COMPLETENESS

For a split restriction category, $\mathbb{X}$, the subcategory of total maps is an $\mathcal{M}$-category, where $m \in \mathcal{M}$ if and only if it is a restriction monic.

Why do pullbacks of restriction monics exist?



In that case $\mathrm{Par}(\mathrm{Total}(\mathbb{X}), \mathcal{M})$ is isomorphic to $\mathbb{X}$!!

Theorem (Completeness)

Every restriction category is a full subcategory of a partial map category.

REPRESENTATION

BTW there is also a representation theorem:

Theorem (Representation: Mulry)

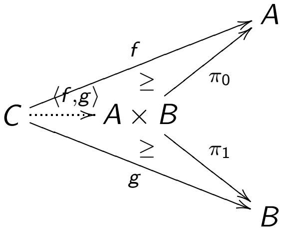
Any restriction category $\mathbb{C}$ has a full and faithful restriction-preserving embedding into a partial map category of a presheaf category

$$\mathbb{C} \longrightarrow \mathsf{Par}(\mathbf{Set}^{\mathsf{Total}(\mathsf{split}_r(\mathbb{C}))^{\mathrm{op}}}, \widehat{\mathcal{M}})$$

CARTESIAN RESTRICTION CATEGORIES

A **cartesian restriction category** is a restriction category with partial products:

- It has a restriction final object 1:
  - Each A has a total map $! : A \longrightarrow 1$
  - If $A \xrightarrow{f} 1$ then $f = \overline{f}!$.
- It has binary restriction products in case for every A and B there is a cone $(A \times B, \pi_0, \pi_1)$ such that given any other cone there is a unique comparison map



such that $\overline{g}f = \langle f, g\rangle\pi_0$ and $\overline{f}g = \langle f, g\rangle\pi_1$.

CARTESIAN RESTRICTION CATEGORIES Partial products are examples of restriction limits ...

The following equations hold in any cartesian restriction category:

- Letting $\Delta = \langle 1, 1 \rangle$ then Δ is total, $\Delta \pi_i = 1$
- $\overline{h}\langle f, g \rangle = \langle \overline{h} f, g \rangle = \langle f, \overline{h} g \rangle$
- $\overline{\langle f, g \rangle} = \overline{f}\,\overline{g}$.

In the total category the partial products become ordinary products:

Theorem

If restriction idempotents split then $\mathbb{X}$ is a cartesian restriction category if and only if $\mathrm{Tot}(\mathbb{X})$ *is a cartesian category.*

Sets and partial maps form a cartesian restriction category ...

MEETS

A restriction category has **meets**

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B}{A \xrightarrow[f \cap g]{} B}$$

if the following are satisfied:

- $f \cap g \leq f$ and $f \cap g \leq g$,
- $f \cap f = f$,
- $h(f \cap g) = hf \cap hg$.

This makes $f \cap g$ the meet of f and g in the hom-set lattice.

In sets and partial maps $f \cap g$ is the intersection of the relations.

DISCRETENESS

An object X in a *cartesian* restriction category is **discrete** in case its diagonal map

$$\Delta : X \longrightarrow X \times X$$

is a partial isomorphism. A cartesian restriction category is **discrete** in case very object is discrete.

In $\mathrm{Par}(\mathrm{Top}, \mathrm{Open})$ the discrete objects are precisely discrete topological spaces.

Sets may be viewed as the discrete objects in Top!

DISCRETENESS

Theorem

A cartesian restriction category is discrete if and only if it has meets.

PROOF: Note $\Delta(\pi_0 \cap \pi_1) = \Delta\pi_0 \cap \Delta\pi_1 = 1 \cap 1$ while

$$\begin{aligned} \overline{\pi_0 \cap \pi_1} &= \overline{\pi_0 \cap \pi_1}\langle \pi_0, \pi_1 \rangle = \langle \overline{\pi_0 \cap \pi_1}\pi_0, \overline{\pi_0 \cap \pi_1}\pi_1 \rangle \\ &= \langle \pi_0 \cap \pi_1, \pi_0 \cap \pi_1 \rangle = (\pi_0 \cap \pi_1)\Delta \end{aligned}$$

Conversely set $f \cap g = \langle f, g \rangle \Delta^{(-1)}$. $\square$

JOINS

A restriction category has a **restriction zero** in case there is a zero map between every pair of objects $A \xrightarrow{0} B$ (with $f0 = 0$ and $0g = 0$) such that $\overline{0_{A,B}} = 0_{A,A}$.

A restriction category has **joins** if

- It has a restriction zero
- Whenever $f \smile g$ there is a join of the maps, $f \vee g$, such that
  - $f, g \leq f \vee g$ and whenever $f, g \leq h$ then $f \vee g \leq h$
  - The join is "stable" in the sense that $h(f \vee g) = hf \vee hg$.

NOTE: stability implies that the join is also "universal" in the sense that $(f \vee g)h = fh \vee gh$.

Sets and partial maps have joins given by the union of relations.

JOINS

In a join restriction category *coproducts are absolute* (i.e. preserved by any join preserving restriction functor)

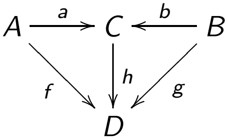
Theorem

In any restriction category with joins $A \xrightarrow{a} C \xleftarrow{b} B$ *is a coproduct iff a and b are restriction monics such that* $\overline{a^{(-1)}}\,\overline{b^{(-1)}} = 0$ *and* $\overline{a^{(-1)}} \vee \overline{b^{(-1)}} = 1_C$*.*

PROOF: To define the copairing map $\langle f|g\rangle := (a^{(-1)}f) \vee (b^{(-1)}g)$ where $a^{(-1)}f \smile b^{(-1)}g$ as $\overline{a^{(-1)}}\,\overline{b^{(-1)}} = 0$. Then

$$a((a^{(-1)}f) \vee (b^{(-1)}g)) = (aa^{(-1)}f) \vee (ab^{(-1)}g) = f \vee 0 = f.$$

It remains to show this map is unique:



then $a^{(-1)}f = a^{(-1)}ah = \overline{a^{(-1)}}h$ and

$$h = 1h = (\overline{a^{(-1)}} \vee \overline{b^{(-1)}})h = (\overline{a^{(-1)}}h) \vee (\overline{b^{(-1)}}h) = (a^{(-1)}f) \vee (b^{(-1)}g).$$

JOINS AND MEETS

A remarkable fact of nature:

Lemma

In any meet restriction category with joins the meet distributes over the join:

$$h \cap (f \vee g) = (h \cap f) \vee (h \cap g).$$

PROOF:

$$\begin{aligned} h \cap (f \vee g) &= \overline{(f \vee g)} h \cap (f \vee g) \\ &= (\overline{f} \vee \overline{g}) h \cap (f \vee g) \\ &= (\overline{f}(h \cap (f \vee g))) \vee (\overline{g}(h \cap (f \vee g))) \\ &= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g)) \\ &= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g)) \\ &= (h \cap f) \vee (h \cap g) \end{aligned}$$

□

JOINS

Another remarkable fact of nature:

Lemma

In any cartesian restriction category with joins $(f \vee g) \times h = (f \times h) \vee (g \times h)$.

PROOF: We shall first prove $\langle f \vee g, h\rangle = \langle f, h\rangle \vee \langle g, h\rangle$:

$$
\begin{aligned}
\langle f \vee g, h\rangle &= \overline{\langle f \vee g, h\rangle}\langle f \vee g, h\rangle = \overline{f \vee g}\,\overline{h}\langle f \vee g, h\rangle \\
&= \overline{f} \vee \overline{g}\langle f \vee g, h\rangle = (\overline{f}\langle f \vee g, h\rangle) \vee (\overline{g}\langle f \vee g, h\rangle) \\
&= \langle \overline{f}(f \vee g), h\rangle \vee \langle \overline{g}(f \vee g), h\rangle = \langle f, h\rangle \vee \langle g, h\rangle
\end{aligned}
$$

Now

$$(f \vee g) \times h = \langle \pi_0(f \vee g), \pi_1 h\rangle = \langle (\pi_0 f) \vee (\pi_0 g), \pi_1 h\rangle = \langle \pi_0 f, \pi_1 h\rangle \vee \langle \pi_0 g, \pi_1$$

$\square$

COPRODUCTS

Why is this all so remarkable?

A restriction category is a **distributive** in case it has a restriction coproducts and the products distribute over the coproducts.

In a join restriction category $\mathbb{X}$ as coproducts are absolute and $A \times \_$ as a functor preserves joins it follows that if $\mathbb{X}$ has coproducts it is *necessarily* distributive.

Local structure (joins) implies global structure (distributivity).

PROSPECT ...

We are interested in categories which express computability. Some properties are:

A. They are restriction categories ....
B. They are *cartesian* restriction categories ...
C. They have joins ...
D. They are discrete (they have meets) ...
E. They have coproducts.

We now know this structure together has some surprisingly pleasant consequences!

What else .... Turing categories.