Proof of Cut Elimination Termination

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Introduction

The $\sigma\pi$logic [1] is a simple logic with cuts, we show here that a process for eliminating non-essential cuts in proofs terminates.

1 Brief overview of $\sigma\pi$logic

1. Atoms
$$\frac{f \in \mathbb{A}}{A \vdash B}\ \mathbb{A} \text{ is a category}$$

2. Sums $\sum_{i \in I} X_i$
$$\frac{\{X_i \vdash Y\}_{i \in I}}{\sum_{i \in I} X_i \vdash Y}\ \textit{cotuple}$$
$$\frac{}{\sum_{i \in \emptyset} \vdash Y}\ 0\ \textit{unit}$$
$$\frac{X \vdash Y_k}{X \vdash \sum_{i \in I \cup \{k\}} Y_i}\ \textit{injection}$$

3. Products $\prod_{i \in I} X_i$
$$\frac{\{X \vdash Y_i\}_{i \in I}}{X \vdash \prod_{i \in I} Y_i}\ \textit{tuple}$$
$$\frac{}{Y \vdash \prod_{i \in \emptyset}}\ 1\ \textit{unit}$$
$$\frac{X_k \vdash Y}{\prod_{i \in I \cup \{k\}} X_i \vdash Y}\ \textit{projection}$$

4. Cut
$$\frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z}$$

2 Cut elimination rules

Any proof in this system involving cuts can be transformed to one with only essential cuts(cuts between axioms) remaining.

To prove that a terminating process for performing such a transformation exists, we provide a set of proof rewrites that is guaranteed to eliminate all non essential cuts in finitely many steps.

We define the height of a proof as follows:

$$\begin{array}{rcl}
height(\frac{}{A \vdash A}) & = & 1 \\
height(\frac{}{\sum_{i\in\emptyset} \vdash Y}) & = & 1 \\
height(\frac{\{\frac{\vdots\pi_i}{X_i\vdash Y}\}_{i\in I}}{\sum_{i\in I} X_i \vdash Y}) & = & max\left\{height\left(\frac{\vdots\pi_i}{X_i \vdash Y}\right) \mid i \in I\right\} + 1 \\
height(\frac{}{Y \vdash \prod_{i\in\emptyset}}) & = & 1 \\
height(\frac{\{\frac{\vdots\pi_i}{Y\vdash X_i}\}_{i\in I}}{Y \vdash \prod_{i\in I} X_i}) & = & max\left\{height\left(\frac{\vdots\pi_i}{Y \vdash X_i}\right) \mid i \in I\right\} + 1 \\
height(\frac{\frac{\vdots\pi}{X\vdash Y_k}}{X \vdash \sum_{i\in I\cup\{k\}} Y_i}) & = & height(\frac{\vdots\pi}{X \vdash Y_k}) + 1 \\
height(\frac{\frac{\vdots\pi}{X_k\vdash Y}}{\prod_{i\in I\cup\{k\}} X_i \vdash Y}) & = & height(\frac{\vdots\pi}{X_k \vdash Y}) + 1 \\
height(\frac{\frac{\vdots\xi}{X\vdash Y} \quad \frac{\vdots\pi}{Y\vdash Z}}{X \vdash Z}) & = & 2\left(height\left(\frac{\vdots\xi}{X \vdash Y}\right) + height\left(\frac{\vdots\pi}{Y \vdash Z}\right)\right)
\end{array}$$

Note that we can treat the empty set tupling and cotupling cases uniformly if we use $max_0(A) = max\{0\} \cup A$ instead of max in the definition.

Lemma 2.1 *The height of a proof is always greater than that of its subproofs.*

PROOF:

We can see this by using structural induction and the definition of height.

□

Lemma 2.2 *If a subproof π in a proof P is replaced by a proof π' of equal or lesser height then the height of the rewritten proof P' is less than or equal to the height of P*

PROOF:

We can see by using structural induction, in the cases of tupling and cotupling, if $height(\pi) = max\{\pi_i | i \in I\}$ then $max((\{\pi_i | i \in I\} - \pi) \cup \pi') \leq height(\pi)$ since $height(\pi') \leq height(\pi)$, the remaining heights are monotonically increasing functions of the subproof heights.

□

Following Dershowitz, et al [2], we define a well founded set $(bag(\mathbb{N}), \ll)$ and a termination function:

$$\tau : \sigma\pi \to bag(\mathbb{N}); P \longmapsto \{height(\frac{\frac{\vdots \pi}{X \vdash Y} \quad \frac{\vdots \xi}{Y \vdash Z}}{X \vdash Z}) \mid \frac{\frac{\vdots \pi}{X \vdash Y} \quad \frac{\vdots \xi}{Y \vdash Z}}{X \vdash Z} \text{ is in P}\}$$

Now we present the rewriting rules.

Note that the logic and the defintion of height has sum and product symmetry so we can obtain dual proofs by reversing sequent directions and changing sums for products.

We handle all non essential cut cases, let P denote the proof before the rewrite and P' to be the proof after rewrite:

Sequent-Identity $A \vdash B \;\; B \vdash B$, B atomic

$$\frac{\frac{\vdots \pi}{A \vdash B} \quad \overline{B \vdash B}}{A \vdash B} \quad \Longrightarrow \quad \frac{\vdots \pi}{A \vdash B}$$

We can see that $height(\frac{\frac{\vdots \pi}{A \vdash B} \quad \overline{B \vdash B}}{A \vdash B})\} > height(\frac{\vdots \pi}{A \vdash B})$ so $\tau(P') \ll \tau(P)$.

Identity-Sequent $B \vdash B \;\; B \vdash A$, B atomic (dual to Sequent-Identity)

Cotuple-Sequent $\sum_{i \in I} A_i \vdash B \;\; B \vdash C$

$$\frac{\frac{\left\{ \frac{\vdots \pi_i}{A_i \vdash B} \right\}_{i \in I}}{\sum_{i \in I} A_i \vdash B} \quad \frac{\vdots \xi}{B \vdash C}}{\sum_{i \in I} A_i \vdash C} \quad \Longrightarrow \quad \frac{\left\{ \frac{\frac{\vdots \pi_i}{A_i \vdash B} \quad \frac{\vdots \xi}{B \vdash C}}{A_i \vdash C} \right\}_{i \in I}}{\sum_{i \in I} A_i \vdash C}$$

Let $\alpha_i = height\left(\frac{\vdots \pi_i}{A_i \vdash B}\right)$, $\beta = height\left(\frac{\vdots \xi}{B \vdash C}\right)$ then

$$height\left(\frac{\frac{\left\{ \frac{\vdots \pi_i}{A_i \vdash B} \right\}_{i \in I}}{\sum_{i \in I} A_i \vdash B} \quad \frac{\vdots \xi}{B \vdash C}}{\sum_{i \in I} A_i \vdash C}\right) - height\left(\frac{\left\{ \frac{\frac{\vdots \pi_i}{A_i \vdash B} \quad \frac{\vdots \xi}{B \vdash C}}{A_i \vdash C} \right\}_{i \in I}}{\sum_{i \in I} A_i \vdash C}\right)$$

$$= 2\left(\left(max\left\{\alpha_i \mid i\in I\right\}+1\right)+\beta\right)-\left(max\{2(\alpha_i+\beta)\mid i\in I\}+1\right)$$

$$= 2max\left\{\alpha_i \mid i\in I\right\}-2max\left\{\alpha_i \mid i\in I\right\}+2\beta-2\beta+2-1=1$$

We see that the height of the subproof is reduced by this rewrite, so $\tau(P') \ll \tau(P)$.

Sequent-Tupling $C\vdash B\ \ B\vdash \prod_{i\in I} A_i$ (dual to Cotuple-Sequent)

Sequent-Injection $A\vdash B\ \ B\vdash \sum_{i\in I} C_i$

$$\dfrac{\dfrac{\vdots\pi}{A\vdash B}\quad \dfrac{\dfrac{\vdots\xi}{B\vdash C_k}}{B\vdash \sum C_i}}{A\vdash \sum C_i} \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\vdots\pi}{A\vdash B}\quad \dfrac{\vdots\xi}{B\vdash C_k}}{A\vdash C_k}}{A\vdash \sum C_i}$$

Let $\alpha = height\left(\dfrac{\vdots\pi}{A\vdash B}\right)$, $\beta = height\left(\dfrac{\vdots\xi}{B\vdash C_k}\right)$ then

$$height\left(\dfrac{\dfrac{\vdots\pi}{A\vdash B}\quad \dfrac{\dfrac{\vdots\xi}{B\vdash C_k}}{B\vdash \sum C_i}}{A\vdash \sum C_i}\right)-height\left(\dfrac{\dfrac{\dfrac{\vdots\pi}{A\vdash B}\quad \dfrac{\vdots\xi}{B\vdash C_k}}{A\vdash C_k}}{A\vdash \sum C_i}\right)$$

$$= 2\left(\alpha+\beta+1\right)-\left(2\alpha+\beta+1\right)=1$$

We can see that the height of the subproof is reduced so $\tau(P') \ll \tau(P)$.

Projection-Sequent $\prod_{i\in I} C_i\vdash B\ \ B\vdash A$ (dual to Sequent-Injection)

Tupling-Projection $A\vdash \prod_{i\in I} B_i\ \ \prod_{i\in I} B_i\vdash C$

$$\dfrac{\dfrac{\left\{\dfrac{\pi_i}{A\vdash B_i}\right\}_{i\in I}}{A\vdash \prod B_i}\quad \dfrac{\dfrac{\xi}{B_k\vdash C}}{\prod B_i\vdash C}}{A\vdash C} \quad\Longrightarrow\quad \dfrac{\dfrac{\pi_k}{A\vdash B_k}\quad \dfrac{\xi}{B_k\vdash C}}{A\vdash C}$$

Let $\alpha_i = height\left(\dfrac{\vdots\pi_i}{A\vdash B_i}\right)$, $\beta = height\left(\dfrac{\vdots\xi}{B_k\vdash C}\right)$ then

$$height\left(\dfrac{\dfrac{\left\{\dfrac{\pi_i}{A\vdash B_i}\right\}_{i\in I}}{A\vdash \prod B_i}\quad \dfrac{\dfrac{\xi}{B_k\vdash C}}{\prod B_i\vdash C}}{A\vdash C}\right)-height\left(\dfrac{\dfrac{\pi_k}{A\vdash B_k}\quad \dfrac{\xi}{B_k\vdash C}}{A\vdash C}\right)$$

$$= 2\left(max\left\{\alpha_i \mid i\in I\right\}+1+\beta\right)+1)-2\left(\alpha_k+\beta\right)=2\left(max\left\{\alpha_i \mid i\in I\right\}-\alpha_k\right)+4>4$$

We can see that so $\tau(P') \ll \tau(P)$.

Injection-Cotuple $C\vdash \sum_{i\in I} B_i\ \ \sum_{i\in I} B_i\vdash A$ (dual to Tupling-Projection)

3 Proof of termination

Lemma 3.1 *If $t_1 \Rightarrow t_2$ via the rewrites above then $\tau(t_1) \gg \tau(t_2)$*

PROOF:

For each of the above rewrites $\tau(t_1) \gg \tau(t_2)$.

$\square$

Proposition 3.2 *Cut elimination done in any order on $\sum\prod$ terms terminates.*

PROOF:

Since $(bag(\mathbb{N}), \ll)$ is well ordered, there cannot be an infinite sequence of rewrites $\tau(t_1) \gg \tau(t_2) \gg \tau(t_3)$.... so the process eventually terminates.

$\square$

References

[1] J. Cockett and R. Seely. Finite sum-product logic, 2001.

[2] Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Commun. ACM*, 22(8):465–476, 1979.