DEFINING WITH EXAMPLES A TOPOS AND THE MONADICITY THEOREM

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Abstract. In this paper I aim to summarize the important concepts, definitions, examples and theorems in sections A2.1 and A2.2 in the book sketches of an elephant a topos theory compendium by Peter Johnstone.

1. Preliminary information

In this section I include some concepts in previous sections that I felt for completeness sake to include, this also has the advantage of letting the reader have consistent notation with the new and previous material (that other authors may differ from). Also I believe it will be a nice review for the reader to the recall the important notions that this section will be using as prerequisite knowledge.

Definition 1.1. Let $\mathcal{C}$ be a category with finite products, we say an object $A \in \mathcal{C}_0$ is **Exponentiable** if we are given an operation assigning to each object $B \in \mathcal{C}_0$ an object $B^A \in \mathcal{C}_0$ equipped with a morphism $ev : B^A \to B$ such that $\forall h : C \times A \to B$, $\exists! \bar{h} : C \to B^A$ such that the following diagram commutes

$$\begin{tikzcd} C \times A \arrow[r, "\bar{h} \times 1_A"] \arrow[dr, "h"'] & B^A \times A \arrow[d, tail, "ev"] \\ & B \end{tikzcd}$$

$\bar{h}$ is called the exponential transpose of h

Definition 1.2. A category $\mathcal{C}$ is called **Cartesian closed** if every object is exponentiable.

Definition 1.3. A Cartesian closed category is called **Properly Cartesian closed** if the category also has equalizers.

Definition 1.4. Let $\mathcal{C}$ be a category with pullbacks. A **generic subobject** in $\mathcal{C}$ is a monic $\top : 1 \rightarrowtail \Omega$ (1 denotes the terminal object in $\mathcal{C}$) such that, given any other monic $m : A' \rightarrowtail A$ there exists a unique $f : A \to \Omega$ making the following diagram a pullback

$$\begin{tikzcd} A' \arrow[r, "!_{A'}"] \arrow[d, tail, "m"] & 1 \arrow[d, tail, "\top"] \\ A \arrow[r, "f"] & \Omega \end{tikzcd}$$

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f is called the **Classifying Map** of the subobject m and Ω is called the **subobject classifier**. It should be noted that classifying maps classify subobjects uniquely up to isomorphism.

2. Defining with Examples a Topos

Johnstone informally describes what a topos is as: a category with all the nice **Set** like features one would want out of a category. This is a nice definition and can give the reader an intuition of what a topos is like. But what fascinated myself with topos's is how many ways one can look at a topos differently. Johnstone starts the book with giving 14 descriptions of "what a topos is like" and goes on to list them, here are some of them: 1. A topos is a category of sheaves on a site 2. A topos is the embodiment of an intuitionistic higher order theory 3. A topos is a generalized space 4. A topos is a semantics for intuitionistic formal systems 5. A topos is a setting for synthetic differential geometry. The formal definition Johnstone give is as follows.

Definition 2.1. A **Topos** is a properly Cartesian closed category with a subobject classifier.

Let $\mathcal{E}$ be a topos, we denote the exponential $\Omega^A = PA$ and $\in_A \rightarrowtail PA \times A$ for the subobject by classified by the evaluation map $ev : PA \times A \to \Omega$. This subobject has the following universal property: given an object B and a subobject $R \rightarrowtail B \times A$ $\exists! r : B \to PA$ for which there is a pullback square

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \in_A \arrow[d, tail, "ev"] \\ B \times A \arrow[r, "r \times 1_A"] & PA \times A \end{tikzcd}$$

This phenomenon is a central notion in what power objects are.

Definition 2.2. In a Cartesian category $\mathcal{E}$, a **power object** of an object $A \in \mathcal{E}_0$ is an object PA equipped with a subobject $\in_A \rightarrowtail PA \times A$ such that given an object B and subobject $R \rightarrowtail B \times A$ $\exists! r : B \to PA$ such that the following diagram is a pullback.

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \in_A \arrow[d, tail, "ev"] \\ B \times A \arrow[r, "r \times 1_A"] & PA \times A \end{tikzcd}$$

Definition 2.3. We say a Cartesian category $\mathcal{E}$ has **power objects** if there exists an operation assigning each object a power object.

Notice that by the discussion before these definitions Topos's have power objects and conversely if we are given a Cartesian category with power objects we have a subobject classifier as this is (up to isomorphism) the same thing as a power object for 1. It will be shown later in the book that a category which is Cartesian and has power objects will also be Cartesian closed but for now we distinguish between

Cartesian categories with power objects and Topos's (even though later on we will see that these are the same) with the following definition

Definition 2.4. A category $\mathcal{E}$ which is Cartesian and has power objects is a **Weak Topos**

It will be shown in later sections that weak topos's are topos's and vice versa but for now to avoid confusion (as this has not been shown yet) we use this terminology.
The first example of a topos is the category of sets. Which better be the case as $\underline{\textbf{Set}}$ is the informal basis for what we want a topos to be like.

Example 2.5. The category $\underline{\textbf{Set}}$ is a Topos

Proof. In order to show that $\underline{\textbf{Set}}$ is a topos I need to show 2 things.
1. Properly Cartesian closed
In order to show this I need to show 3 things: a) $\underline{\textbf{Set}}$ has finite products, this is indeed true and these products are just Cartesian products. b) Has equalizers, this is also true as given two maps $f, g : X \to Y$ we define the equalizer of these two maps to be $\{x \in X | f(x) = g(x)\}$ with the equalizing map being the inclusion of this set into X it is straight forward to verify that this satisfies the desired universal property. c) Every object is exponentiable, to show this let $A \in \underline{\textbf{Set}}_0$ I need to show that $\forall B \in \underline{\textbf{Set}}_0$ that there exists $B^A \in \underline{\textbf{Set}}_0$ with a map $ev : B^A \times A \to B \in \underline{\textbf{Set}}_1$ satisfying that $\forall h : C \times A \to B \in \underline{\textbf{Set}}_1$ $\exists! \bar{h} : C \to B^A \in \underline{\textbf{Set}}_1$ making the following diagram commute

$$\begin{tikzcd}
C \times A \arrow[r, "\bar{h}\times 1_A"] \arrow[dr, "h"'] & B^A \times A \arrow[d, "ev"] \\
& B
\end{tikzcd}$$

We define the exponential $B^A = [A, B] = hom_{\underline{\textbf{Set}}}(A, B)$ with evaluation map $ev : [A, B] \times A \to B$ defined element wise by $ev(f : A \to B, a \in A) = f(a) \in B$, given that map $h : C \times A \to B$ we define the unique $\bar{h} : C \to [A, B]$ by $\bar{h}(c) = h(c, a)$ which we can verify make the diagram commute by

$$\begin{tikzcd}
(c,a) \arrow[r, "\bar{h}\times 1_A"] \arrow[dr, "h"'] & (\bar{h}(c), a) \arrow[d, "ev"] \\
& h(c,a) = \bar{h}(c)(a) = h(c,a)
\end{tikzcd}$$

To check that this $\bar{h}$ is unique suppose there exists another map $\bar{h}' : C \to [A, B]$ making the following diagram commute

$$\begin{tikzcd}
C \times A \arrow[r, "\bar{h}' \times 1_A"] \arrow[r, bend left, "\bar{h} \times 1_A"] \arrow[dr, "h"] & B^A \times A \arrow[d, "ev"] \\
& B
\end{tikzcd}$$

Such that $\bar{h} \neq \bar{h}' \iff \exists c \in C$ such that $\bar{h}(c) : A \to B \neq \bar{h}'(c) : A \to B$ but this is only true $\iff \exists a \in A$ such that $\bar{h}(c)(a) \neq \bar{h}'(c)(a)$ but then the following diagram must not commute

$$\begin{tikzcd}
(c, a) \arrow[r, "\bar{h}' \times 1_A"] \arrow[dr, "h"] & (\bar{h}(c), a) \arrow[d, "ev"] \\
& h(c, a) = \bar{h}(c)(a) \neq \bar{h}'(c)(a)
\end{tikzcd}$$

Contradicting our assumption that $\bar{h}'$ made the diagram commute in the first place so $\bar{h}$ is unique. This shows that **Set** is properly Cartesian closed

2. The second thing I have to show is that **Set** has a subobject classifier. To show that **Set** has a subobject classifier I need to show that it has a generic subobject $1 \rightarrowtail \Omega$ such that given any monic (injective set function) $m : A \rightarrowtail B$ that $\exists! f : B \to \Omega$ making the following diagram commute and a pullback

$$\begin{tikzcd}
A \arrow[r, "!_A"] \arrow[d, tail, "m"] & 1 \arrow[d, tail] \\
B \arrow[r, "f"] & \Omega
\end{tikzcd}$$

I will define $\Omega = \{\triangle, \square\}$ (any two element set) and let $1 = \{\star\}$ (any one element set) and define the generic subobject $1 \rightarrowtail \Omega$ by $\star \to \square$ (which is injective). Now given any monic $m : A \rightarrowtail B$ I will define the unique $f : B \to \Omega$ pointwise by $f(b) = \square$ if $b \in im(m)$ and $f(b) = \triangle$ if $b \notin im(m)$. With these defined we can check that these maps indeed make the diagram commute as

$$\begin{tikzcd}
a \arrow[r, "!_A"] \arrow[d, tail, "m"] & \star \arrow[d, tail] \\
m(a) \in im(m) \arrow[r, "f"] & \square
\end{tikzcd}$$

Commutes, and we can verify that this forms a pull back as given the following diagram commuting

$$\begin{tikzcd}
A' \arrow[drr, bend left, "!_{A'}"] \arrow[dr, dotted, "\exists!\theta"] \arrow[ddr, bend right, "f'"'] & & \\
& A \arrow[r, "!_A"] \arrow[d, "m"] & 1 \arrow[d] \\
& B \arrow[r, "f"] & \Omega
\end{tikzcd}$$

We can define the unique theta as $\theta : A' \to A$ by $\theta(a') = a$ such that $m(a) = f'(a')$. I first must show that this is a set function, but first an observation is needed, Since the outer square commutes and elements from A' will be mapped to $\Box \in \Omega$ going across and the down this implies that going down and then across from A' everything will also be mapped to $\Box$ which is true $\iff im(f') \subset im(m)$ by the construction of f. To show that θ is a set function I need to show two things 1. Well defined: suppose $a' = a'' \in A'$ then $\theta(a')$ is equal to the $a \in A$ such that $m(a) = f'(a')$ but m is monic (injective) which implies there is only one such $a \in A \implies \theta(a'') = a$ as there is no choice but the one element dictated by the monic m. 2. Total: Suppose $a' \in A'$ then since $im(f') \subset im(m)$, $\forall a' \in A\ \exists a \in A$ such that $f'(a') = m(a)$ but this implies that $\theta(a') = a$ so θ is total and well defined and therefore a function. In order to show that this θ makes the diagram commute it is sufficient to show that $\theta m = f'$ commutes, which we can see by $m \circ \theta(a') = m(a) = f'(a')$ by definition of θ. This θ is unique as if there was another θ' making the following diagram commute

$$\begin{tikzcd}
A' \arrow[drr, bend left, "!_{A'}"] \arrow[dr, bend left, "\theta'"] \arrow[dr, bend right, "\theta"'] \arrow[ddr, bend right, "f'"'] & & \\
& A \arrow[r, "!_A"] \arrow[d, "m"] & 1 \arrow[d] \\
& B \arrow[r, "f"] & \Omega
\end{tikzcd}$$

Such that $\theta \neq \theta' \iff \exists a' \in A'$ such that $\theta(a') \neq \theta'(a')$ but then since m is monic

$$f'(a') = m \circ \theta(a') \neq m \circ \theta'(a')$$

Which shows that θ' does not make the diagram commute contradicting out assumption that θ is not unique so θ must be unique which implies the category **Set** has a sub object classifier and is hence a Topos. $\Box$

Remark 2.6. The category $\underline{\mathbf{Set}}_f$ is a topos as well. The proof is the same by replacing sets with finite sets, our power object $[A, B]$ is also finite sets as A, B are finite and our subobject classifier is a finite set already so nothing really changes.

Proving that a category is a topos takes a lot of work (as seen above) and it is because of that I will not go into as much details as I did for **Set** when showing something is a topos. As demonstrated in the next example.

Example 2.7. Let $\mathcal{C}$ be a small category then the functor category $[\mathcal{C}, \underline{\mathbf{Set}}]$ is a topos.

Proof. This proof is in the book under two statements covered in previous sections: proposition 1.5.5 and lemma 1.6.6, I will not go over them fully but to give you a brief picture, our subobject classifier $\Omega \in [\mathcal{C}, \underline{\mathbf{Set}}]_0$ will be a functor. What this does to objects is, elements of $\Omega(A)$ correspond to morphisms $\mathcal{C}(A, -) \to \Omega$ and exponentials G^F by the Yoneda lemma implies that elements of $G^F(A)$ must correspond bijectively to morphisms $\mathcal{C}(A, -) \to G^F$. Details are in the previous sections mentioned at the beginning. □

We will see later on in the book that this example generalizes substantially and that if $\mathcal{C}$ is a finite category and $\mathcal{E}$ is any topos then $[\mathcal{C}, \mathcal{E}]$ is a topos.
This example is used mostly as a stepping stone to introduce some notions of presheaves and sheaves and to generalize these concepts further. We start with the definition of a presheaf.

Definition 2.8. Let X be a topological space with open set lattice $\mathcal{O}(X)$ (considered a small preorder), then a **Presheaf** on X is a functor $F : \mathcal{O}(X)^{op} \to \underline{\mathbf{Set}}$

Example 2.9. Since $\mathcal{O}(X)$ is a small preorder $\mathcal{O}(X)^{op}$ is a small preorder which can be considered a category (objects: Open sets, morphisms: Superset relations) and by the previous example since $\mathcal{O}(X)^{op}$ is a small category the the functor category or the cateogry of presheaves $[\mathcal{O}(X)^{op}, \underline{\mathbf{Set}}]$ is a topos.

Some terminology regarding sheaves and presheaves is that if $U \in \mathcal{O}(X)^{op}_0$ and $F \in [\mathcal{O}(X)^{op}, \underline{\mathbf{Set}}]$ then elements of $F(U)$ are called sections and maps $F(U) \to F(V)$ induced by $V \subset U$ are restrictions of sections. The next definition tells us what a sheaf is.

Definition 2.10. A presheaf $F \in [\mathcal{O}(X)^{op}, \underline{\mathbf{Set}}]$ is a **Sheaf** if given any open covering $(U_i | i \in I)$ of an open set $U \in \mathcal{O}(X)^{op}_0$ and any family $(s_i | i \in I)$ of elements of $F(U_i)$ which are compatible (in the sense that for each pair (i, j) for $i, j \in I$ the restrictions of s_i and s_j are equal in $F(U_i \cap U_j)$) $\exists! s \in F(U)$ whose restriction to each U_i equals $s_i, \forall i \in I$

Example 2.11. The category of sheaves on a topologacal space denoted $Sh(X)$ is a Topos

Proof. Johnstone leaves this proof later in the book as well but for the idea of the proof is that we can get this category is Cartesian from the category being closed under finite limits. It can be show that if G is a sheaf and F is a presheaf then G^F is a sheaf and so in the category of sheaves G and F are sheaves and so there exponential is a sheaf. So $Sh(X)$ is Cartesian closed. The subobject classifier Ω will be a functor which send objects to the set of all open subsets of U. □

What I found interesting in this section is that we can generalize this notion of presheaves and sheaves to arbitrary small categories. To this end we need to define a notion of a covering to objects of this small category described below.

Definition 2.12. Let $\mathcal{C}$ be a small category. A **Coverage** of $\mathcal{C}$ is a function, assigning each object $A \in \mathcal{C}_0$ a collection $T(A)$ of families $(f_i : A_i \to A | i \in I)$ called a T-covering family, such that if $g : B \to A \in \mathcal{C}_1$ then $\exists$ a T-covering family $(h_j : B_j \to B | j \in J)$ such that each gh_j factors through some f_i

Definition 2.13. A **Site** is a small category with a coverage

The "old name" as Johnstone refers to it is a Grothendieck pre-topology, and with this coverage we can form a notion of sheaves (just like what we did before) which is given in the following definition.

Definition 2.14. Let $\mathcal{C}$ be a small category we say that a functor $f \in [\mathcal{C}, \underline{\mathbf{Set}}]$ **satisfies the sheaf axiom** for a family of morphisms $(f_i : A_i \to A | i \in I)$ if, whenever we are given a family of elements $s_i \in F(A_i)$ which are compatible in the sense that, whenever $g : B \to A_i$ and $h : B \to A_j$ satisfies $f_i g = f_j h$ (i and j not necessarily distinct) then $\exists! s \in F(A)$ such that $F(f_i)(s) = s_i$ for each $i \in I$.

Lemma 2.15. If $(\mathcal{C}, T)$ is a site ($\mathcal{C}$ a category and T a coverage) then $sh(\mathcal{C}, T)$ which is the category of functors satisfying the sheaf axiom, is a topos

Proof. Left till a later section where Johnstone proves a more general result. $\square$

An interesting consequence of this lemma is that given any regular category one can put a covering on it (called a regular covering) and then embed it (fully and faithfully) into a category of sheaves which is a topos.

3. Monadicity theorem

In this section the book covers 5 results leading up to the main theorem called the Monadicity theorem. We first start with a definition of a special functor relating to power objects.

Definition 3.1. Let $\mathcal{E}$ be a weak topos, then the assignment $A \to PA = \Omega^A$ can be made into a functor $P : \mathcal{E}^{op} \to \mathcal{E}$

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "P(B)"] \arrow[d, "f"] & PB \arrow[d, "P(f)"] \\ A \arrow[r, "P(A)"] & PA \end{tikzcd}$$

$$in\ \mathcal{E} \qquad in\ \mathcal{E}^{op} \xrightarrow{P} in\ \mathcal{E}$$

Where $P(f) : PB \to PA$ is the name of the relation $E_f \rightarrowtail PB \times A$ where E_f and the map is defined by the following pullback.

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, tail] & \epsilon_B \arrow[d, tail] \\ PB \times A \arrow[r, "1_{PB} \times f"] & PB \times B \end{tikzcd}$$

It has been awhile since we talked about names of relations but to remind the reader $P(f)$ being the name of the relation $E_f \rightarrowtail PB \times A$ means that $P(f)$ is the unique map making the following a pullback.

$$\begin{tikzcd}
E_f \arrow[r] \arrow[d, tail] & \epsilon_A \arrow[d, tail, "ev"] \\
PB \times A \arrow[r, "P(f)\times 1_A"] & PA \times A
\end{tikzcd}$$

Here P defined above is a contravariant functor, but we can define a covariant functor which is identical to P with what it does to objects (which will be used later on).

Definition 3.2. Let $\mathcal{E}_m$ be the subcateogry of monomorphisms in $\mathcal{E}$ (A topos) define $\exists : \mathcal{E}_m \to \mathcal{E}$ where if $m : A \rightarrowtail B$ is monic then $\exists m : PA \to PB$ is the name of the composite (thought of as a relation) of

$$\epsilon_A \rightarrowtail PA \times A \rightarrowtail PA \times B$$

Lemma 3.3. The functor $P : \mathcal{E}^{op} \to \mathcal{E}$ has a left adjoint namely $P : \mathcal{E} \to \mathcal{E}^{op}$

Proof. In order to show that these two functors are adjoints it is sufficient to set up natural bijections between $A \to PB$ and $B \to PA$. $B \to PA$ is in natural bijection to $B \to \Omega^A$ by definition, which is in natural bijection with $A \times B \to \Omega$ since our topos is Cartesian. Which is in natural bijection with Subobject$(A \times B)$ since we have a subobject classifier. Which is in natural bijection to $Q \rightarrowtail A \times B$ (said subobject). Which is in natural bijection to $Q \rightarrowtail B \times A$ by applying the twist isomorphism $A \times B \to B \times A$, and by reserving the steps with $Q \rightarrowtail B \times A$ we get that this is in natural bijection with $A \to PB$ $\square$

Now for a definition that will be used later on.

Definition 3.4. $\{-\} : A \to PA$ is the name of the relation $(1_A, 1_A) : A \rightarrowtail A \times A$

The reason for this notation is that in **<u>Set</u>** this morphism send elements $a \in A$ to $\{a\} \in PA$.

Lemma 3.5. Let $f : A \to B$ be a morphism in a weak topos, then
i) $\{-\}f : A \to B \to PB$ names the relation $(1_A, f) : A \rightarrowtail A \times B$
ii) $Pf\{-\} : B \to PB \to PA$ names the relation $(f, 1_A) : A \rightarrowtail B \times A$

Proof. i) Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "(1_A,f)"] & B \arrow[r] \arrow[d, tail, "(1_B,1_B)"] & \epsilon_B \arrow[d, tail] \\
A \times B \arrow[r, "f\times 1_B"] & B \times B \arrow[r, "\{-\}\times 1_B"] & PB \times B
\end{tikzcd}$$

Both of these squares are pullbacks and it is clear from considering the composite of the bottom row of arrows that $\{-\} \times 1_B \circ f \times 1_B = \{-\} \circ f \times 1_B$ names the relation $(1_A, f)$.
ii) follows from i) and Lemma 3.3 specifically the natural bijection (call it $\theta_{A,B}$) between $A \to PB$ and $B \to PA$ and so $\{-\}f : A \to PB$ is in natural bijection with $\theta_{A,B}(\{-\}f) : B \to PA = \theta_{A,B}(f)\theta_{B,B}(\{-\}) = Pf\{-\}$ as theta maps $\{-\}$ to itself. $\square$

Corollary 3.6. i) $\{-\} : B \to PB$ is monic for any object B
ii) the functor $P : \mathcal{E}^{op} \to \mathcal{E}$ is conservative.

Proof. i) Suppose $f, g : A \to B$ such that $\{-\}f = \{-\}g$, then by the previous lemma there names must be the same up to isomorphism which implies $(1_A, f) \cong (1_A, g)$ in $\mathrm{Sub}(A \times B)$ but this implies that there is an isomoprhism $i : A \to A$ where $gi = f$ and $1_A i = 1_A$ but $1_A i = i = 1_A$ so $gi = g1_A = g = f$ so $g = f$ and hence $\{-\}$ is monic.
ii) To prove this statement we first need a lemma specifically Corollary 1.6.2 which states that: in a category with a subobject classifier, every monomorphism is regular. In particular such a category is balanced. Since $\mathcal{E}$ and hence $\mathcal{E}^{op}$ has a subobject classifier, these categories must be balanced and so to show that P is a conservative functor it is sufficient to show that P is a faithful functor. Let $f, g : A \to B$ such that $Pf = Pg$ then $Pf\{-\} = Pg\{-\}$ and so (as in i)) $(f, 1_A) \cong (g, 1_A)$ in $\mathrm{Sub}(B \times A)$ but this implies (just as in i)) that $f = g$ and so P is faithful and therefor P is conservative. $\square$

The next lemma allows us to form a relationship between $\exists f$ and Pf. This is also our last lemma until the Monadicity theorem.

Lemma 3.7. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "g"] & B \arrow[d, tail, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

Be a pullback square in a weak topos with g, h being monomorphisms. The the following square commutes.

$$\begin{tikzcd} PB \arrow[r, "Pf"] \arrow[d, "\exists h"] & PA \arrow[d, "\exists g"] \\ PD \arrow[r, "Pk"] & PC \end{tikzcd}$$

Proof. We can see that the above square commutes (aka $\exists g \circ Pf = Pk \circ \exists h$) by showing that they both name the same relation, and since names of relations are unique if we can show this then these would have to be the same morphism. To show that these name the same relation consider the following diagram.

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, tail] & \epsilon_B \arrow[d, tail] \\ PB \times A \arrow[r, "1_{PB} \times f"] \arrow[d, tail, "1_{PB} \times g"] & PB \times B \arrow[d, tail, "1_{PB} \times h"] \\ PB \times C \arrow[r, "1_{PB} \times k"] & PB \times D \end{tikzcd}$$

Both squares in this diagram are a pullback, the top one by definition and the bottom one is a pullback since $PB \times (-)$ preserves pullbacks, and this bottom square is just the image of our original pullback. The left vertical composite is named by both $\exists g \circ Pf$ and $Pk \circ \exists h$ □

This lemma allows us to make some connection with P and $\exists$ but it is quite abstract and not very intuitive but a final corollary of this lemma lets us make a more concrete connection to these functors.

Corollary 3.8. If $m : A \rightarrowtail B$ is monic then $Pm \circ \exists m = 1_{PA}$

Proof. By applying the previous lemma to the following pullback square

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, tail, "1_A"] & A \arrow[d, tail, "m"] \\
A \arrow[r, "m"] & B
\end{tikzcd}$$

We obtain that the following diagram commutes

$$\begin{tikzcd}
PA \arrow[r, "P(1_A)=1_{PA}"] \arrow[d, tail, "\exists m"] & PA \arrow[d, "\exists 1_A = 1_{PA}"] \\
PB \arrow[r, tail, "Pm"] & PA
\end{tikzcd}$$

Which clear implies the desired equality. □

Now that we have obtained all of these results we are finally ready to state the punchline of this section and really what we have been building towards throughout this paper.

Theorem 3.9. Let $\mathcal{E}$ be a weak topos, then the functor $P : \mathcal{E}^{op} \to \mathcal{E}$ is monadic. In other words this functor induces an equivalence between $\mathcal{E}^{op}$ and the category of algebras induced by the adjunction $(P \dashv P)$

Proof. In order to prove this statment we use a theorem proved from an earlier section of the book,

Theorem 3.10. Let $U : \mathcal{D} \to \mathcal{C}$ be a functor and suppose,
1. U has a left adjoint
2. U is conservative
3. $\mathcal{D}$ has and U preserves coequalizers of reflexive pairs, then U is monadic.

We have already shown 1. in Lemma 3.3 and we showed 2. in Corollary 3.6 so it suffices to show 3. first thing we need to check is that $\mathcal{E}^{op}$ has coequalizers, but since $\mathcal{E}$ has equalizers $\mathcal{E}^{op}$ will have coequalizers. What is left to show is that P preserves coequalizers of reflexive pairs. Let

$$\begin{tikzcd}
E \arrow[r, tail, "m"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B
\end{tikzcd}$$

Be an equalizer diagram in $\mathcal{E}$ such that the pair (f, g) is coreflexive. Then by the following lemma (which appeared earlier in the book as well)

Lemma 3.11. let $f, g : A \to B$ be a corelfexive pair in $\mathcal{C}$ Then a morphism $e : E \to A$ is an equalizer of f and g if and only if the square

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"] & A \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback.
Using this lemma the following diagram must be a pullback.

$$\begin{tikzcd} E \arrow[r, tail, "m"] \arrow[d, tail, "m"] & A \arrow[d, "g"] \\ A \arrow[r, tail, "f"] & B \end{tikzcd}$$

Since this square is a pullback applying Lemma 3.7 we obtain that $Pf \circ \exists g = \exists m \circ Pm$ but by Corollary 3.8 we have $Pm \circ \exists m = 1_{PE}$ and $Pg \circ \exists g = 1_{PA}$. Since the diagram above commutes and since P is a functor it must be the case that $Pm \circ Pf = Pm \circ Pg$ and so the diagram below

$$\begin{tikzcd} PB \arrow[r, bend left, "Pf"] \arrow[r, bend right, "Pg"'] & PA \arrow[r, "Pm"] & PE \end{tikzcd}$$

Commutes and with the morphisms $\exists g$ and $\exists m$ will form the desired split coequalizer system in $\mathcal{E}$ and so this diagram is a coequalizer diagram. ☐

Corollary 3.12. A (weak) topos is cocartesian

Proof. This is due to any category which is monadic over a cartesian cateogyr is cartesian due to the forgetful functor from the category of algebras creating limits ☐

When people first started studying topos's they required that for a category to be considered a topos it must be both cartesian and cocartesian but after discovering this corolory they realized that the condition of cocartesian is redundant, so the definition was modified to just require a topos be Cartesian.
This concludes the two sections.

Acknowledgements

References

[1] Peter Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, 2002
[2] https://ncatlab.org/nlab/show/HomePage

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