Relation Categories and Allegories

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Leaving behind our previous definition of a topos, we now move on to define the notion of an allegory. Allegories give us a different perspective with which we can approach the construction of a topos, but this approach will not be discussed in these notes. Instead, we will just define an allegory and go through some significant properties. In order to see what sorts of properties we want in an allegory, we first look at how we can define categories of relations in regular categories, and then define an allegory as a category which satisfies similar nice properties.

Notational notes: in this set of notes, a "subobject" X of Y will be considered to be an isomorphism class of monomorphisms $X \rightarrowtail Y$ where $m : X \to Y$ and $m' : X' \to Y$ are equivalent if and only if $X \cong X'$. Another note to make is that these notes will be using applicative notation. That is, compositions of $f : X \to Y$ and $g : Y \to Z$ will be written as $gf : X \to Z$.

1 Relations in Regular Categories

Recall first that a regular category is a Cartesian category which has images, and whose covers are stable under pullbacks. Now we can begin to define relations in such categories, starting with the weaker condition of just Cartesian.

Definition 1.1. Let $\mathcal{C}$ be a Cartesian category and A, B be objects of $\mathcal{C}$. We say a *relation* from A to B is a subobject of $A \times B$. We denote a relation from A to B by $A \looparrowright B$.

We can fully determine any relation $A \looparrowright B$ by a monomorphism $T \to A \times B$ taken as a representative of the isomorphism class. Using this, we can define the idea of a tabulation of a relation.

Definition 1.2. Suppose $f : T \to A$ and $g : T \to B$ are morphisms in $\mathcal{C}$ such that the induced morphism $(f, g) : T \to A \times B$ is monic. Then if (f, g) belongs to the isomomorphism class of the relation $\phi : A \looparrowright B$, we say (f, g) is a *tabulation* of ϕ.

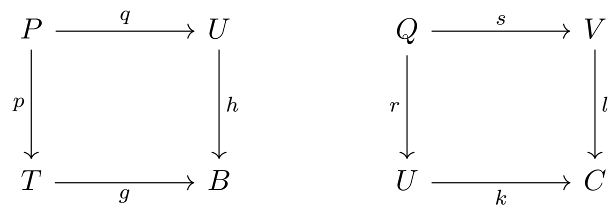
We can use tabulations of relations to define a composition of these relations now. Let $\mathcal{C}$ be a Cartesian category with images, with $\phi : A \looparrowright B$ and $\psi : B \looparrowright C$ relations in $\mathcal{C}$ tabulated by $(f, g) : T \to A \times B$ and $(h, k) : U \to B \times C$ respectively. We form the pullback

$$\begin{array}{ccc} P & \xrightarrow{q} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow{g} & B \end{array}$$

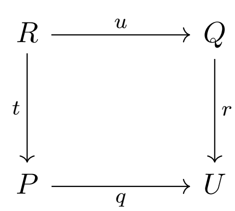
and define the composition $\psi\phi$ to be the relation tabulated by the image of $(fp, kq) : P \to A \times C$. This composition is independent of choice of tabulations, since the choice of tabulation will affect the p and q we get in our diagram, and ultimately give us the same relation in the end.

Lemma 1.3. *Let $\mathcal{C}$ be a Cartesian category with images. The composition of relations is associative if and only if $\mathcal{C}$ is regular.*

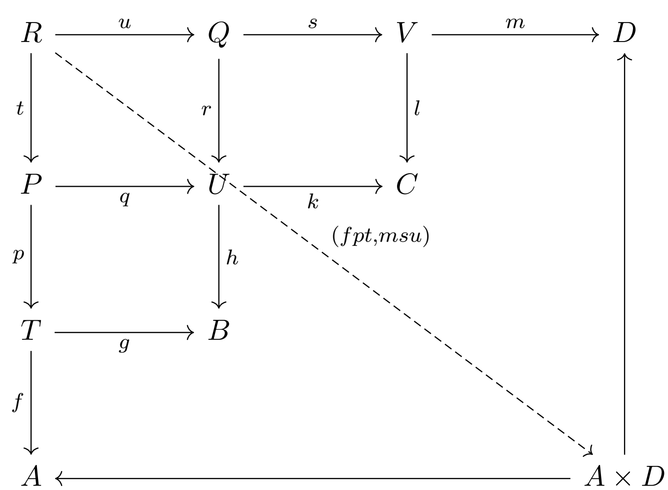
Proof. Suppose $\phi : A \nrightarrow B$, $\psi : B \nrightarrow C$, and $\chi : C \nrightarrow D$ are tabulated by $(f, g) : T \to A \times B$, $(h, k) : U : B \to C$, and $(l, m) : C \nrightarrow D$ respectively.
($\Leftarrow$) Assume $\mathcal{C}$ is regular. Then we can form the pullback diagrams



and with these objects and morphisms now defined, we can now form the pullback



using the Q, P, q, and r from the pullback diagrams above. Stacking these two diagrams together, we get the diagram



where we get the induced morphism $(fpt, msu) : R \to A \times D$ since $fpt : R \to A$ and $msu : R \to D$. We want to show that the image of (fpt, msu) tabulates both of the relations $\chi(\psi\phi)$ and $(\chi\psi)\phi$.

($\Rightarrow$) Conversely, suppose that composition is associative. We just need to show that covers are stable under pullback to show that $\mathcal{C}$ is regular. Let $e : A \to B$ be a cover in $\mathcal{C}$, and $f : C \to B$ be any arbitrary morphism in $\mathcal{C}$. Now we can define the relations $\phi : C \nrightarrow B$, $\psi : B \nrightarrow A$, and

$\chi : A \nrightarrow 1$ tabulated by $(1_C, f)$, $(e, 1_A)$, and $(1_A, !_A)$ respectively. So taking the composite $\chi\psi$

Then since $(e, !_A) : A \to B \times 1 \cong B$ is a cover, the composite $\chi\psi$ is tabulated by $(1_B, !_B)$, and so $(\chi\psi)\phi$ is tabulated by $(1_C, !_C)$.

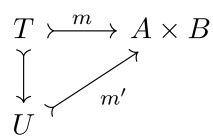
□

Corollary 1.4. *If $\mathcal{C}$ is a regular category, then relations in $\mathcal{C}$ form the morphisms of a category Rel($\mathcal{C}$) having the same objects of $\mathcal{C}$.*

Proof. We already have objects, arrows, and an associative composition from above. All that is left to prove is that there exists an identity morphism for each object. But the relation $\iota_A : A \nrightarrow A$ tabulated by the diagonal morphism $(1_A, 1_A)$ gives us exactly what we want, since pulling back along an identity morphism just gives us back an identity, so left or right composition by ι_A is composition with identities.

□

We have now that Rel($\mathcal{C}$) is a category, but in fact it is more than just a category. The category Rel($\mathcal{C}$) is actually a locally ordered 2-category, which just means that between any two relations, we can define a partial order by $\phi \leq \psi$ if and only if we have



whenever m is in the isomorphism class of ϕ and m' is in ψ. That is, a monomorphism in ϕ is the composition of a monomorphism in ψ and another monomorphism. In fact, we can notice that this partial order is preserved under composition, which is to say that if $\phi \leq \psi$, then $\phi\chi \leq \psi\chi$ and $\theta\phi \leq \theta\psi$ whenever these compositions make sense.

Since we have a 2-category, we can also talk about adjoints. In particular, a relation $\phi : A \nrightarrow B$ is left adjoint to $\psi : B \nrightarrow A$ if and only if $\phi\psi \leq \iota_B$ and $\iota_A \leq \psi\phi$. Going through to check this is rather boring, as the triangle identities fall out directly since our 2-morphisms define a poset structure. This is going to be relevant when we start looking at opposite relations and graphs of $\mathcal{C}$ morphisms, which we will begin to define now.

Definition 1.5. Let $\phi : A \nrightarrow B$ be a relation tabulated by (f, g). Then the *opposite relation* $\phi^o : B \nrightarrow A$ is the relation tabulated by (g, f).

Definition 1.6. If $f : A \to B$ is a morphism in $\mathcal{C}$, we call the induced morphism $(1_A, f) : A \to A \times B$ the *graph* of f. This morphism tabulates a relation denoted $f_\bullet : A \nrightarrow B$. We also write $f^\bullet = (f_\bullet)^o$.

Proposition 1.7.

(i) For any morphism f in $\mathcal{C}$, $f_\bullet$ is left adjoint to $f^\bullet$ in Rel($\mathcal{C}$).

(ii) Let $\phi : A \nrightarrow B$ be a morphism in Rel($\mathcal{C}$) having a right adjoint. Then there is a unique morphism f in $\mathcal{C}$ such that $\phi = f_\bullet$.

Proof.

(i) Let $f : A \to B$ be a morphism in $\mathcal{C}$. The relation $f^\bullet f_\bullet$ is tabulated by the kernel pair of f. This relation is reflexive, which exactly gives us $\iota_A \le f^\bullet f_\bullet$. The other composition, $f_\bullet f^\bullet$ is tabulated by $(f, f) : A \to B \times B$, which factors through the diagonal morphism, so $f_\bullet f^\bullet \le \iota_B$, therefore $f_\bullet$ is left adjoint to $f^\bullet$.

(ii) (Sketch) Let $\psi : B \nrightarrow A$ be the right adjoint of ϕ. Say ϕ is tabulated by monomorphism $(a, b) : T \to A \times B$ and ψ is tabulated by monic $(b', a') : T' \to B \times A$. Going through the appropriate pullback diagrams with the a and a', as well as b and b', $\iota_A \le \psi\phi$ tells us that a and a' are covers. This, along with the other inequality $\phi\psi \le \iota_B$ forces the equality $\psi = \phi^o$. Then by setting $f = ba^{-1}$, we get that $\phi = f_\bullet$. The uniqueness comes from the fact that the functor which takes $f \mapsto f_\bullet$ is faithful.

□

Definition 1.8. We call any morphism $f : A \to B$ in $\mathcal{C}$ a *map* if it has a right adjoint.

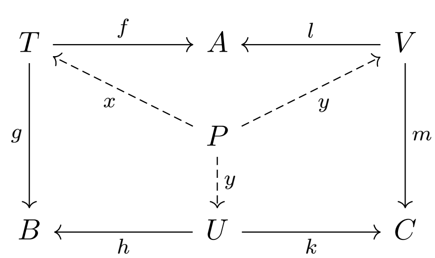
The proposition above tells us that we can actually recover $\mathcal{C}$ entirely (up to isomorphism) from $\mathrm{Rel}(\mathcal{C})$ if we look at the subcategory of $\mathrm{Rel}(\mathcal{C})$ with all objects of $\mathrm{Rel}(\mathcal{C})$, but containing only maps in $\mathrm{Rel}(\mathcal{C})$ as its morphisms.

For our next proposition, we must first note that since a regular category $\mathcal{C}$ has all of the appropriate limits, it has meets, which we can denote with $\phi \cap \psi$ for the meet of ϕ and ψ.

Proposition 1.9. *Let $\phi : A \nrightarrow B$, $\psi : B \nrightarrow C$, and $\chi : A \nrightarrow C$ be three relations in a regular category. Then*

$$\psi\phi \cap \chi \le (\psi \cap \chi\phi^o)\phi.$$

Proof. (Sketch) Let $(f, g) : T \rightarrowtail A \times B$, $(h, k) : U \rightarrowtail B \times C$, $(l, m) : V \rightarrowtail A \times C$ be tabulations of ϕ, ψ, and χ respectively. We form the limit P of the diagram



so that the x, y, and z make the diagram commute and are the universal such morphisms.

In the case where $\mathcal{C}$ is the category of sets, the object P would be the set of all triples $(a, b, c) \in A \times B \times C$ where a and b are related by ϕ, b and c are related by ψ, and a and c are related by χ. The idea behind this proof is that if you start with $(a, c) \in \psi\phi \cap \chi$, there must be some b such that $(a, b) \in \phi$ and $(b, c) \in \psi$, so $(a, b, c) \in P$. But then we get that $(b, c) \in \psi \cap \chi\phi^o$ and $(a, b) \in \phi$, so $(a, c) \in (\psi \cap \chi\phi^o)\phi$. Obviously the more general proof requires the use of more general constructions such as pullbacks, and uses the fact that image factorizations are stable under pullbacks, but the details of this are just a generalization of the same ideas we just looked at in the case of sets.

□

The above proposition is commonly referred to as the *modular law*, alluding to the similar structure we find in modular lattices. It is not quite as strong as having a distributive law, but this inequality can give us a lot of what we could hope for in relation categories.

2 Allegories and Tabulations

In this section, we will define a new type of category referred to as an allegory. Essentially, we will be forgetting everything we learned in the previous chapter, and then building all of the same structure up from scratch. The relation categories discussed in the previous section give us an idea of what shape of structures should be considered when we define allegories, and it will be immediately clear from the definition that any relation category $\mathrm{Rel}(\mathcal{C})$ on a regular category $\mathcal{C}$ is an allegory.

Definition 2.1. An *allegory* is a locally ordered 2-category $\mathcal{A}$ whose hom-posets have binary intersections equipped with an anti-involution $\phi \mapsto \phi^o$ and satisfying the modular law

$$\psi\phi \cap \chi \leq (\psi \cap \chi\phi^o)\phi$$

whenever these compositions make sense.

In an allegory $\mathcal{A}$, we denote the 1-morphisms by $\phi : A \nrightarrow B$ as in the previous section. We also retain the definition that a *map* in $\mathcal{A}$ is a morphism with a right adjoint. Note here that we will have to be wary of when the word "morphism" or "map" is used. We also continue to use Greek letters in this section for morphisms and latin letters for maps. However, we now denote the identity morphisms with 1 rather than ι.

Since we are using essentially the same definition of "maps" as in the previous section, we can again

Because of the anti-involution we have in $\mathcal{A}$, the modular law is actually equivalent to

$$\psi\phi \cap \chi \leq \psi(\phi \cap \psi^o\chi).$$

Lemma 2.2. *For any morphism ϕ in an allegory, we have $\phi \leq \phi\phi^o\phi$.*

Proof. By application of the modular law, $\phi = 1\phi \cap \phi \leq (1 \cap \phi\phi^o)\phi \leq \phi\phi^o\phi$.

□

Now let's look at one example. Consider a lattice L regarded as a locally ordered 2-category with one object, taking composition to be the join operation. Since join is commutative, the identity $\lambda^o = \lambda$ is an anti-involution. So L satisfies the conditions to be an allegory. In this case, the modular law is exactly equivalent to the more common notion of the modular law in lattice-theory, where

$$\lambda\mu \cap \nu \leq \lambda(\mu \cap \nu) \quad \text{whenever} \quad \lambda \leq \nu.$$

Now we can go into some more depth about maps in allegories, which has already been proved (partially) in proposition 1.7 for relation categories.

Lemma 2.3.

(i) In an allegory, the right adjoint of a map f is necessarily f^o.

(ii) The maps in an allegory are discretely ordered (i.e. for any maps f and g, if $f \leq g$ then $f = g$).

Proof.

(i) Let $f : A \to B$ be a map in an allegory. We can assume its left adjoint is g^o for some map g, since any adjunction $f \dashv \phi$ gives an adjunction $\phi^o \dashv f^o$. Now applying our modular law, we have

$$1_A = g^o f \cap 1_A \leq (g^o \cap 1_A f^o)f = (f^o \cap g^o)f.$$

It is not too difficult to see that $f(f^o \cap g^o) \leq fg^o$, and $fg^o \leq 1_B$ since they are adjoint, so $f(f^o \cap g^o) \leq 1_B$. But then we have that f is left adjoint to $f^o \cap g^o = (f \cap g)^o$, but adjoints are unique so $(f \cap g)^o = g^o$, so $f \cap g = g$. By a totally symmetric argument, we can find $f \cap g = f$, and therefore $f = g$, and thus $f^o = g^o$ as required.

(ii) Let $f \leq g$ be maps in an allegory. This also means that $f^o \leq g^o$, so

$$g \leq gf^o f \leq gg^o f \leq f,$$

therefore $f = g$.

Now in keeping with the theme of generalizing $\mathrm{Rel}(\mathcal{C})$, we want to define a notion of tabulations in an allegory the way we had tabulations of relations in a regular category.

Definition 2.4. Let $\phi : A \nrightarrow B$ be a morphism in $\mathcal{A}$. A *tabulation* of ϕ is a pair of maps $f : T \to A$, $g : T \to B$ with a common domain satisfying $\phi = gf^o$ and $f^o f \cap g^o g = 1_T$. If every morphism in $\mathcal{A}$ has a tabulation, we call $\mathcal{A}$ a *tabular* allegory.

We can notice that this notion of tabulation is equivalent to the one we defined in the previous section in the case of $\mathrm{Rel}(\mathcal{C})$ modulo the isomorphism $\mathrm{Map}(\mathrm{Rel}(\mathcal{C})) \cong \mathcal{C}$ mentioned in the remark after definition 1.8. The second condition in definition 2.4 is exactly what tells us that the (f, g) in a tabulation in $\mathrm{Rel}(\mathcal{C})$ is jointly monic. So the category $\mathrm{Rel}(\mathcal{C})$ is tabular.

□

Lemma 2.5. *Suppose $\phi : A \nrightarrow B$ has a tabulation (f, g), and let $x : C \to A$, $y : C \to B$ be maps. Then $yx^o \leq \phi$ if and only if there exists a map h such that $x = fh$ and $y = gh$. Moreover, if such an h exists, it is unique.*

Proof. First suppose $x = fh$ and $y = gh$. Then $yx^o = ghh^o f^o \leq gf^o = \phi$ since h is a map. Conversely, suppose $yx^o \leq \phi$. Define another morphism $\theta = f^o x \cap g^o y : C \nrightarrow T$. The idea here will be to show that θ will be a map which we will denote h which satisfies the condition required. Since x and y are maps, we have

$$1_C \leq y^o y 1_C \leq y^o y x^o x \leq y^o \phi x = y^o g f^o x.$$

So we get

$$\begin{aligned} \theta^o \theta &= (x^o f \cap y^o g)(f^o x \cap g^o y) \\ &= (x^o f \cap y^o g)(f^o x \cap (f^o x \cap g^o y)) \\ &\geq (x^o f \cap y^o g) f^o x \cap 1_C \\ &\geq 1_C \cap y^o g f^o x \cap 1_C \\ &= 1_C \end{aligned}$$

by applying the modular law twice. Similarly, we can use the modular law again to get that

$$\begin{aligned} \theta \theta^o &= (f^o x \cap g^o y)(x^o f \cap y^o g) \\ &\leq f^o x x^o f \cap g^o y y^o g \\ &\leq f^o f \cap g^o g \\ &= 1_T. \end{aligned}$$

This gives us that θ is a map, so we can denote it $\theta = h$. Now we observe

$$fh = f(f^o x \cap g^o y) \leq f f^o x \leq x$$

by the modular law, and since fh and x are both maps, we get $fh = x$ by lemma 2.3. By a similar argument we get that $gh = y$.

Finally, we must show that such an h is unique. Let k be another map which satisfying $fk = x$ and $gk = y$. Then we get

$$h = (f^o f \cap g^o g)k \leq f^o f k \cap g^o g k = f^o x \cap g^o y = h,$$

so again by lemma 2.3 we have $h = k$.

□

Corollary 2.6. *Any two tabulations of a given morphism in an allgeory are uniquely isomorphic.*

Proof. Let (f, g) and (f', g') be two tabulations of a morphism $\phi : A \nrightarrow B$, where $f : T \to A$, $g : T \to B$, $f' : T' \to A$, and $g' : T' \to B$ are maps. Applying lemma 2.5 to (f, g), and substituting $x = f'$ and $y = g'$, we get $f' = fh$ and $g' = gh$. But we can apply the lemma the other way around to get $f = f'h'$ and $g = g'h'$. Moreover, this h and h' are unique, giving us that (f, g) and (f', g') are uniquely isomorphic.

□

As in the case of relation categories, we can again define a subcategory of $\mathcal{A}$, consisting of the same objects as $\mathcal{A}$, but limiting the morphisms to only containing maps in $\mathcal{A}$. We denote this category by Map($\mathcal{A}$).

Lemma 2.7. *Let $f : A \nrightarrow B$ be a map in a tabular category $\mathcal{A}$. The following are equivalent:*

(i) *f is monic in Map($\mathcal{A}$),*

(ii) *$f^o f = 1_A$,*

(iii) *(f, f) is a tabulation of $f f^o$.*

Proof. First we show that (i) ⇒ (ii). Suppose f is monic, and let (h, k) be a tabulation of $f^o f$. Then by the definition of a tabulation, we have $f^o f = kh^o$. Using this identity, we have $fk \leq fkh^o h = ff^o fh$ and by the inequality from lemma 2.2, $ff^o f \geq fh$. But since all of these are maps, any inequalities become equalities by lemma 2.3, so we have that $fk = fh$. Now we can use that f is monic to say that in fact $k = h$. So, $1_A \leq f^o f = hh^o \leq 1_A$, again using that f and h are maps, so $f^o f = 1_A$.

Now we show that (ii) ⇒ (iii). Suppose $f^o f = 1_A$. Then, $f^o f \cap f^o f = f^o f = 1_A$, so (f, f) is a tabulation of $f f^o$.

Finally, we show that (iii) ⇒ (i). Suppose (f, f) is a tabulation of $f f^o$. Now let h, k be maps such that $fh = fk$. Then we can define maps $x = y = fh = fk$, so by lemma 2.5 this h is unique, so $h = k$ as required.

□

Proposition 2.8. *Let $\mathcal{A}$ be a tabular allegory. Then Map($\mathcal{A}$) is locally regular (that is, it has pullbacks and images, and covers are stable under pullback).*

The proof of this proposition just combines previous results using modular law and results about tabulations. However, what we really want to note from all of this is that tabular categories are an analogue to categories of relations in locally regular categories. But we already saw that Rel($\mathcal{C}$) was a tabular allegory for some regular category $\mathcal{C}$, so we want to see if we can find an analogue for Rel($\mathcal{C}$) in the case that $\mathcal{C}$ is regular, not just locally regular. So we just add the following final piece of the puzzle to serve to bridge the gap between locally regular and regular. In particular, we want to add some condition to $\mathcal{A}$ so that Map($\mathcal{A}$) has a terminal object

Definition 2.9. An object U of $\mathcal{A}$ is called a *unit* if 1_U is the largest morphism $U \nrightarrow U$, and for every object A there is a morphism $\phi : A \nrightarrow U$ with $\phi^o\phi \geq 1_A$.

As we can see in this definition, it is almost just explicitly demanding the existence of a terminal object. However, this is worth stating as a lemma so that we know that it's important enough to reference back to.

Lemma 2.10. *If U is a unit in an allegory $\mathcal{A}$, then it is a terminal object in Map($\mathcal{A}$). The converse holds if $\mathcal{A}$ is tabular.*

Now we get to the big theorem that we would really want. We can finally categorize when an allegory is really essentially Rel($\mathcal{C}$) for a regular category $\mathcal{C}$.

Theorem 2.11.

(i) If $\mathcal{C}$ is a regular category, then Rel($\mathcal{C}$) is a tabular category with a unit, and $\mathcal{C} \cong$ Map(Rel($\mathcal{C}$)).

(ii) If $\mathcal{A}$ is a tabular allegory with a unit, then Map($\mathcal{A}$) is a regular category, and $\mathcal{A} \cong$ Rel(Map($\mathcal{A}$)).

All of this has already been mostly verified or sketched previously, except the isomorphism $\mathcal{A} \cong$ Rel(Map($\mathcal{A}$)). But even for this detail, all the work has already been done in previous results. This theorem tells us finally how to really generalize the idea of Rel($\mathcal{C}$) for a regular category $\mathcal{C}$, by looking specifically at allegories which are tabular and have a unit.