Categorical proof theory: basic exercises on $\Sigma\Pi$

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1. Write out in detail the proof that cut elimination (done in any order) terminates on $\Sigma\Pi$-terms. (Use the bag of cut-heights as defined in class). Then go through the argument that cut-elimination is confluent modulo equations.

2. Explain how the $\Sigma\Pi$-terms can be interpreted as categorical maps in a category with products and coproducts by giving a translation. Prove carefully that all the proof equivalences of $\Sigma\Pi$ hold in any category with products and coproducts.

3. Explain why a proof $1 \vdash T$ (i.e a *point* $1 \longrightarrow P$) in $\Sigma\Pi$ is regarded as a player strategy and a proof $T \vdash 0$ (a copoint $T \longrightarrow 0$) is regarded as an opponent strategy. Describe an algorithm for determining whether a type has a point.

4. Prove that no type has both a point and a copoint in $\Sigma\Pi(\mathbb{A})$. Give an algorithm for determining whether there are proofs $T_1 \longrightarrow T_2$ in $\Sigma\Pi(\emptyset)$

5. In $\Sigma\Pi(\mathbb{A})$ how many different proofs are there 0×0 to 0×0? How many proofs are there from 0×0 to $1 + 1$ (or from 0×0 to $\Sigma_{i=0}^{n-1} 1 = n \cdot 1$)?

6. In $\Sigma\Pi(\mathbb{A})$ When is an object an initial object (i.e. it has an initial *role*). Note that, for example all the following are initial: 0, 1×0, $0 + 0$ and $(1 \times 0) + (0 \times 1)$

7. Which of the following categories have products and coproducts and what are they:

   (a) Sets and maps;

   (b) Sets and partiel maps;

   (c) Sets and relations;

   (d) Finite dimensional vector spaces and linear maps.

   Does $\Sigma\Pi(\emptyset)$ embed fully and faithfully into any of these categories? If not why not?