Categorical proof theory: exercises on monidal and linearly distributive categories

Robin Cockett

November 2, 2006

A (non-symmetric) **monoidal category** is a category $\mathbb{C}$ with a bifunctor $_ \otimes _$, an object $\top$

$$_ \otimes _ : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \top : 1 \longrightarrow \mathbb{C}$$

and natural isomorphisms

$$\begin{array}{rcl} a_{\otimes} : (X \otimes Y) \otimes Z & \longrightarrow & X \otimes (Y \otimes Z) \\ u^R_{\otimes} : X \otimes \top & \longrightarrow X & \\ u^L_{\otimes} : \top \otimes X & \longrightarrow X & \end{array}$$

such that

$$\begin{array}{ccc} (X \otimes \top) \otimes Z & \xrightarrow{a_{\otimes}} & X \otimes (\top \otimes Z) \\ & {\scriptstyle u^R_{\otimes}\otimes 1} \searrow \quad \swarrow {\scriptstyle u^L_{\otimes}} & \\ & X \otimes Y & \end{array}$$

$$\begin{array}{ccccc} ((X \otimes Y) \otimes Z) \otimes W & & \xrightarrow{a_{\otimes}} & & (X \otimes Y) \otimes (Z \otimes W) \\ \downarrow {\scriptstyle a_{\otimes}\otimes 1} & & & & \downarrow {\scriptstyle a_{\otimes}} \\ (X \otimes (Y \otimes Z)) \otimes W & \xrightarrow[a_{\otimes}]{} & X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow[1\otimes a_{\otimes}]{} & ((X \otimes Y) \otimes Z) \otimes W) \end{array}$$

(1) Prove that:

(a) Any category with products is a monoidal category (with monoidal structure the product);

(b) Any category with coproducts is a monoidal category (with monoidal structure the coproduct);

(c) If $\mathbb{X}$ and $\mathbb{Y}$ are monoidal categories then $\mathbb{X} \times \mathbb{Y}$ is a monoidal category (with $(X, Y) \otimes (X', Y') = (X \otimes X', Y \otimes Y')$);

(d) That **Set** $\times$ **Set** is a monoidal category with tensor $(S, S') \otimes (T, T') = (S \times T, S' + T')$;

(e) That **Set** with the Koslowski tensor:

$$X \otimes Y = \begin{cases} 1 & \text{if neither } X \text{ or } Y \text{ are empty} \\ X & \text{if } Y \text{ is empty} \\ Y & \text{if } X \text{ is empty} \end{cases}$$

is a monoidal category.

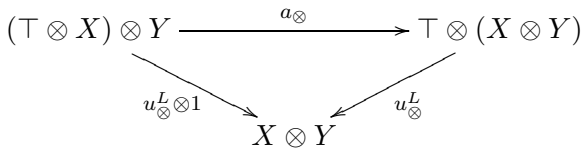
(2) Prove that in any monoidal category:

(a) The two arrows

$$\top \otimes \top \underset{u^R_\otimes}{\overset{u^L_\otimes}{\rightrightarrows}} \top$$

are equal;

(b) The following diagram commutes in any monoidal category

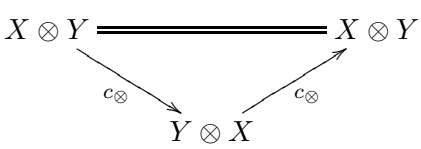


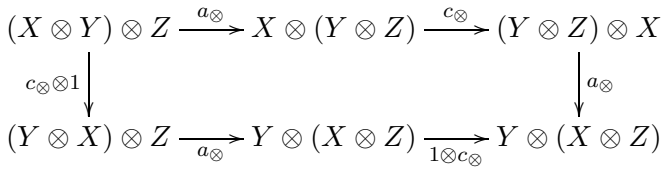
(3) Given any category $\mathbb{X}$ the category of endofunctors and natural transformations is a category. Prove that with respect to composition of functors that it is a monoidal category.

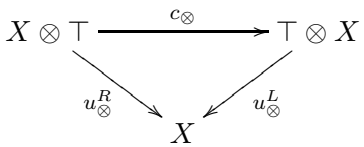
A monoidal category is **symmetric** in case there is a natural transformation

$$c_\otimes : X \otimes Y \longrightarrow Y \otimes X$$

such that







(4) Prove that any category with either products or coproducts is a symmetric monoidal category.

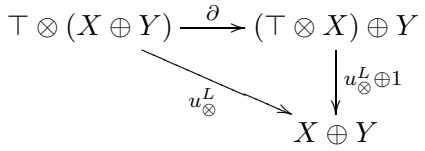
(5) Prove that **Set** with Koslowski tensor is a symmetric monoidal category.

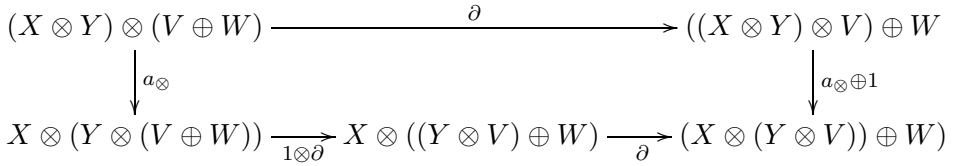
(6) Prove that the category of relations **Rel** with respect to the product of sets is a symmetric monoidal category. Show that this is neither a product nor a coproduct on this category.

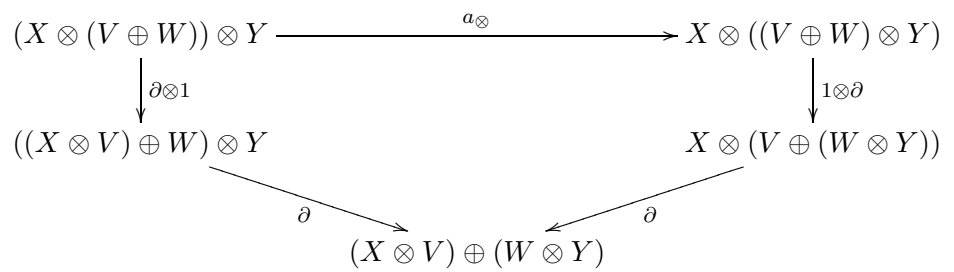
A **linearly distributive** category is a category with two monoidal structures, a "tensor", $(\otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes)$ and a "par" $(\oplus, \top, a_\oplus, u^L_\oplus, u^R_\oplus)$. These are linked by two natural transformation called the linear distributions:

$$\partial^L : X \otimes (Y \oplus Z) \longrightarrow (X \otimes Y) \oplus Z \qquad \text{and} \qquad \partial^R : (X \oplus Y) \otimes Z \longrightarrow X \oplus (Y \otimes Z)$$

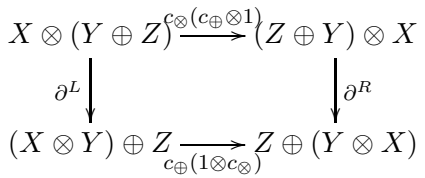
Linearly distributive categories have a left right symmetry in their axiomatization (∂^L flips to ∂^R, $u^L_\otimes$ to $u^R_\otimes$, $u^L_\oplus$ to $u^R_\oplus$, $a_\otimes$ to $a^{-1}_\otimes$, and $a_\oplus$ to $a^{-1}_\oplus$) and also an arrow direction flipping symmetry (this is a duality in the sense that the opposite of a linearly distributive category is a linearly distributive category: here the tensor and par are flipped, ∂_L and ∂_R do not change while all the monoidal map change tensor structure and become their inverse). These symmetries allow a more succinct description of the coherences:







A linearly distributive category is symmetric in case the two tensors are symmetric and



(7) Write out explicitly all the coherences implied by the two symmetries applied to these diagrams! How many are there?

(8) Prove that **Set** with respect to the product and the Koslowski tensor is a (symmetric) linearly distributive category.

(9) Prove that a distributive lattice (with respect to $\wedge$ and $\vee$) is a linearly distributive category.

(10) Prove that any monoidal category is a "compact" linearly distributive category where tensor and par coincide.

An object B in a linearly distributive category is a right linear adjoint to an object A (and A is left linear adjoint to B) in case there are maps $\eta : \top \longrightarrow A \oplus B$ and $\epsilon : B \otimes A \longrightarrow \bot$ such that:

$$\begin{array}{ccccc} A & \xrightarrow{(u^L_\otimes)^{-1}} & \top \otimes A & \xrightarrow{\eta\otimes 1} & (A \oplus B) \otimes A \\ \| & & & & \downarrow \partial \\ A & \xleftarrow[u^R_\oplus]{} & A \oplus \bot & \xleftarrow[1\oplus\epsilon]{} & A \oplus (B \otimes A) \end{array} \qquad \begin{array}{ccccc} B & \xrightarrow{(u^R_\otimes)^{-1}} & A \otimes \top & \xrightarrow{1\otimes\eta} & B \otimes (A \oplus B) \\ \| & & & & \downarrow \partial \\ B & \xleftarrow[u^L_\oplus]{} & \bot \oplus A & \xleftarrow[1\oplus\epsilon]{} & (B \otimes A) \oplus B \end{array}$$

(11) Prove that $\top$ is always a right linear adjoint to $\bot$.

(12) In a distributive lattice what is a right linear adjoint?

(13) In **Set** seen (as above in **(8)**) as a linearly distributive category which objects have a linear right adjoint?

(14) Prove that if B and B' are right linear adjoints to A then B is isomorphic to B'.

(15) Prove that if B is right linear adjoint to A and B' is right linear adjoint to A' that $B' \oplus B$ is right linear adjoint to $A \otimes A'$.

(16) Prove that if B is right linear adjoint to A and B' is right linear adjoint to A' then $B + B'$ is right linear adjoint to $A \times A'$ whenever tensor (i.e all functors $A \otimes _$ and $_ \otimes A$) preserves coproducts and par (i.e all functors $A \oplus _$ and $_ \oplus A$) preserves products.

(17) Prove that in a symmetric linearly distributive category A is left linear adjoint to B implies B is left linear adjoint to A!

A $*$-autonomous category is a symmetric monoidal closed category together with a dualizing object $\bot$ such that the natural map $A \longrightarrow (A \multimap \bot) \multimap \bot$ is an isomorphism:

(18) Prove that a symmetric linearly distributive category in which every object has a linear left adjoint is a $*$-autonomous category.

(19) Conversely prove that every $*$-autonomous category is such a linearly distributive category.

(20) (Girard) A coherence space is a set together with two reflexive symmetric relations $(S, \smile, \frown)$ such that

- Whenever $x \smile y$ and $x \frown y$ then $x = y$
- For each pair x and y either $x \smile y$ or $x \frown y$.

A map between coherence spaces is a relation R between the underlying sets $R : (S, \smile, \frown) \longrightarrow (S', \smile', \frown')$ such that

- $x \smile y \wedge xRx' \wedge yRy' \Rightarrow x' \smile' y'$
- $x' \frown' y' \wedge xRx' \wedge yRy' \Rightarrow x \frown y'$

(a) Prove that a coherence space is determined by specifying *either* $\smile$ or $\frown$;

(b) Prove that this is a category Coh;

(c) Show that there is an involution on Coh that is a functor $(_)^* : \mathsf{Coh}^{\mathrm{op}} \longrightarrow \mathsf{Coh}$ such that $(f^*)^* = f$ and has $(S, \smile, \frown)^* = (S, \frown, \smile)$ (hint: what happens on the maps ... try the converse relation).

(d) Show that there is a tensor of coherent spaces defined by:

$$(S, \smile, \frown) \otimes (S', \smile', \frown') = (S \times S', \smile \times \smile', _)$$

where the relation missing is determined by the smile!

(e) Show the involution, using de Morgan duality, determines a second tensor: the par;

(f) Show that the resulting structure is a linearly distributive category (what is ∂ try $a_\times$!);

(g) Show that $(S, \smile, \frown)$ is left linear adjoint to $(S, \frown, \smile)$;

(h) Conclude that Coh is a $*$-autonomous category!