Exercises: restriction categories

November 20, 2006

1. Prove carefully that in any restriction category:

   (a) $\overline{f} = \overline{\overline{f}}$, $\overline{f} = \overline{ff}$;

   (b) $\overline{fg} = \overline{f\overline{g}}$;

   (c) If $f : A \longrightarrow B$ is monic that $\overline{f} = 1_A$;

   (d) The composition of two total maps (a map $f : A \longrightarrow B$ is total if $\overline{f} = 1_A$) is total;

   Conclude that the total maps form a subcategory of any restriction category.

2. In what sense is the restriction operator a *combinator* recall that a combinator is usually expected to satisfy a "naturality" condition:

$$\frac{F(X) \xrightarrow{h} G(X)}{F'(X) \xrightarrow[c[h]]{} G'(X)}$$

   such that if $F(f)h' = hG(f)$ then $F'(f)c[h'] = c[h]G'(f)$. Show that for a restriction

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "x"'] & X \arrow[d, "x"] \\ X' \arrow[r, "\overline{g}"'] & X' \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd}$$

   commutes whenever x and y are total.

3. A map $f : A \longrightarrow B$ is a restricted isomorphism (or partial isomorphism) in case there is a a map $f^{(-1)} : B \longrightarrow A$ with $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$. Prove that:

   (a) If f is a restricted isomorphism $f^{(-1)}$ is unique.

   (b) A restricted monic (a section of a restriction idempotent) is a restricted isomorphism.

   (c) Restricted isomorphisms compose.

   (d) In any restriction category the full subcategory of restricted inverses forms a restriction category $\mathsf{Inv}(\mathbb{X}) \subseteq \mathbb{X}$.

4. Prove that with respect to any set of idempotents E, that if $\mathbb{X}$ is a restriction category then $K_E(\mathbb{X})$ is a restriction category. Prove that when the restriction idempotents are chosen the total category has pullbacks the splittings of the idempotents form a system of $\mathcal{M}$-maps and that $K_r(\mathbb{X}) \equiv \mathsf{Par}_{\mathcal{M}}(\mathsf{Total}(\mathbb{X}))$.

5. Prove that for any restriction category defining for parallel arrows $f \leq g \Leftrightarrow \overline{f}g = f$ provides an enrichment into posets. Specifically prove that:

   (a) $f \leq g$ so defined produced a partial order on the arrows;

   (b) The composition of arrows respects the partial order:

   $$f \leq f' \;\;\&\;\; g \leq g' \Rightarrow fg \leq f'g'.$$

   Conversely (harder) Pieter conjectured that if a category has a restriction combinator and that the above definition of inequality gives an enrichment, that the category is necessarily a restriction category. Is he right?

6. Prove that a category with a restriction operator satisfying $\overline{1} = 1$ and $\overline{\overline{f}g}h\overline{k} = \overline{g}\,\overline{hkf}h$ is precisely a restriction category.

7. A closure system on a restriction category $\mathbb{X}$ is a second restriction operator:

   $$\frac{A \xrightarrow{f} B}{A \xrightarrow[\mathsf{j}(f)]{} B}$$

   such that

   [Rj.1] $\overline{f}\mathsf{j}(f) = \overline{f}$

   [Rj.2] $\overline{\mathsf{j}(f)} = \mathsf{j}(f)$

   [Rj.3] $h\mathsf{j}(f) = \mathsf{j}(hf)h$

   [Rj.4] $\mathsf{j}(f)\mathsf{j}(g) = \mathsf{j}(\overline{f}g)$.

   (a) Prove that $\mathsf{j}(1_A) = 1_A$, $\mathsf{j}(\mathsf{j}(f)) = \mathsf{j}(f)$ and whenever $f \leq g$ then $\mathsf{j}(f) \leq \mathsf{j}(g)$.

   (b) A map f is said to be j-closed if $\overline{f} = \mathsf{j}(f)$: prove that the j-closed maps form a restriction category $\mathbb{X}_{\mathsf{j}}$ and that there is a restriction preserving inclusion $\mathbb{X}_{\mathsf{j}} \longrightarrow \mathbb{X}$;

   (c) Show that

   $$\frac{A \xrightarrow{f} B}{A \xrightarrow[1_A]{} A} \qquad \frac{A \xrightarrow{f} B}{A \xrightarrow[\overline{f}]{} A} \qquad \frac{A \xrightarrow{f} B}{A \xrightarrow[\mathsf{j}_1(f)\mathsf{j}_2(f)]{} A}$$

   are all closure systems (where j_1 and j_2 are closure systems).

   (d) Show that the objoius inclusion functor $\mathcal{I} : \mathbb{X}_{\mathsf{j}} \longrightarrow \mathbb{X}$ preserves the restriction.

8. A Cartesian restriction category is a restriction category with "partial products" and a "partial terminal object".

We shall say that $A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$, where π_0 and π_1 are total, is a partial product in a restriction category in case given any other $A \xleftarrow{f} C \xrightarrow{g} B$ there is a unique map $\langle f, g\rangle : C \longrightarrow A \times B$ such that $\langle f, g\rangle\pi_0 \leq f$, $\langle f, g\rangle\pi_1 \leq g$, and $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$.

We shall say that an object 1 in a restriction category is terminal in case for each A there is a unique total map $! : A \longrightarrow 1$ and any map $g : A \longrightarrow 1$ has $g = \overline{g}!$.

Prove that;

(a) $\langle f, g\rangle\pi_0 = \overline{g}f$ and $\langle f, g\rangle\pi_1 = \overline{f}g$. Furthermore, show that having a unique map $\langle f, g\rangle$ (as above) satisfying these two conditions equivalently determines the partial product.

(b) Prove that partial products and partial terminals in the total subcategory are products and terminals respectively. Conclude that partial products and partial terminals are determined up to a unique isomorphism.

(c) In a cartesian restriction category prove that $\_ \times \_ : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ is a restriction functor (i.e. a functor with the additional property that it preserves restrictions) and $\Delta = \langle 1, 1\rangle$ is a natural transformation all of whose components are total. Show that in generat the projections are *not* natural.

(d) In a Cartesian restriction category prove that $\overline{f} = \langle 1, f\rangle\pi_0$.

(e) In a cartesian restriction category prove that $\mathbb{X}(A, 1) = \{e = \overline{e} | e : A \longrightarrow A\}$.

(f) Prove that if $\mathbb{X}$ is a cartesian restriction category that $\mathbb{X}_{\mathrm{j}}$ is for any closure system.

9. A cartesian restriction category is said to be **discrete** in case $\Delta = \langle 1, 1\rangle : A \longrightarrow A \times A$ is a restricted isomorphism. Prove that:

(a) The full subcategory of all discrete objects is a (discrete) cartesian restriction category;

(b) The total subcategory of a discrete caretsian restriction category has all finite limits;

(c) An arrow f in a discrete cartesian category is monic if and only if $\overline{(f \times f)\Delta^{(-1)}} = \overline{\Delta^{(-1)}}$.

10. A cartesian closed restriction category is a cartesian restriction category $\mathbb{X}$ with the property that to each pair of objects there is an object $A \rightharpoonup B$ with a map

$$\bullet_{A,B} : (A \rightharpoonup B) \times A \longrightarrow B$$

such that given any $g : C \times A \longrightarrow B$ with $(e \times 1)g = g$, where $e = \overline{e}$ then there is a unique map $\widehat{g}^e$, with $e\widehat{g}^e = \widehat{g}^e$ such that

$$\begin{tikzcd}
C \times A \arrow[r, "g"] \arrow[d, dotted, "\hat{g}\times 1"'] & B \\
(A \rightharpoonup B) \times A \arrow[ur, "\bullet"'] &
\end{tikzcd}$$

commutes. (This is the relativized definition).

Prove that:

(a) $e'\widehat{g}^e = \widehat{g}^{ee'}$ for any $\overline{e'} = e'$. Conclude that $\widehat{g}^e = e\widehat{g}^1$ (we shall drop the superscripted identity map so $\widehat{g}^1$ will be written $\widehat{g}$) and $\widehat{g}$ is total.

(b) (Harder) Define the following combinator:

$$\frac{A \times X \xrightarrow{f} B}{A \times X \xrightarrow{\overline{f}^{[X]}} A}$$

where $\overline{f}^{[X]} = \langle \pi_0, f \rangle \pi_0$. Prove that there is a total map

$$\overline{(\_)} : A \rightharpoonup B \longrightarrow A \rightharpoonup A$$

such that $\widehat{f(\_)} = \widehat{\overline{f}^{[x]}}$. (i.e. this is the internal representation of restriction).

(c) (Harder) Prove that $1 \rightharpoonup A \xrightarrow{\bullet_{1,A}} A$ has a section $A \xrightarrow{\eta} 1 \rightharpoonup A$ which induces a restriction idempotent $1 \rightharpoonup A$ and that this makes $1 \rightharpoonup A$ a partial map classifier.

11. Prove that sets and partial maps form a cartesian closed restriction category.

12. By DCPO is meant the category of directed complete posets with Scott-open partial maps. It has the following description:

Objects: These are posets (partially ordered sets), P, which are directed complete: this means that given any directed set $D \subseteq P$ there is a least upper bound $\bigvee D$ called the directed join. A set is directed in case it is non-empty and for each $x, y \in D$ there is a $z \in D$ with $x \leq z$ and $y \leq z$. Note that $\bigvee$ is a partial operation which is only defined on (non-empty) directed sets.

Maps: These are *partial* maps $f : P_1 \longrightarrow P_2$ which preserve the directed joins $f(\bigvee D) = \bigvee f(D)$. Note that if $f(D)$ is empty then $f(\bigvee D)$ cannot be defined.

(a) Prove that these partial maps preserve the partial order on their domain of definition.

(b) Prove that the domain of definition of any map is a Scott-open subset, that is a subset which cannot be entered by a directed join of elements outside it. Show that

i. Finite intersections, and arbitrary unions of Scott-open sets are Scott-open.

ii. The set of all subsets of a set $\mathcal{P}(A)$ is a poset with the usual inclusion of sets. It is also directed complete (in fact it is complete). Show that in here:

A. $\{\{a\} | a \in A\}$ is not Scott-open;

B. $\{U \subseteq A | U \neq \emptyset\}$ is Scott-open;

C. $\{U \subseteq A || U| \geq \infty\}$ is not Scott-open;

D. Characterize the Scott-open subsets of $\mathcal{P}(A)$ in terms of open sets: (Hint show $\uparrow U = \{V | U \subseteq V\}$ where U is finite gives a basis for the open sets.)

iii. Show that the partial maps on the natural numbers $\mathbb{N} \rightharpoonup \mathbb{N}$ form a directed complete poset but not a complete poset.

(c) Prove that the above definition yields a category which is a restriction category. Prove that it is actually a cartesian closed restriction category.

(d) What are the discrete object in DCPO.

(e) Show that in the Engeler graph model (essentially as presented by Pieter) $\mathcal{F}(U)$ is an object in this category such that $\mathcal{F}(U) \rightharpoonup \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is a section.

(f) Prove that $\mathcal{F}(U)$ is a partial combinatory algebra.

(g) Describe a fixed point combinator Y such that $\mathsf{Y} \bullet x = x \bullet (\mathsf{Y} \bullet x)$ on $\mathcal{F}(U)$.