Regular Categories

We adopt some notation convention from Dr. Cockett's online category theory notes. Unless stated otherwise, mathboldface letters such as $\mathbf{X}$ are categories, $\mathbf{X}_0$ denotes $\mathbf{X}$'s class of objects, and $\mathbf{X}_1$ denotes its class of arrows. We write composition diagrammatically and indicate applicative compositiion with '$\circ$.'

According to Peter Johnstone...

Definition 1. $\mathbf{X}$, is *regular* if it *is cartesian*, *has images*, and *covers* in $\mathbf{X}$ are stable under pullback.

A *cartesian* category is defined to be a category satisfying any one of the following equivalent conditions.

Lemma 2. *Let $\mathbf{X}$ be a category. The following are equivalent:*

(1) $\mathbf{X}$ *has finite limits*
(2) $\mathbf{X}$ *has finite products and equalizers of pairs of morphisms.*
(3) $\mathbf{X}$ *has a terminal object, products of pairs of objects, and equalizers of pairs of morphisms.*
(4) $\mathbf{X}$ *has terminal objects and pullbacks.*

Proof. Here's a sketch, details can be found in Robin's category theory notes.

It's clear that (1) implies (2) because finite products and equalizers of pairs of morphisms are examples of finite limits. More precisely, products are given as limits of functors $\mathbf{I} \longrightarrow \mathbf{X}$ where $\mathbf{I}$ is a finite discrete category, and equalizers are given by limits of functors $\mathbf{K} \longrightarrow \mathbf{X}$, where $\mathbf{K}$ is the category with two objects and a single pair of parallel non-identity arrows.

To see (2) implies (3) it's enough to notice that terminal objects are given by the empty product, and finite products imply binary products.

For (3) implies (4) it suffices to show how to construct pullbacks from binary products and equalizers of pairs of morphisms. Namely, the pullback of two maps f, g, is given by taking the product of the their domains, post composing each of the projections with f and g respectively, and equalizing the resulting pair of compositions.

Finally, (4) implies (1) is a bit of work. It helps to recall that the existence of terminal objects and pullbacks implies the existence of finite products, and therefore binary products and pullbacks. In turn, the existence of binary products and pullbacks implies the existence of equalizers. This is given as a proposition in Robin's notes in the section about (co)limits and (co)completeness.

From here the limit of a finite diagram, $\mathbf{J} \xrightarrow{D} \mathbf{X}$, can be shown to be the equalizer:

$$E \xrightarrow{\quad e \quad} \prod_{X\in\mathbf{J}_0} D(X) \underset{\langle \pi_{\partial_1(f)} \rangle}{\overset{\langle \pi_{\partial_0(f)} D(f) \rangle}{\rightrightarrows}} \prod_{f\in\mathbf{J}_1} \partial_1(f)$$

That is, precomposing each projection from the middle product with e gives a family of maps in $\mathbf{X}$, $(e\pi_{D(X)})_{X\in\mathbf{J}_0}$, which make E a cone over D (as a consequence of e equalizing the parallel arrows between the products above). Moreover, for any other cone C (with maps $(c_{D(X)})_{X\in\mathbf{J}_0}$) over D, we can see

$$c_{D(\partial_0(f))} D(f) = c_{D(\partial_1(f))}$$

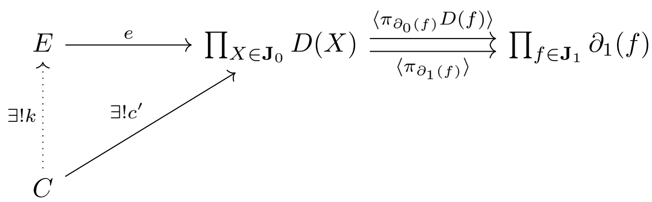
for any f in $\mathbf{J}_1$. The universal property of the product over $\mathbf{J}_0$ gives a unique map c' from C into the product such that,

$$c'\pi_{D(X)} = c_X$$

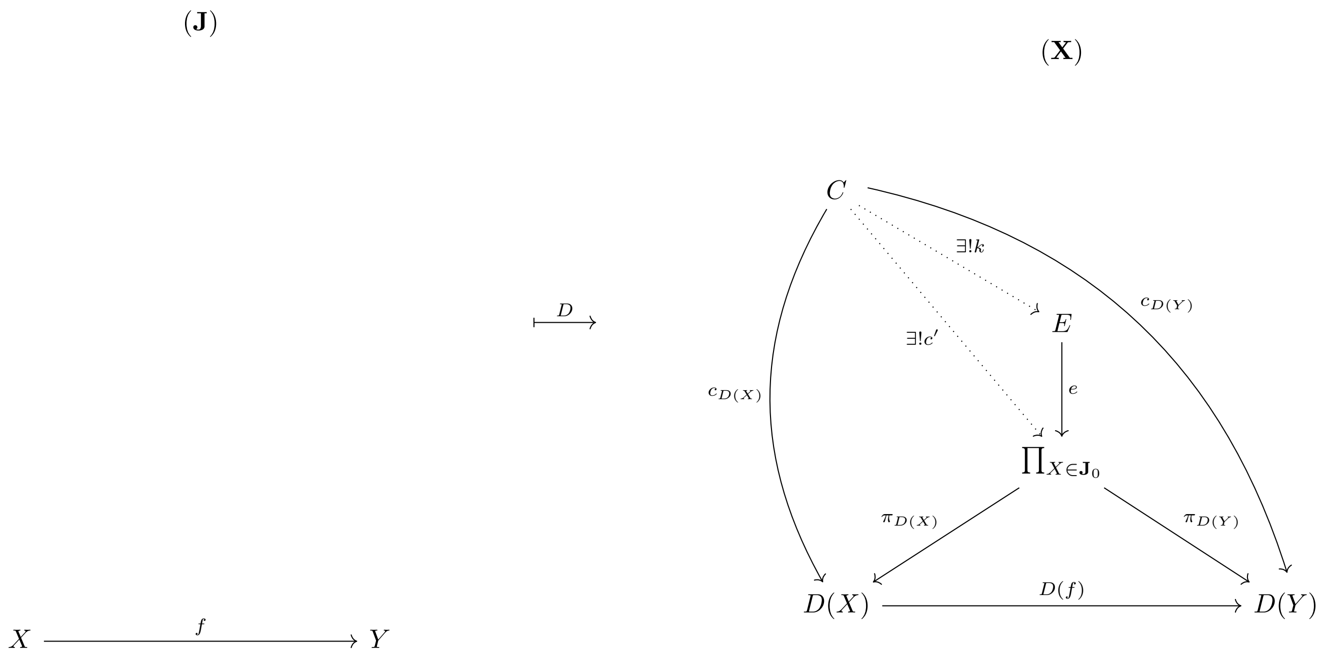
for each $X \in \mathbf{J}_0$. This implies

$$c'\pi_{\partial_0(f)} D(f) = c_{\partial_0(f)} D(f) = c_{\partial_1(f)} = c'\pi_{\partial_1(f)},$$

which shows that C, along with c', also equalize the parallel pair of arrows. Hence there's a unique map k from C to E making the following diagram commute:



It's clear from this diagram commuting that k is a morphism of cones and this the universal property of the equalizer of this parallel pair of arrows is exactly the universal property of the limit of D. The right side of the following diagram summarizes the argument we just used to show that E is the universal cone over D, and it should be read as, "if the outside commutes (for some cone C over D), then the dotted maps on the inside are guaranteed by the universal property of the product and equalizer respectively."



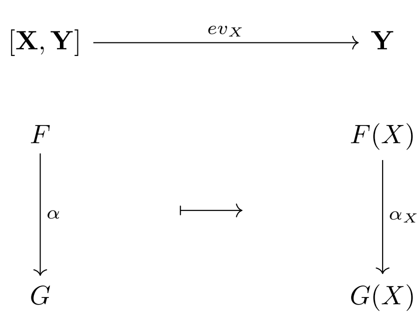
□

Definition 3. Johnstone calls a functor $F : \mathbf{X} \to \mathbf{Y}$ between cartesian categories *cartesian* if it preserves the cartesian structure up to isomorphism. That is, if F preserves finite limits.

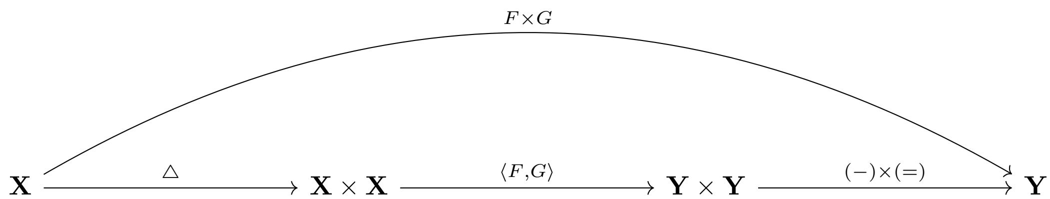
Example 4. The category of sets with functions as maps, denoted **Set**, is a cartesian category with any singleton set as a terminal object and the fibered product as the pullback. To be more precise, for any functions $f : X \to Z$, $g : Y \to Z$, their pullback is the set $\{(x, y) \in X \times Y : f(x) = g(y) \in Z\}$ with projections to X or Y given by subset inclusion into the product $X \times Y$ followed by projecting into X or Y respectively.

Example 5. Other categories structured over **Set** such as the categories of groups or topological spaces are also cartesian, where the finite limits are limits in **Set** of the underlying sets with some extra structure that needs to be accounted for. I think this is what Johnstone means when he says the cartesian structure can be chosen so that the underlying-set functor preserves it strictly; one can choose these limits as representatives because they're unique up to isomorphism, and their underlying sets will recover them exactly when forgetting the extra structure in **Set**.

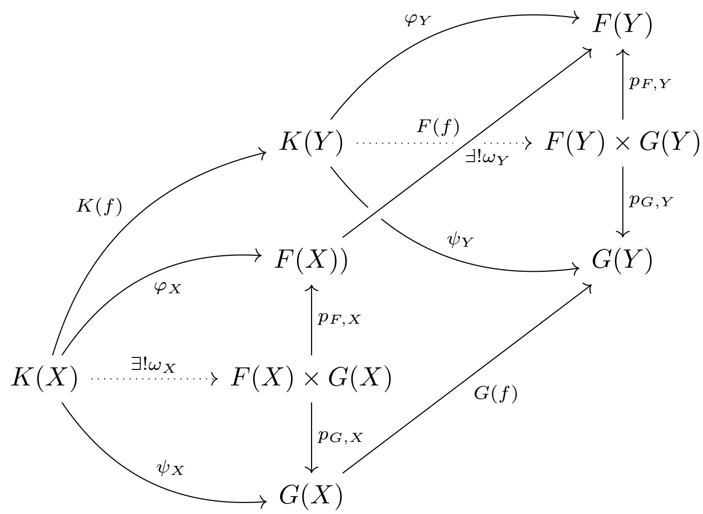
Example 6. Let $\mathbf{Y}$ be a cartesian category. Then for each $X \in \mathbf{X}_0$, the evaluation functor,



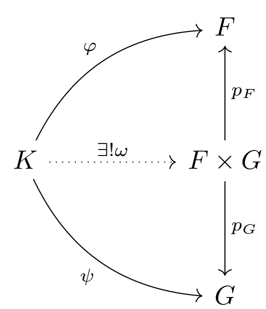
is cartesian. Where the cartesian structure in the functor category, $[\mathbf{X}, \mathbf{Y}]$, may be defined pointwise to pick up the structure from $\mathbf{Y}$ and therefore preserve it strictly. To elaborate, it makes sense to define the product of two functors F and G to be another functor which sends an object X in $\mathbf{X}$, to the product $F(X) \times G(X)$ in $\mathbf{Y}$:



The collections of pointwise projections in each variable determine natural transformations from $F \times G$ to F and G respectively, and for any other functor K with natural transformations φ and ψ from K to F and G respectively, we have the following commuting diagram:

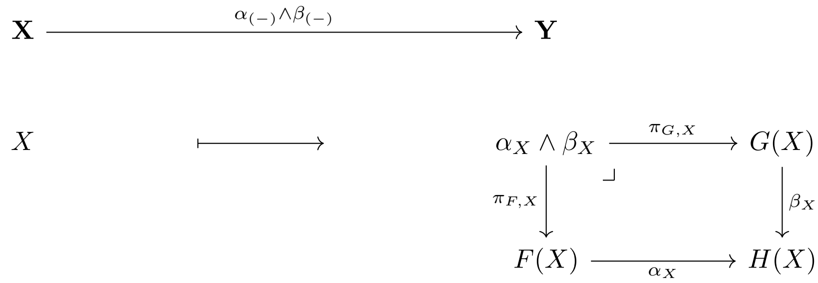


The unique ω's come from the pointwise products in $\mathbf{Y}$ and together they determine a unique natural transformation from K to $F \times G$. This gives the following product diagram in $[\mathbf{X}, \mathbf{Y}]$:

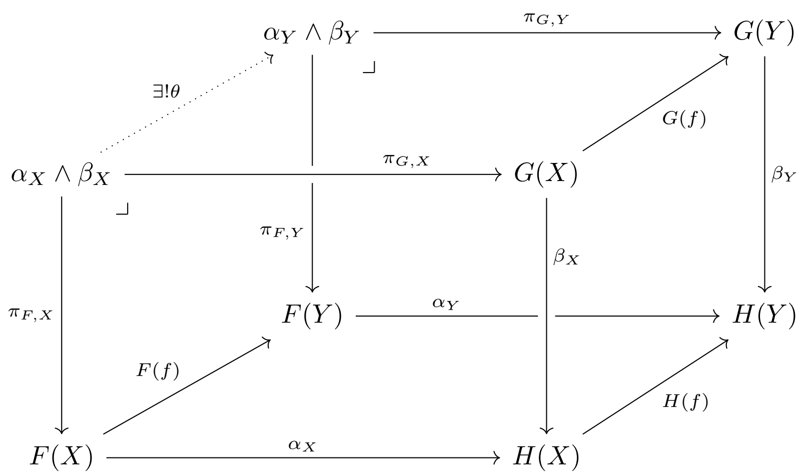


Applying the evaluation functor, for some object $X \in \mathbf{X}_0$, to this product diagram picks out the pointwise product diagram in $\mathbf{Y}$ seen at the front of the funny looking cylinder above which determined natural transfromation ω. This shows that products are strictly preserved by each evaluation functor when the functor category is given pointwise product structure.

The pullback of two natural transfromations $\alpha : F \to G$, $\beta : G \to H$, is a functor which maps an object $X \in \mathbf{X}_0$ to the pullback of the X-components of α and β in $\mathbf{Y}$:

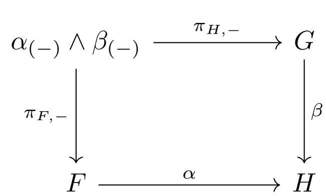


Note this is only the object assignment for the functor. The collections of projections to F and G over all objects $X \in \mathbf{X}_0$ are the natural transformations which can be seen on the top and left faces of the following commuting cube:



The front and back faces are pullbacks, while the bottom and right faces commute by naturality of α and β respectively. The dotted arrow is given by the universal property of the pullback in the back and the top and left faces show that the projections are natural transformations between the $\alpha - \beta-$pullback functor and F or H respectively.

Hence there's a commuting square in $[\mathbf{X}, \mathbf{Y}]$,



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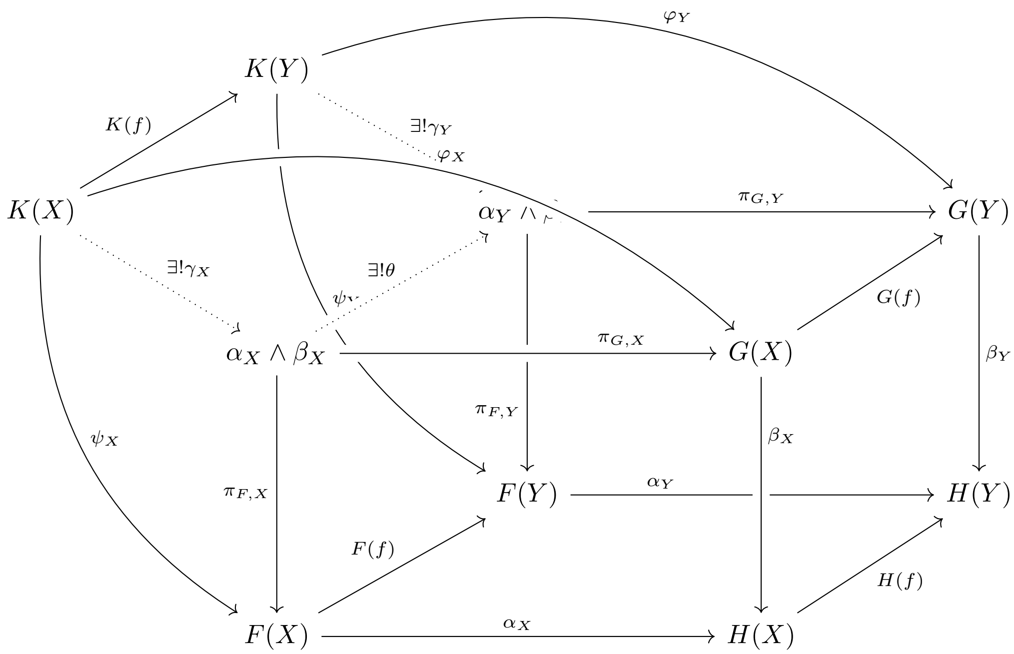
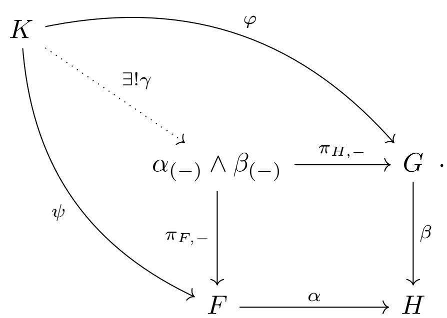
and for any other functor K and natural transformations φ, ψ which make the square commute in place of the pointwise pullback we have that for any $f : X \to Y$ in $\mathbf{X}_1$, the following diagram commutes:



The family of unique γ's is given by the universal properties of each of the pointwise pullbacks since the bent faces on the top and left side of the cube commute by naturality of φ and ψ respectively. This determines a unique natural transformation γ which shows the square above is a pullback in $[\mathbf{X}, \mathbf{Y}]$:



It's clear that evaluating this pullback diagram at an object $X \in \mathbf{X}_0$ (by applying the ev_X functor) just picks out a particular pointwise pullback as pictured on the front of the cube above. Hence the evaluation functors all strictly preserve pullbacks and are cartesian.

Definition 7. A *conservative* functor is functor which reflects isomorphisms. That is, $F : \mathbf{X} \to \mathbf{Y}$ reflects isomorphism if and only if $f \in \mathbf{X}_1$ is an isomorphism whenever $F(f) \in \mathbf{Y}_1$ is an isomorphism.

Lemma 8. *Let $\mathbf{X}$ be a small cartesian category. Then there exists a set S and a functor, $\mathbf{X} \xrightarrow{F} \mathbf{Set}/S$, from $\mathbf{X}$ into the slice category of sets over S such that F is cartesian and conservative.*

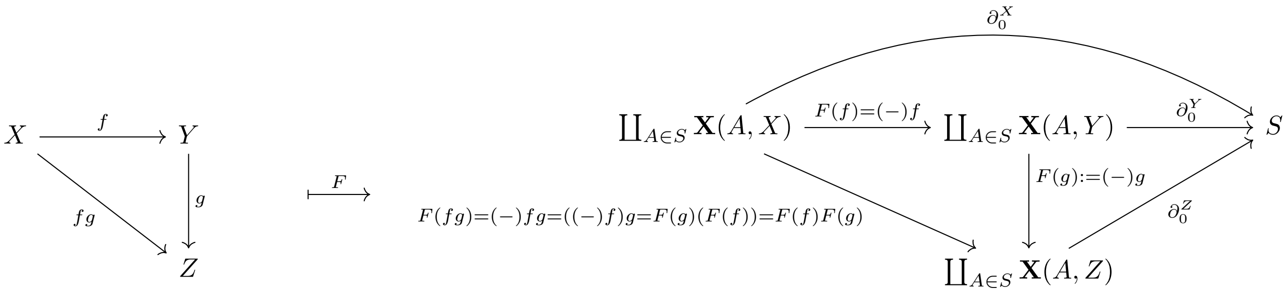
Proof. Let $S = \mathbf{X}_0$ (since $\mathbf{X}$ is small) and let F denote the functor which sends an object X to the disjoint union of hom-sets, $\coprod_{A \in \mathbf{X}_0} \mathbf{X}(A, X)$. The slice map to S is induced by the collection of functions

$$\mathbf{X}(A, X) \to S; f \mapsto \partial_0(f) = A$$

since the disjoint union is the coproduct in **Set**,

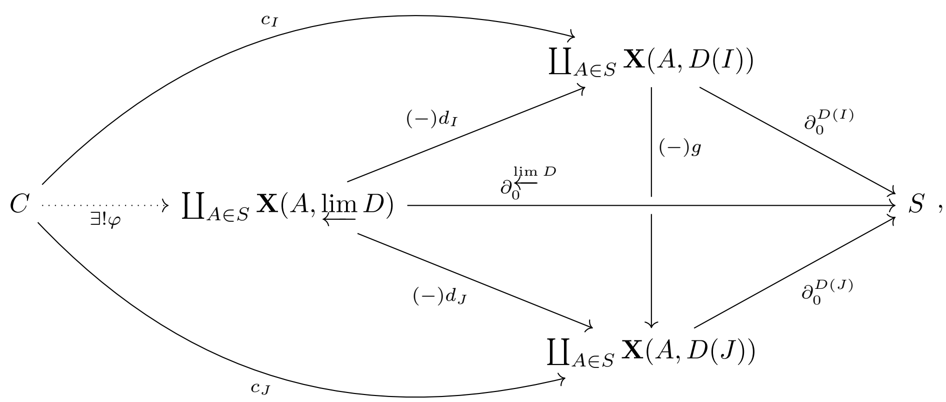
$$F(X) := \partial_0^X : \coprod_{A \in \mathbf{X}_0} \mathbf{X}(A, X) \to S.$$

To see F is functorial notice what happens when applying F to a triangle in $\mathbf{X}$:



where $F(f)$ is defined by taking a map from A to X and post composing it with f to give a map from A to Y (similarly for $F(g)$). In this way $F(f)$ restricts to a family of functions, $F_A(f) : \mathbf{X}(A, X) \to \mathbf{X}(A, Y)$, between the fibers of the coproduct. It's clear that the identity map on an object $X \in \mathbf{X}_0$ would get mapped to the identity arrow between the coproducts because post-composition with the identity arrow on the codomain leaves the original function unchanged and implies identity maps between each of the respective components/fibers. It follows that F is a functor.

To see F is cartesian, take any finite diagram $D : \mathbf{J} \to \mathbf{X}$. Since $\mathbf{X}$ is cartesian the limit of D exists and to show this limit is preserved by F take any $I \xrightarrow{g} J$ in $\mathbf{J}$, and notice the inside of the following diagram commutes:



where $d_I : \varprojlim D \to D(I)$ and $d_J : \varprojlim D \to D(J)$ such that $d_I g = d_J$. If there exists a set C with arrows c_I for each $I \in \mathbf{J}$ which makes the rest of the diagram commute, then we have that for each $x \in C$

$$c_I(x)g = c_J(x) : A \to D(J).$$

Hence the collection of maps $\{c_I(x)\}_{I\in\mathbf{J}}$ makes A a cone over D and there exists a unique $\theta_x : A \to \varprojlim D$. Doing the same for all $x \in C$ yields a map

$$\theta : C \to \coprod_{A\in S}\mathbf{X}(A, \varprojlim D); x \mapsto \theta_x,$$

which is unique because each θ_x is unique. This shows the coproduct satisfies the same universal property as the limit of the diagram $DF : \mathbf{J} \to \mathbf{Set}/S$, and the isormophism follows.

Finally, to see F reflects isomorphisms, suppose the following arrow, $F(f)$, is an isomorphism:

$$X \xrightarrow{f} Y \quad \overset{F}{\longmapsto} \quad \coprod_{A\in S}\mathbf{X}(A,X)) \xrightarrow{(-)f} \coprod_{A\in S}\mathbf{X}(A,Y))$$

Then it's necessarily an isomorphism between each of the fibres, so for any $A \in S$, given a map $g : A \to X$, there's a unique $h : A \to Y$ such that $gf = h$. Conversely given a map h, there's a unique g satisfying the same equation.

So first taking $g = 1_X$, there exists a unique map such that $f : X \to Y$ such that $1_X f = f$. Next, taking $h = 1_Y$ gives a unique $g : Y \to X$ such that $gf = 1_Y$. Moreover,

$$1_X f = f = f(gf) = (fg)f$$

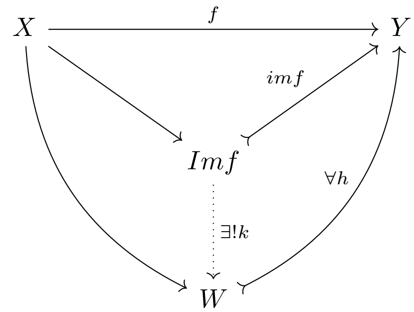
shows that 1_X and fg are both mapped to the same arrow by the bijection $(-)f$. It follows they're equal and F reflects isomorphisms. □

Example 9. Another example of a cartesian functor is the yoneda embedding. For a locally small cartesian category $\mathbf{X}$, the yoneda embedding $\mathbf{X}(=,-) : \mathbf{X} \to [\mathbf{X}^{op}, \mathbf{Set}]$ is a full and faithful cartesian embedding because the 'hom' functor preserves (finite) limits in its second argument.

Definition 10. Let $\mathcal{M}$ denote the class of monomorphisms in $\mathbf{X}$. We say $\mathbf{X}$ *has images* if there exists an assignment

$$\mathbf{X}_1 \to \mathcal{M}; (f : X \to Y) \mapsto (imf : Imf \rightarrowtail Y)$$

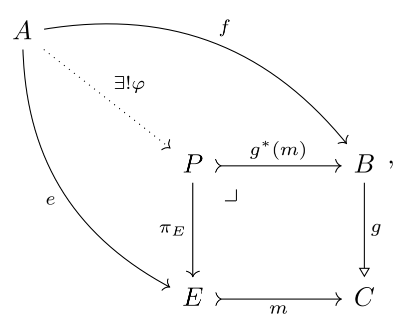
such that imf is the smallest proper subobject of Y through which f factors, as shown by the universal property in the diagram below:



Definition 11. A *cover* is a morphism which can't factor through a proper subobject of its codomain, and is denoted by the open headed triangle arrow: $\longrightarrow\!\!\triangleright$.

Proposition 12. *In any category with binary pullbacks, covers are closed under composition.*

Proof. Let $\mathbf{X}$ be a category with binary pullbacks, f and g composable covers, and a factorization of $fg = em$ through some monic m. Then consider the pullback of m along g,



which shows that $f = \varphi g^*(m)$ factors through a monomorphism $g^*(m)$ since monics are stable under pullback. This implies $g^*(m)$ is an isomorphism since f is a cover and therefore $g = (g^*(m)^{-1}\pi_E)m$ factors through m. But g is a cover so m is monic and it follows that fg is also a cover. □

Lemma 13. *For any category* $\mathbf{X}$*, the following are equivalent.*

(1) $\mathbf{X}$ *has images*

(2) $\forall A \in \mathbf{X}_0$ *, the inclusion functor*

$$i : \mathbf{Sub}_{\mathbf{X}}(A) \to \mathbf{X}/A$$

has a left adjoint

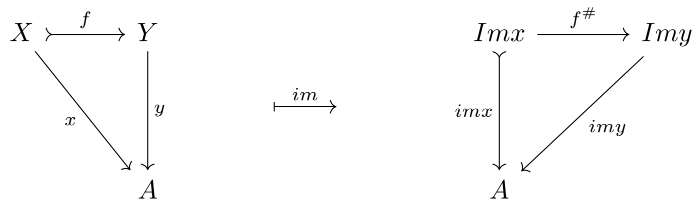
(3) *If* $\mathbf{X}$ *has pullbacks, then for each* $f : A \to B$ *in* $\mathbf{X}_1$*, the pullback functor*

$$f^* : \mathbf{Sub_X}(B) \to \mathbf{Sub_X}(A)$$

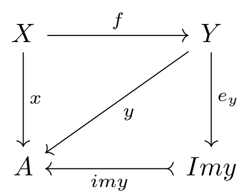
has a left adjoint.

Proof. (1) $\implies$ (2)
Define a functor

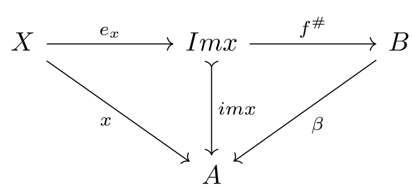
$$\mathbf{X}/A \xrightarrow{\quad im \quad} \mathbf{Sub_X}(A)$$



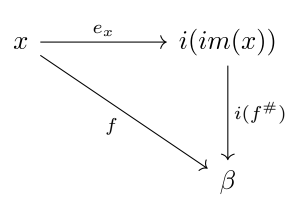
where $f^\#$ is induced by the universal property of the image since x also factors through Imy as follows:



To show $im \dashv i$, we'll exhibit the universal property. Let $x : X \rightarrowtail A$ be a subobject and suppose that $f : x \to i(\beta)$ is a map of subobjects of A for some $\beta : B \rightarrowtail$. Since $f\beta = x$, there's a unique map $f^\# : Imx \rightarrowtail B$ such that $f^\#\beta = imx$ given by the universal property of the image. Then we have



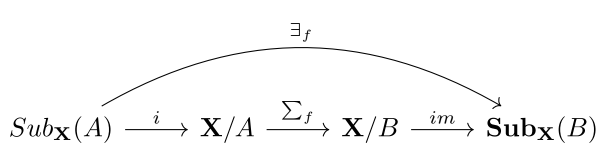
which shows the triangle,



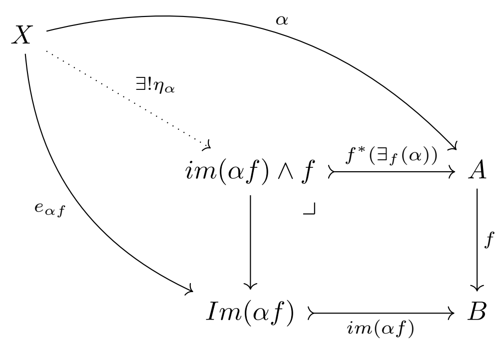
commutes. Hence $im \dashv i$.

(2) $\implies$ (3)

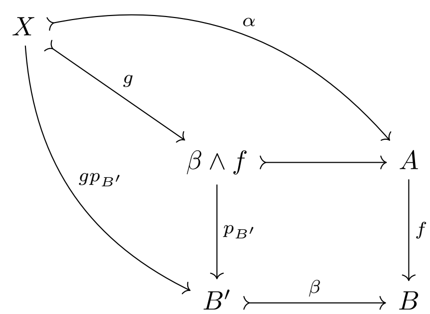
Assume $im \dashv i$ and $f : A \to B$. Define



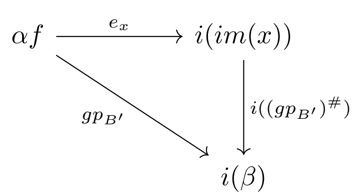
where $\sum_f$ denotes the base change functor that post-composes a subobject of A with f and makes it a subobject of B. To see $\exists_f \dashv f^*$, first let $\alpha : X \rightarrowtail A$ be a subobject of A. Post-composing α with f, taking the image of the composition, and then taking the pullback of the image along f gives the map $f^*(\exists_f(\alpha))$:



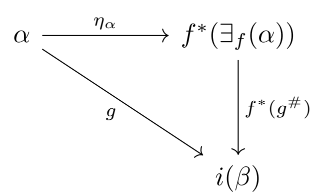
where the bottom of the outer part of the diagram is just the image factorization of the top part, so the outer square commutes. Now if $\beta : B' \rightarrowtail B$, and $g : \alpha \to f^*(\beta)$, that is:



then we get the following diagram in $\mathbf{X}/B$,



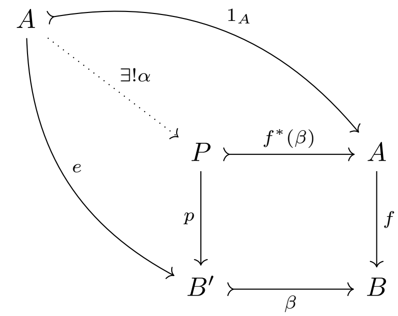
where $(gp_{B'})^{\#} : Im(\alpha f) \rightarrowtail B$ is the unique map from the adjunction $im \dashv i$. Since $im(\alpha f) = \exists_f(\alpha)$, this defines a unique $g^{\#} : \exists_f(\alpha) \to \beta$ such that



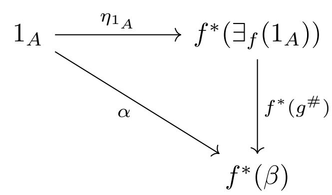
commutes. Hence $\exists_f \dashv f^*$, which shows (2) $\implies$ (3).

(3) $\implies$ (1)

Suppose $\mathbf{X}$ has pullbacks there's an adjunction, $\exists_f \dashv f^*$. Let $f : A \to B \in \mathbf{X}_1$ and suppose $f = e\beta$ for some $\beta : B' \rightarrowtail B$ in $\mathbf{Sub_X}(B)$. Then consider the pullback diagram



and in particular notice that the top triangle shows $\alpha : 1_A \to f^*(\beta)$ is a map in $\mathbf{Sub}_{\mathbf{X}}(A)$. The universal property of the adjunction then says



commutes for a unique map $g^{\#} : \exists_f(1_A) \to \beta$ in $\mathbf{Sub}_{\mathbf{X}}(B)$. This is exactly the universal property which justifies defining the image of f to be $\exists_f(1_A)$, where $A = \partial_0(f)$ more generally.

□

Note 14. Notice the unit of the adjunction, $im \dashv i$, in (2) $\implies$ (3) between the domain of a map and the domain of its image cannot factor through a proper subobject of its codomain, so it's a cover.

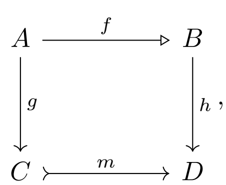
Lemma 15. *Suppose* $\mathbf{X}$ *has pullbacks and* $f : A \to B \in \mathbf{X}_1$*. Then the following are equivalent,*

(1) *f is a cover*

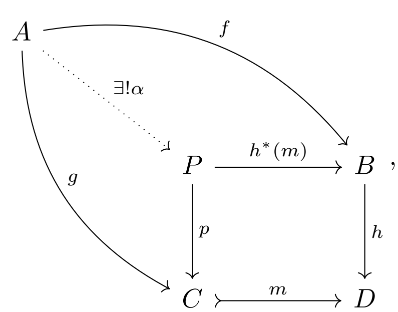
(2) *f is orthogonal to $\mathcal{M}$ (the class of monics in $\mathbf{X}$).*

(3) *If covers are stable under pullback, then the pullback functor, $f^* : \mathbf{X}/B \to \mathbf{X}/A$, is conservative.*

Proof. (1) $\implies$ (2)
Suppose $f : A \to B$ is a cover and the square,

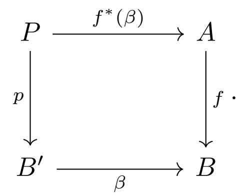


commutes. Then the cover, f, factors through $h^*(m)$,

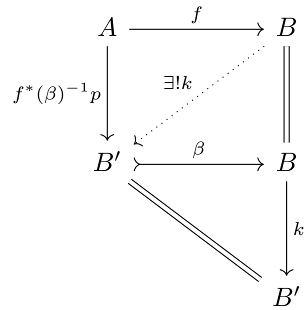


and therefore $h^*(m)$ must be an isomorphism. Let $k := h^*(m)^{-1}p : B \to C$ and notice it splits the first square, and uniqueness follows immediately from the fact that m is monic.

(2) $\implies$ (3)
Suppose f is is orthogonal to $\mathcal{M}$ and $f^*(\beta)$ is an isomorphism for some $\beta : B' \to B$. Consider the pullback:



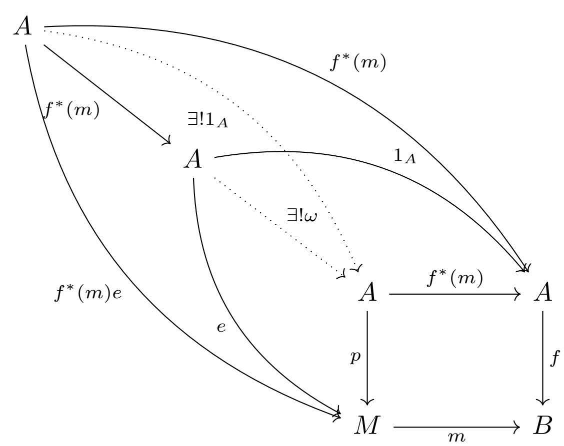
The top arrow is an isomorphism, so the following diagram commutes:



This shows $k\beta = 1_B$ and $\beta k = 1_{B'}$, therefore β is an isomorphism and f^* is conservative.

(3) $\implies$ (1)

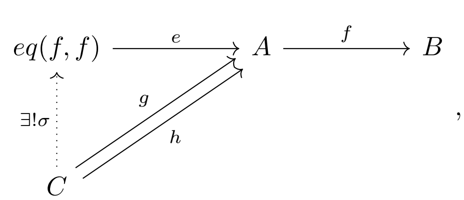
Suppose f factors as $f = em$ for some monic m. Then pulling m back along f, considering the universal property from the factorization, and precomposing with $f^*(m)$ gives the commuting diagram:



where the dotted arrows show that $\omega f^*(m) = 1_A = f^*(m)\omega$. Since $f^*(m)$ is an isomorphism and f^* is conservative, it follows that m is an isomorphism. Hence f doesn't factor through an proper subobjects of its codomain, ie. it's a cover. $\square$

Definition 16. A family of maps which cannot factor through proper subobjects of a common codomain is called a *covering family.*

Notes 17. Maps which are orthogonal to the class of monomorphisms are usually referred to as *strong epimorphisms.* If $\mathbf{X}$ has equalizers, then for any cover f, if $gf = hf$ for some g, h, then taking the equalizer of f and itself in the middle between gives,



which shows there exists σ such that $g = \sigma e = h$, and f is epic. This means every cover is epic but in order for an epic map to be a cover we need regularity.

Another property of interest is the stability of images under pullback. Since monics are stable under pullback, it's enough to ask that covers are stable under pullback. This is a key part of the definition of a regular category, which we recall to be a cartesian category with images whose covers are stable under pullback.

Now, let's look at some important monomorphisms, because they will be relevant in later sections.

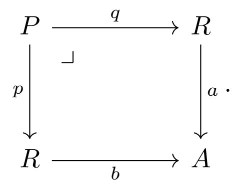
Definition 18. Let $R \underset{b}{\overset{a}{\rightrightarrows}} A$ be a pair of parallel arrows in a cartesian category $\mathbf{X}$.

(1) (a, b) is a *relation* if $(a, b) : R \to A \times A$ is monic.

(2) (a, b) is *reflexive* if there exists a retraction r of both a and b.

(3) (a, b) is *symmetric* if there exists a section s of both a and b.

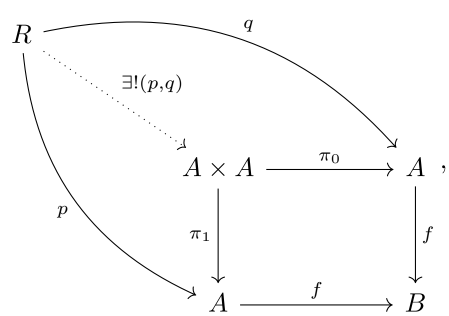
(4) (a, b) is *transitive* if there exists a map $t : P \to R$ where P is the pullback of a along b and $ta = pa$, $tb = qb$ where



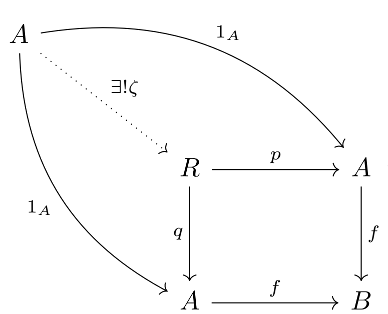
(5) (a, b) is an equivalence relation if it satisfies all of the above.

Notice $(2), (3), (4)$ and (5) only make sense if (1) is already satisfied.

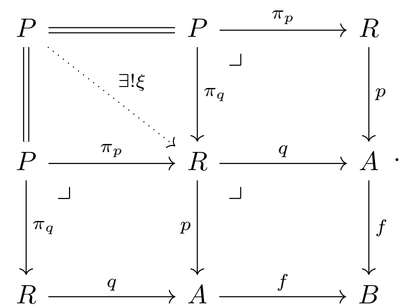
Example 19. Taking the pullback, R, p, q, of any morphism, f along itself gives a relation $(p, q) : R \to A \times A$ from the universal property of the product as follows,



where the outer square is a pullback. Since we can interchange p and q in the pullback diagram without consequence, its universal property shows it's reflexive and symmetric by taking A and its identity as shown:



Taking the pullback of p along q on both sides of the original pullback square gives a commuting diagram,



where the induced map ξ shows (p,q) is transitive, since $\xi q = \pi_p q$ and $\xi p = \pi_q p$. It follows that kernel pairs of morphisms are equivalence relations.

Definition 20. An equivalence relation is called *effective* if it occurs as the kernel pair of its coequalizer.

Definition 21. An *effective regular* category is a regular category in which all equivalence relations are effective.

Example 22. Not every regular category is effective. Consider equivalence relation

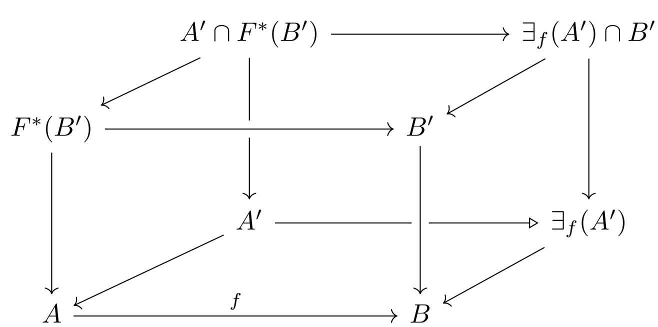
$$R := \{(a,b) : a \equiv b \mod 2\} \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} \mathbb{Z}$$

in the category of free abelian groups. Its coequalizer in this category is the zero group, and the kernel pair of this coequalizer is all of $\mathbb{Z} \oplus \mathbb{Z}$ so it can't be effective. Effectiveness fails here because the quotient group $\mathbb{Z}/2\mathbb{Z}$ should be the coequalizer (and it is in the category of abelian groups), but it's not free, so it can't be.

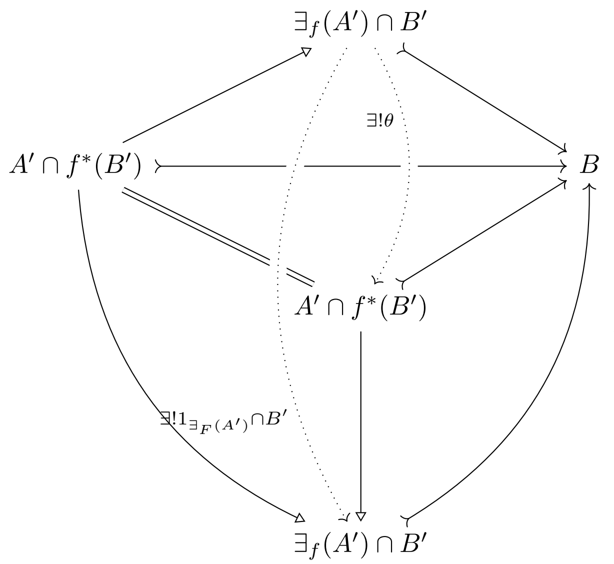
Lemma 23. *Let* $\mathbf{X}$ *be a regular category and* $f : A \to B$ *in* $\mathbf{X}$. *For any* $a : A' \to A$, $b : B' \to B$,

$$\exists_f(A' \cap f^*(B')) \cong \exists_f(A') \cap B'$$

Proof. Consider the commuting square,



where the front, left, and right faces are all pullbacks. This means that the rectangle made up of the left and front face is also a pullback and therefore the back face is a pullback as well, where the universal property is induced by the universal properties of the other three pullbacks and chasing around the square a little. Since covers are stable under pullback, this means the top edge of the back face is a cover. The diagonal of the right face is a monomorphism so $\exists_f(A') \cap B'$ is (the domain of) the image of the composite maps from the back left top to the front right bottom. Then we have the following commuting diagram of subobjects of B:

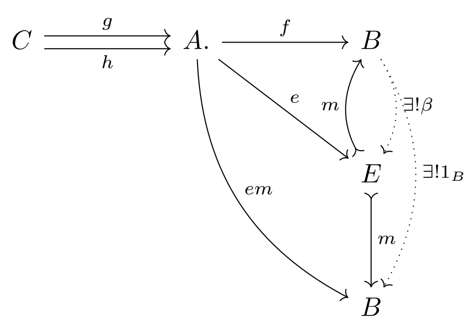


The top left triangle involving θ shows that θ is a post-compositional inverse of the cover on the top of the back face of the cube above, while the middle composition involving both dotted arrows shows θ is also a pre-compositional inverse for it. This θ is the necessary isomorphism. □

Note 24. Johnstone says this is sometimes called *Frobenius Reciprocity.*

Proposition 25. *In a regular category, the regular epimorphisms are exactly the covers.*

Proof. First going forwards, suppose $f : A \to B$ is a regular epimorphism, and it factors as $f = em$ for some monic m. Say f is the coequalizer of g and h, then

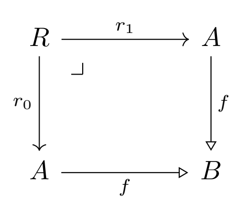


shows that $\beta m = 1_B$ for some β. On the other hand, we have that

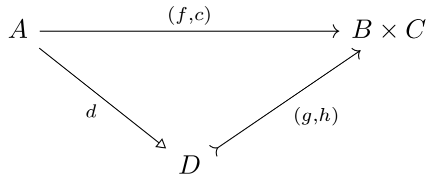
$$(m\beta)m = m(\beta m) = m1_B = 1_E m$$

and since m is monic, $m\beta = 1_E$. This shows m is an isomorphism and so f is a cover.

Conversely, let $f \in \mathbf{X}(A, B)$ be a cover, and let $r_0, r_1 : R \to A$ be its kernel pair. To see f is the coequalizer of r_0 and r_1, we just need to show that c factors through f because it coequalizes the kernel pair by definition and uniqueness follows immediately from f being epic (as seen in **Notes 16**).



Suppose there exists some $c : A \to C$ which coequalizes the kernel pair, and let $d(g, h)$ denote the (cover-)image factorization of the pairing map (f, c):



If g is an isomorphism, then

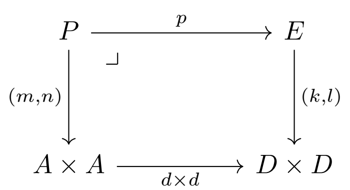
$$fg^{-1} = (f,c)\pi_0 g^{-1} = d(g,h)\pi_0 g^{-1} = dgg^{-1} = d$$

implies

$$c = (f,c)\pi_1 = d(g,h)\pi_1 = dh = (fg^{-1})h = f(g^{-1}h)$$

which shows that c factors through f. Since $f = dg$ factors through g after applying $\pi_0 : B \times C \to B$, it suffices to show g is monic to show it's an isomorphism.

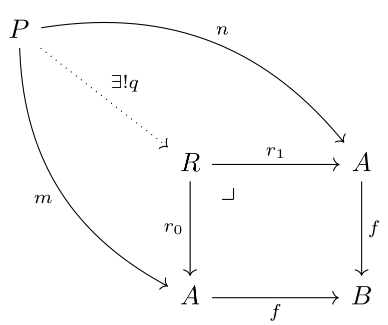
Suppose $kg = lg$ for some $k, l : E \to D$ and form the pullback,



Now notice

$$mf = m(dg) = (md)g = (pk)g = p(kg) = p(lg) = (pl)g = (nd)g = nf$$

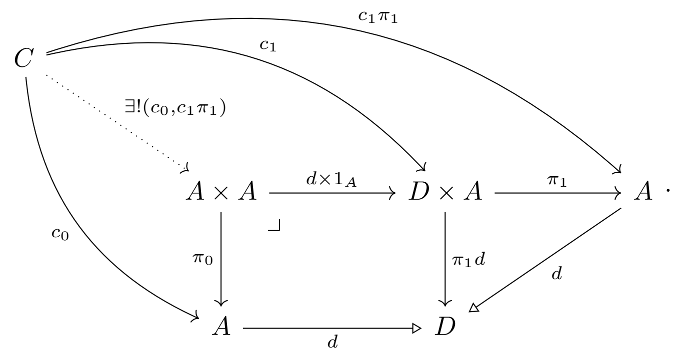
which induces a map $q : P \to R$,



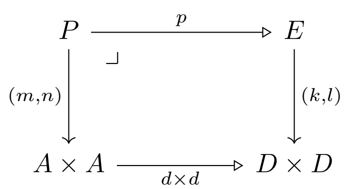
such that

$$p(kh) = (pk)h = (md)h = m(dh) = mc = (qr_0)c = 1(r_0c) = q(r_1c) = (qr_1)c = nc = n(dh) = (nd)h = (pl)h = p(lh)$$

Now $d \times d = (1_A \times d) \circ (d \times 1_A)$, where $d \times 1_A = (\pi_1 d)^*(d)$ is a pullback of d as shown by the following pullback diagram:



Since covers are stable under pullback, $d \times 1_A$ is a cover, and an identical argument shows $1_A \times d$ is a cover. Moreover, covers are closed under composition by proposition 12, so it follows that $d \times d$ is a cover, and stability of covers under pullbacks also tells us $p = (k, l)^*(d \times d)$ is a cover:



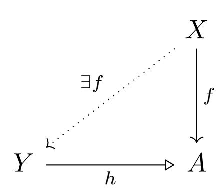
In particular p is epic, so $p(kh) = p(lh)$ implies $kh = lh$. Recalling the assumption that $kg = lh$, we have $k(g, h) = l(g, h)$, which implies $k = l$ since (g, h) is monic. We've shown g is monic, hence it's an isomorphism because $f = dg$ factors through it, and therefore f is the coequalizer of its kernel pair.

□

Note 26. We needed to define and play with cartesian categories a bit before doing the same with regular categories, because every regular category is cartesian. Of course not every cartesian category is regular, so we might ask if it's possible to "regularize" a cartesian category. The main theorem we'll prove at the end answers this question in the affirmative. The general idea behind its proof is to define a regular category $\mathbf{Reg}(\mathbf{X})$ and a functor I so that every object of $\mathbf{Reg}(\mathbf{X})$ is a morphism in $\mathbf{X}$, every embedded object $I(X)$ is projective.

First we'll recall projective objects and define $\mathbf{Reg}(\mathbf{X})$. Then we'll see it's a regular category before we define I and show it's full, faithful, and cartesian. Finally we'll show this embedding is universal by showing that any regular functor out of $\mathbf{X}$ extends to a regular functor out of $\mathbf{Reg}(\mathbf{X})$ which is unique up to canonical isomorphism in the functor category.

Definition 27. An object $X \in \mathbf{X}_0$ is *projective* if for any map $g \in \mathbf{X}(X, A)$ and any cover $h \in \mathbf{X}(Y, A)$, there exists an $f \in \mathbf{X}(X, Y)$ such that $fh = g$. Here's the accompanying diagram:



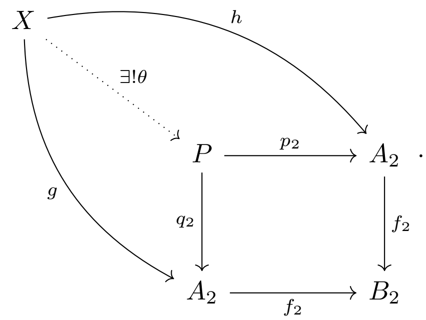
Definition 28. Let $\mathbf{X}$ be a cartesian category. *The regular category generated by* $\mathbf{X}$ is denoted $\mathbf{Reg}(\mathbf{X})$. It has morphisms of $\mathbf{X}$ as objects, ie.

$$\mathbf{Reg}(\mathbf{X})_0 := \mathbf{X}_1,$$

and a morphism $[g] : f_1 \to f_2$ in $\mathbf{Reg}(\mathbf{X})$ is an equivalence class of morphisms

$$\{g : \partial_0(f_1) \to \partial_0(f_2) \mid gf_2 \text{ coequalizes the kernel pair of } f_1\}$$

in $\mathbf{X}$ under the relation $g \equiv h$ if and only if f_2 coequalizes them. An equivalent characterization of this relation asks that g and h factor through the kernel pair of f_2, and this equivalence can be seen by staring at the following diagram:



If f_2 coequalizes g and h, then the outside commutes and both g and h factor through the kernel pair of f_2 via θ. Conversely, if they factor through the kernel pair, then there exists a θ such that the triangles at the top left of the diagram commute, and it follows that f_2 coequalizes g and h:

$$gf_2 = (\theta q_2)f_2 = (\theta p_2)f_2 = hf_2.$$

We'll use both of these characterizations as we see fit.

Define composition of morphisms to be the obvious choice,

$$[g][h] := [gh],$$

and for any object $f : A \to B$ in $\mathbf{Reg}(\mathbf{X})$, define the identity morphism to be equivalence classes of identity morphisms on its domain,

$$1_f := [1_{\partial_0(f)}].$$

Notice that for any composable $[f], [g], [h]$ in $\mathbf{Reg}(\mathbf{X})_1$, associativity,

$$[f][gh] = [f][g][h] = [fg][h],$$

and the identity law,

$$[1_A][f] = [1_A f] = [f] = [f1_{A'}] = [f][1_{A'}],$$

are trivial if composition is well defined. So to justify this definition and show that $\mathbf{Reg}(\mathbf{X})$ is a category it's sufficient to show that $\equiv$ is an equivalence relation and composition is well defined.

Proposition 29. *The relation , $\equiv$, from the previous definition is an equivalence relation.*

Proof. Let $f_i : A_i \to B_i$ for $i \in \mathbb{N}$ as necessary, $g, h : A_1 \to A_2$ as above.

Reflexivity is clear because if we have $gf_2 : A_1 \to A_2 \to B_2$, then $gf_2 = gf_2$ shows $g \equiv g$.

To see symmetry, suppose $g \equiv h$. Then $gf_2 = hf_2$, and equivalently $hf_2 = gf_2$ shows that $h \equiv g$.
Finally to show transitivity, suppose $g \equiv h \equiv k$. Then $gf_2 = hf_2 = kf_2$ shows that $gf_2 = kf_2$, which means $g \equiv h$, and it follows that $\equiv$ is transitive. □

Proposition 30. *Composition, as defined in Definition 28, is well defined.*

Proof. Consider the following arrows in $\mathbf{Reg}(\mathbf{X})$:

$$f_1 \xrightarrow{[g]} f_2 \xrightarrow{h} f_3$$

where

$$\begin{array}{ccccc} A_1 & \xrightarrow{g} & A_2 & \xrightarrow{h} & A_3 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ B_1 & & B_2 & & B_3 \end{array}$$

in **X**. Now suppose $g_1 \equiv g_2 \in [g]$ and $h_1 \equiv h_2 \in [h]$ we need to show $g_1 h_1 \equiv g_2 h_2$. To do this we'll use both of the equivalent characterizations of $\equiv$.

We'll start with the first characterization and deduce that f_3 coequalizes h_1 and h_2,

$$h_1 f_3 = h_2 f_3,$$

and precompose with g_1 to get

$$g_1(h_1 f_3) = g_1(h_2 f_3).$$

Now we'll use the second characterization, that g_1 and g_2 factor through the kernel pair of f_2, (q_2, p_2) by a common first map, to get a map θ such that $g_1 = \theta q_2$ and $g_2 = \theta p_2$. Substituting g_1 on the left side of the equality above we see,

$$g_1(h_2 f_3) = \theta q_2(h_2 f_3) = \theta(q_2 h_2) f_3 = \theta(p_2 h_2) f_3 = (\theta p_2) h_2 f_3 = (g_2 h_2) f_3,$$

where the fifth equality from the left is from the fact that h_2 coequalizes the kernel pair of f_2 (by definition of $[h_2] : f_2 \to f_3$ in $\mathbf{Reg}(\mathbf{X})$). Putting everything together we have

$$(g_1 h_1) f_3 = g_1(h_1 f_3) = g_1(h_2 f_3) = (g_2 h_2) f_3,$$

as desired. □

Corollary 31. $\mathbf{Reg}(\mathbf{X})$ *is a category.*

Proof. See the last two sentences of Definition 28. □

Proposition 32. $\mathbf{Reg}(\mathbf{X})$ *is a cartesian category.*

Proof. Appealing way back to Lemma 2, we'll show that **X** has terminal objects and pullbacks.

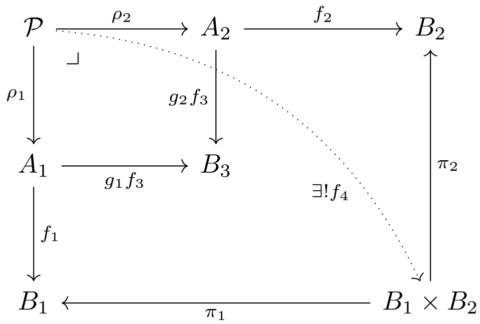
Since **X** is cartesian, it has a terminal object, $\top$, and we'll show its identity morphism, $1_\top$, is the terminal object in $\mathbf{Reg}(\mathbf{X})$.

Take any $f : A \to B$ in **X**. We need a unique morphism $f \to 1_\top$ in $\mathbf{Reg}(\mathbf{X})$, and we claim this is is the equivalence class of the unique morphism from the domain of f to the terminal object in **X** , $[!_A]$. To see $[!_A] : f \to 1_\top$ exists, let p, q be the kernel pair of f and notice $!_A$ coequalizes them:

$$p!_A = !_P = q!_A.$$

For uniqueness, suppose $[g] : f \to 1_\top$. Then $g : A \to \top$ implies $g = !_A$ by uniqueness of $!_A$, hence $[g] = [!_A]$ and $[!_A]$ is unique. It follows that $1_\top$ is terminal in $\mathbf{Reg}(\mathbf{X})$.

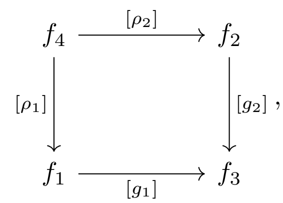
Now we'll define the pullback of arbitrary morphisms $[g_1] : f_1 \to f_3$ and $[g_2] : f_2 \to f_3$, where $f_i : A_i \to B_i$ for $i = 1, 2, 3$ in **X**. This is done by post composing the pullback projections of $g_1 f_3$ and $g_2 f_3$ with f_1 and f_2 respectively, and using the universal property of the product:



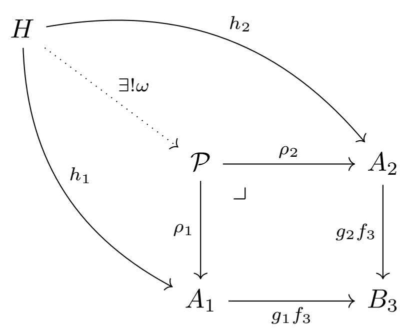
More precisely, $f_4 = (\rho_1 f_1, \rho_2 f_2)$ is the unique pairing map. Now, if $g_1 \equiv h_1$ and $g_2 \equiv h_2$, then $g_j f_3 = h_j f_3$ for $j = 1, 2$ so ρ_1 and ρ_2 don't depend on the choice of representatives for $[g_1]$ and $[g_2]$. The pullback square implies $\rho_1 g_1 \equiv \rho_2 g_2$ so

$$[\rho_1][g_1] = [\rho_1 g_1] = [\rho_2 g_2] = [\rho_2][g_2]$$

and the candidate pullback square in $\mathbf{X}$,



commutes. Suppose there exist $[h_i] : h \to f_i$ such that $[h_1][g_1] = [h_2][g_2]$. Then $[h_1 g_1] = [h_2 g_2]$, which means $h_1 g_1 f_3 = h_2 g_2 f_3$ and induces a unique map ω as shown:



To see $[\omega] : h \to f_4$ in $\mathbf{Reg}(\mathbf{X})$, let (k, ℓ) be the kernel pair of h and compute, $k\omega f_4$ and $\ell \omega f_4$ by post composing with both projections π_1 and π_2:

$$k\omega(f_4 \pi_i) = k\omega(\rho_i f_i) = k(\omega \rho_i) f_i = k h_i f_i = \ell h_i f_i = \ell(\omega \rho_i) f_i = \ell \omega(\rho_i f_i) = \ell \omega(f_4 \pi_i)$$

By the universal property of the product, $B1 \times B_2$, the kernel pair is coequalized by ωf_4,

$$k\omega f_4 \pi_i = \ell \omega f_4 \pi_i.$$

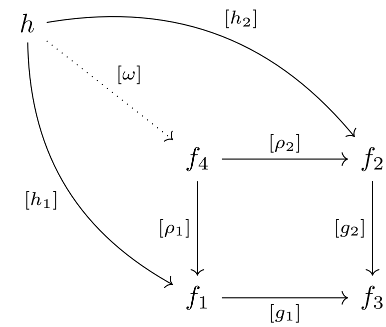
Notice if $h_i \equiv h_i'$ $(i = 1, 2)$, then

$$[h_i g_i] = [h_i][g_i] = [h_i'][g_i]$$

shows that $h_i g_i f_3 = h_i' g_i f_3$, which means the universal map ω in the pullback diagram above doesn't depend on the choice of representative for $[h_i]$. Now we can simply compute,

$$[\omega][\rho_i] = [\omega \rho_i] = [h_i]$$

to get a commuting diagram in $\mathbf{Reg}(\mathbf{X})$:



It remains to show $[\omega]$ is unique, so suppose we can substitute $[\sigma]$ for $[\omega]$ in the last diagram. Then for $i = 1, 2$,

$$[\omega\rho_i g_i] = [\omega][\rho_i][g_i] = [h_i] = [\sigma][\rho_i][g_i] = [\sigma\rho_i g_i]$$

which says,

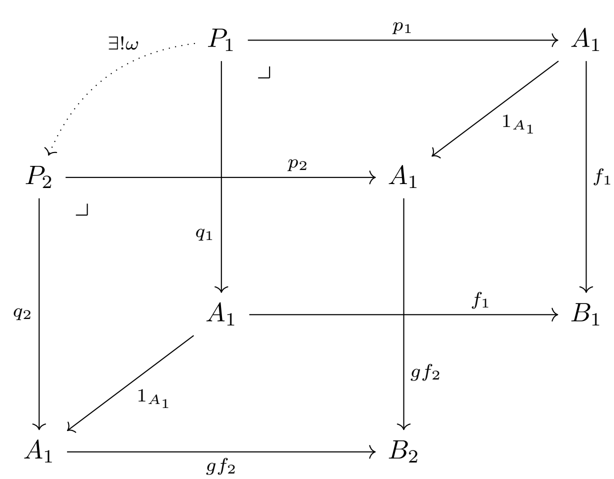
$$\omega\rho_i g_i f_3 = \sigma\rho_i g_i f_3.$$

This means σ can replace $\omega : H \to \mathcal{P}$ in the second last pullback diagram in $\mathbf{X}$. The universal property of the pullback tells us $\sigma = \omega$, which implies $[\sigma] = [\omega]$ of course. Hence $[\omega]$ is unique and $(f_4, [\rho_1], [\rho_2])$ is the pullback of $[g_1]$ and $[g_2]$ in $\mathbf{Reg}(\mathbf{X})$.

Having shown $\mathbf{Reg}(\mathbf{X})$ has a terminal object and pullbacks of pairs of morphisms, we can conclude by Lemma 2 that it satisfies Johnstone's definition of cartesian. □

Proposition 33. *A morphism $[g] : f_1 \to f_2$ in $\mathbf{Reg}_($**X**$)$ is monic if and only if the kernel pairs of gf_2 and f_1 coincide.*

Proof. Suppose $[g]$ is monic and let p_1, q_1 denote the kernel pair of f_1, and p_2, g_2 denote the kernel pair of gf_2. Since gf_2 coequalizes both kernel pairs by definition(s), we can see $[1_{A_1}] : f_1 \to gf_2$ induces a unique map $\omega : P_1 \to P_2$ in the diagram below.

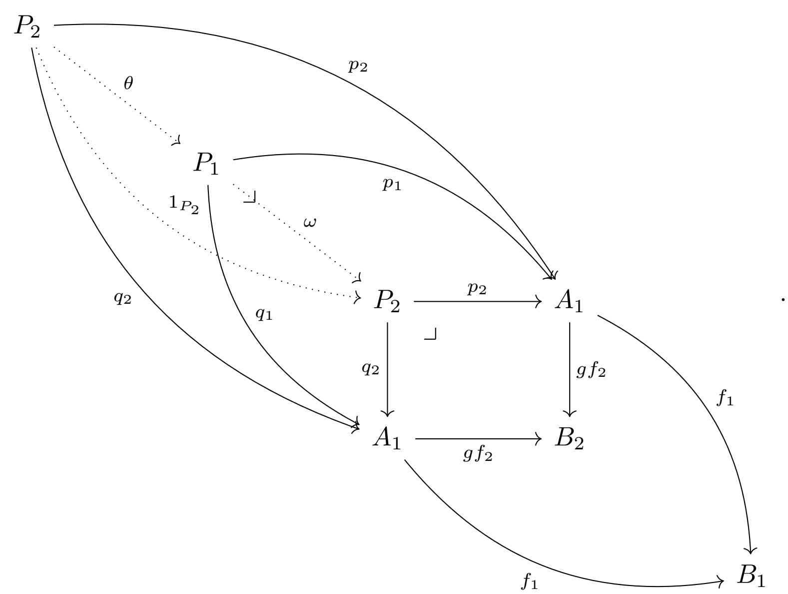


Since p_2 and q_2 are coequalized by gf_2. Then $[q_2], [p_2] : 1_{P_2} \to f_1$ since the kernel pair of the identity map is a pair of identity maps, which are clearly coequalized by $q_2 f_1$ and p_2 and f_1 respectively. Moreover, we have that

$$1_{P_2} \underset{[q_2]}{\overset{[p_2]}{\rightrightarrows}} f_1 \xrightarrow{[g]} f_2$$

,

and since $[g]$ is monic, $[p_2] = [q_2]$. This means $p_2 f_1 = q_2 f_1$, and the universal property of the following pullback diagram guarantees a unique $\theta : C \to P_1$ such that $\theta p_1 = p_2$, $\theta q_1 = q_2$:



Note that the pullback starting at P_1 ends at B_1 in the bottom right. It's clear that precomposing once more with P_1, p_1, q_1, and ω shows $\omega\theta = 1_{P_1}$ by using the universal property of P_1.

From this we can see that the pullbacks are isomorphic. This must be what Johnstone means when he says the kernel pairs "coincide," because nothing in our assumptions guarantees that P_1 and P_2 are equal. For the kernel pairs to be exactly equal we would require that θ and ω are identity maps and consequently for the pullback objects P_i to be equal for $i = 1, 2$.

Conversely, suppose the pullbacks in question are isomorphic via θ and ω as pictured in the previous diagram. Also assume that $[h][g] = [k][g]$ in $\mathbf{Reg}(\mathbf{X})$. Then $hgf_2 = kgf_2$, and the universal property of P_2 gives a unique map $\alpha : A_0 \to P_2$ such that $\alpha p_2 = h$ and $\alpha q_2 = k$. Since $\theta\omega = 1_{P_2}$, we have

$$\alpha\theta p_1 = \alpha\theta\omega p_2 = \alpha p_2 = h \qquad , \qquad \alpha\theta q_1 = \alpha\theta\omega q_2 = \alpha q_2 = k$$

and consequently,

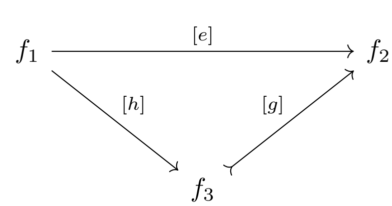
$$kf_1 = \alpha\theta q_1 f_1 = \alpha\theta p_1 f_1 = hf_1 \qquad . \text{ In other words, } [k] = [h].$$

□

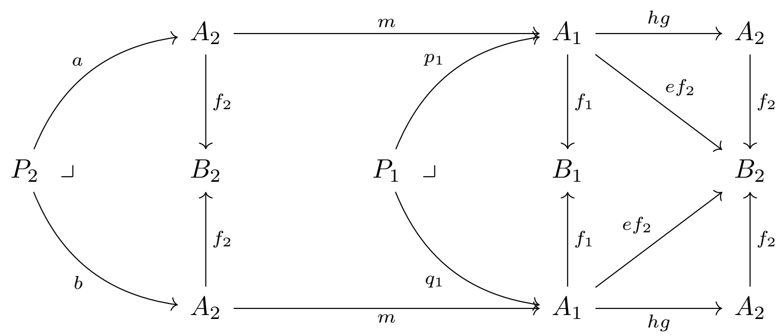
Proposition 34. *Split epimorphisms in* $\mathbf{X}$ *are covers in* $\mathbf{Reg}(\mathbf{X})$.

Proof. Johnstone notes that this condition is unnecessary but sufficient to ensure the equivalence class in $\mathbf{Reg}(\mathbf{X})$ of a given morphism in $\mathbf{X}$ is a cover in $\mathbf{Reg}(\mathbf{X})$.

Let $[e] : f_1 \to f_2$ where e is epic in $\mathbf{X}$. Suppose $m : A_2 \rightarrowtail A_1$ is monic in $\mathbf{X}$ and it splits the epic $e : A_1 \to A_2$, that is $me = 1_{A_2}$. Further suppose that $[e] = [h][g]$ factors through some monic $[g]$:



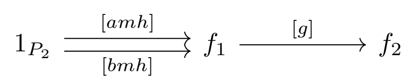
Let a, b denote the kernel pair of f_2, then the diagram,



commutes, because

$$(amh)gf_2 = am(hg)f_2 = a(me)f_2 = a1_{P_2}.f_2 = af_2 = bf_2 = b(me)f_2 = bm(hg)f_2 = (bmh)gf_2$$

This means we have a parallel pair of arrows in $\mathbf{Reg}(\mathbf{X})$,



,

and since $[g]$ is monic, $amhf_1 = bmhf_1$. This shows $[mh] : f_2 \to f_3$ is in $\mathbf{Reg}(\mathbf{X})$, now we just need to show that it's the inverse of $[g]$. First show it's a precompositional inverse,

$$[mh][g] = [m][h][g] = [m][e] = [me] = [1_{A_2}]$$

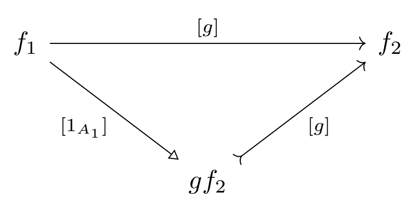
and then use that fact to show

$$([g][mh])[g] = [g]([mh][g]) = [g][1_{A_2}] = [1_{A_1}][g].$$

Since $[g]$ is monic we get $g][mh] = [1_{A_1}]$ and it follows that $[g]$ is an ismorphism. We've shown $[e]$ cannot factor through a proper subobject of its codomain if e is split epic, hence morphisms in $\mathbf{Reg}(\mathbf{X})$ containing split epimorphisms from $\mathbf{X}$ are covers. □

Proposition 35. $\mathbf{Reg}(\mathbf{X})$ *has a cover-image factorization system.*

Proof. Let $[g] : f_1 \to f_2$, $f_i : A_1 \to B_i$ in $\mathbf{X}$ as above. Then we claim that $[g]$ factors through a cover followed by an image as follows:

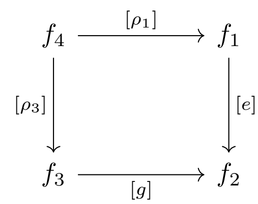


It's clear that the triangle commutes, as $[1_{A_1}][g] = [1_{A_1}g] = [g]$. To see the first arrow is a cover notice that 1_{A_1} is trivially a split epimorphism invoke proposition 34. For the second arrow being monic, use proposition 33 along with the fact that the kernel pair of gf_2 (the domain of the arrow) is equal to the kernel pair of gf_2 (a representative of the arrow post composed with the arrow's codomain). □

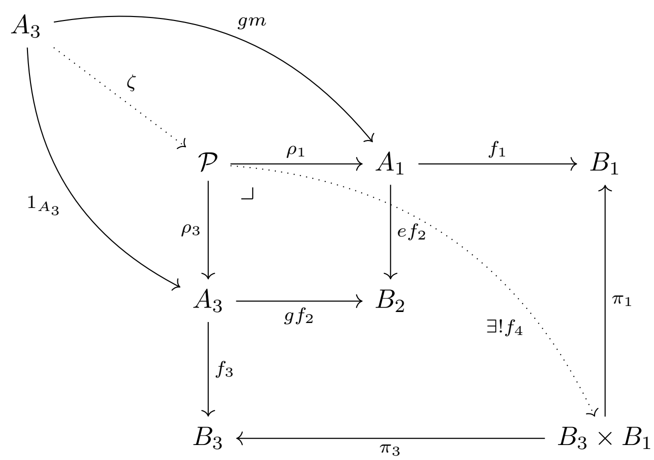
Proposition 36. *Covers are stable under pullback in* $\mathbf{Reg}(\mathbf{X})$.

Proof. We'll show that any morphism in $\mathbf{Reg}(\mathbf{X})$ which can be represented by a split epic in $\mathbf{X}$ is stable under pullback and this will suffice by proposition 33.

Let $e : A_1 \to A_2$ be split by some monic $m : A_2 \rightarrowtail A_1$ in $\mathbf{X}$, and let the following diagram be a pullback square in $\mathbf{Reg}(\mathbf{X})$:



where $f_4 = (\rho_2 f_2, \rho_1 f_1) : P \to B_1 \times B_2$ is the universal pairing map as constructed in proposition 32 by post composing the pullback projections of ρ_1 and ρ_3 with f_1 and f_3 respectively, and taking the universal morphism pairing map into the product as seen from the square(s) in the diagram below.



Moreover, notice ρ_2 splits from the universal property of the pullback. It's easy to see ρ_2 is epic now because if $\rho_2 h = \rho_2 k$, then

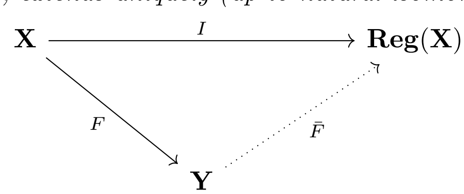
$$h = \zeta \rho_2 h = \zeta \rho_2 k = k.$$

Hence $[\rho_2]$ is a cover by proposition 34, and covers are stable under pullback. □

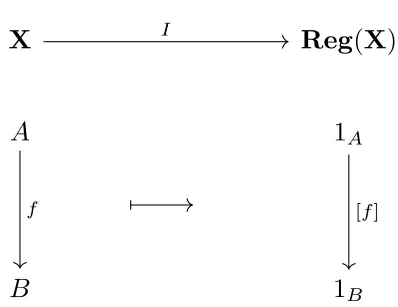
Lemma 37. $\mathbf{Reg}(\mathbf{X})$ *is a regular category*

Proof. Combine corollary 31 and propositions 32, 35, and 36. □

Theorem 38. *There exists a fully faithful cartesian embedding* $I : \mathbf{X} \to \mathbf{Reg}(\mathbf{X})$ *whose image on objects is contained in the class of projective objects in* $\mathbf{Reg}(\mathbf{X})$*. Moreover, this embedding is universal in the sense that any cartesian functor* $F : \mathbf{X} \to \mathbf{Y}$ *where* $\mathbf{Y}$ *is a regular category, extends uniquely (up to natural isomorphism) to a functor* $\bar{F}$*:*



Proof. Define I as follows:



The assignment of morphisms is well defined because kernel pairs of identity arrows are identities which are coequalized by any arrow with a suitable domain, and if $f = g$, then $[f] = [g]$ immediately. To see identities and composition are preserved, let $f : A \to B$, $g : B \to C$, in $\mathbf{X}$ and $[h], [k]$ arbitrary arrows $\mathbf{Reg}(\mathbf{X})$ represented by $h : Z \to A$, $k : A \to B$ in $\mathbf{X}$ respectively.

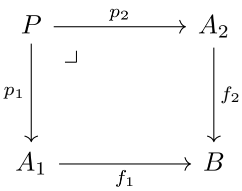
Then we can see in the following equalities that composition is preserved on the left,

$$I(fg) = [fg] = [f][g] = I(f)I(g) \qquad ; \qquad [h]I(1_A) = [h][1_A] = [h1_A] = [h] \qquad ; \qquad I(1_A)[k] = [1_A][k] = [1_A k] = [k],$$

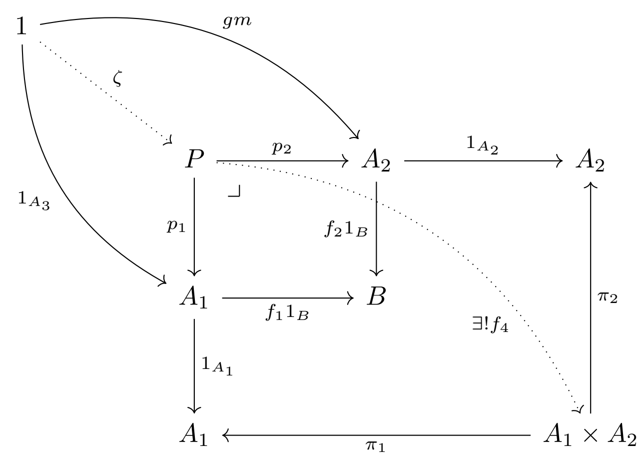
and identities are preserved in the middle and right. It follows that I is a functor.

It's also easy to see I is full and faithful. Suppose $[f] : I(A) \to I(B)$, then any representative $f : A \to B$ is a map in $\mathbf{X}$ by definition, and applying I recovers $[f]$ so I is full. Now suppose $I(f) = I(g) : 1_A \to 1_B$ for two maps $f, g : A \to B$. Then $[f] = [g] : 1_A \to 1_B$ in the regular category and by definition $f = f1_B = g1_B = g$, which shows I is faithful.

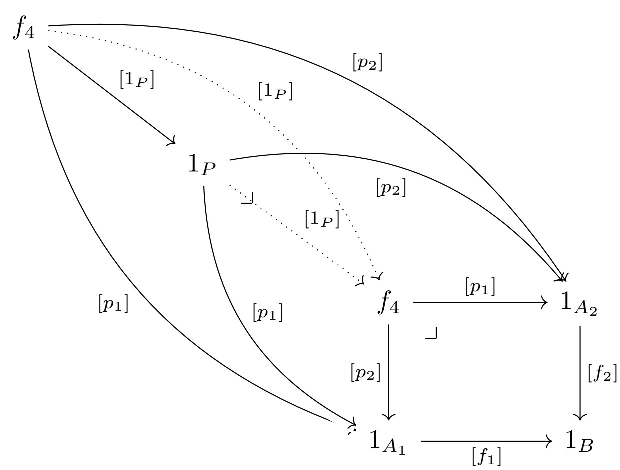
In the first part of the proof of proposition 32 we saw that $1_\top = I(\top)$ is the terminal object in $\mathbf{Reg}(\mathbf{X})$, so I preserves the terminal object. To see pullbacks are preserved up to canonical isomorphism suppose we have a pullback diagram,



in $\mathbf{X}$. Applying I gives a commuting square in $\mathbf{Reg}(\mathbf{X})$ but notice that the pullback candidate in this case is 1_P rather than the pullback object we constructed in proposition 32. That construction is given by the square in the following diagram, and we can see the setup for proving the isomorphism bubbling out from the left corner:



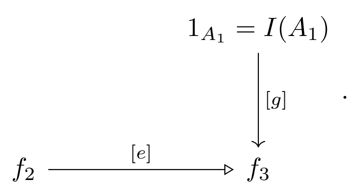
Let ρ_1, ρ_2 denote the kernel pair of f_4. Then $\rho_i f_4 \pi_i = \rho_i p_i$, which shows that $[\rho_i] : f_4 \to 1_{A_i}$ for $i = 1, 2$. Now using the universal property of the pullback we constructed we can see the canonical the isomorphism we were after from one direction:



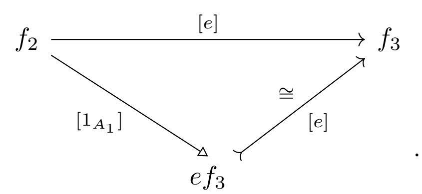
.

Notice that the dotted arrows are unique, and this implies the solid arrow, $[1_p] : f_4 \to 1_P$, is is also unique for if any other $[g]$ did the trick, we would have $[1_P] = [g][1_P] = [g1_P] = [g]$ from the composition involving the dotted arrows in the diagram above, and hence $1_P = g1_P = g$. The other composition is trivial because the entire isomorphism is, so pullbacks are preserved by I up to the most canonical isomorphism we could hope for.

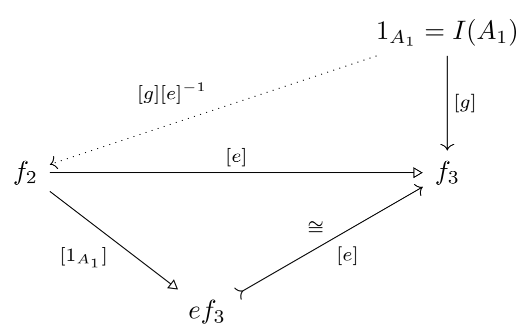
Up to this point, we've shown that I is a fully faithful cartesian functor. To see that the image of every object is projective, consider the following diagram in $\mathbf{Reg}(\mathbf{X})$,



where $[e]$ is a cover. If $[e]$ is not representable by a split epimorphism, then the monomorphism in its cover-image factorization is an isomorphism because it's already a cover:



Then we can post compose $[g]$ with the inverse of this isomorphism to get a commuting diagram,

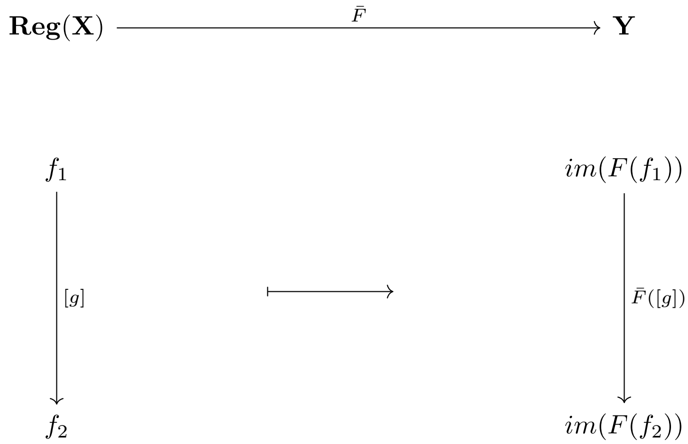


which shows that $I(A_1)$ is projective. On the other hand, if $[e]$ is representable by an epic $e : A_2 \to A_3$ which is split by a monic, $m : A_3 \rightarrowtail A_2$, then since the kernel pair of $I(A_1)$ is a pair of identity morphisms on A_1, they're coequalized by any morphism, in particular by $gm : A_1 \to A_3$. Hence $[gm] : I(A_1) \to f_2$ is an arrow in $\mathbf{Reg}(\mathbf{X})$, and immediately we get

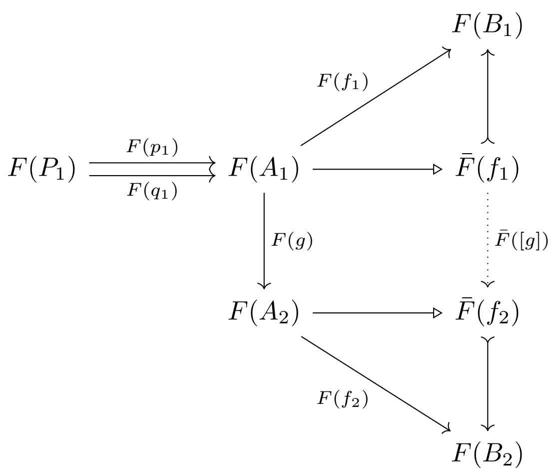
$$[gm][e] = [g(me)] = [g1_{A_3}] = [g]$$

by the splitting of e, which shows that $I(A_1)$ is projective.

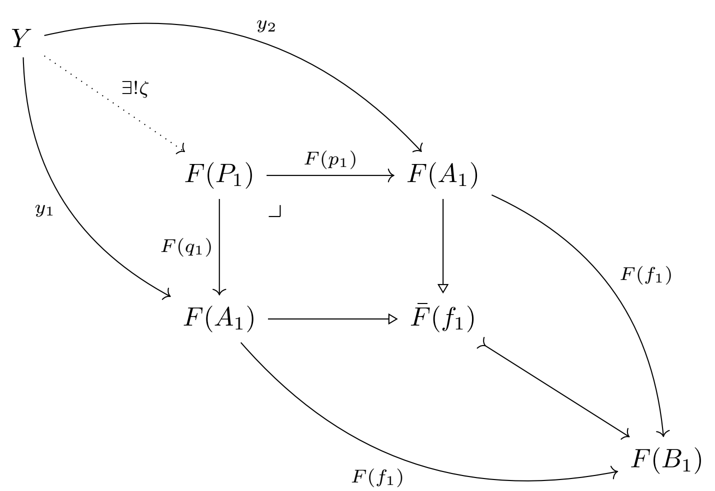
It only remains to show that any other cartesian functor $F : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y}$ is a regular category, extends uniquely to a regular functor $\bar{F} : \mathbf{Reg}(\mathbf{X}) \to \mathbf{Y}$. Define $\bar{F}$ as follows:



where $\bar{F}(g)$ is the unique map given from the following coequalizer diagram:



Since F preserves pullbacks, it preserves kernel pairs and so $F(p_1)$ and $F(q_1)$ are the kernel pair of $F(f_1)$, as shown at the top. By proposition 24 covers are regular epics, in particular the first map in the factorization of $F(f_1)$ is the coequalizer of its kernel pair, and since the second map in the factorization is monic, we can see the cover coequalizes $F(p_1)$ and $F(q_1)$. To see the cover is actually *the* coequalizer of $F(p_1)$ and $F(q_1)$, consider the pullback diagram below:

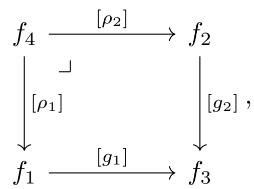


From this diagram it's clear that pulling the cover back along itself gives the same kernel pair as pulling $F(f_1)$ back along itself, because they satisfy the same universal property. It follows that the cover is the coequalizer of the kernel pair of $F(f_1)$, which induces the arrow defined as $\bar{F}([g])$ above. Moreover, uniqueness immediately implies composition and identities are preserved.

Now suppose $g \equiv h$, then $gf_2 = hf_2$ and applying F we see $F(g)F(f_2) = F(gf_2) = F(hf_2) = F(h)F(f_2)$. This means we could sub $F(g)$ for $F(h)$ in the coequalizer diagram above to induce a unique $\bar{F}([h])$. Notice $\bar{F}([g])$ and $\bar{F}(h)$ are both coequalized by the monic in the image factorization of f_2 that follows, so they must be equal. This shows $\bar{F}$ is a well defined functor.

It's immediate that $\bar{F}$ preserves the terminal object because F preserves the terminal object of $\mathbf{X}$. In particular, the image factorization of $F(1_{\top_{\mathbf{X}}}) = 1_{F(\top_{\mathbf{X}})} = 1_{\top_{F(\mathbf{X})}} = 1_{\top_{\mathbf{Y}}}$ is trivial because every morphism into the terminal object is unique.

To show pullbacks are preserved, assume the following is a pullback square,



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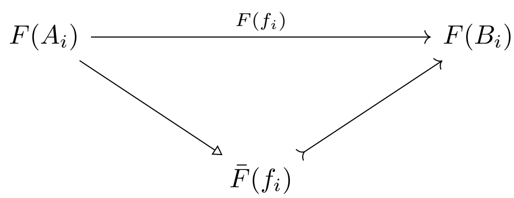
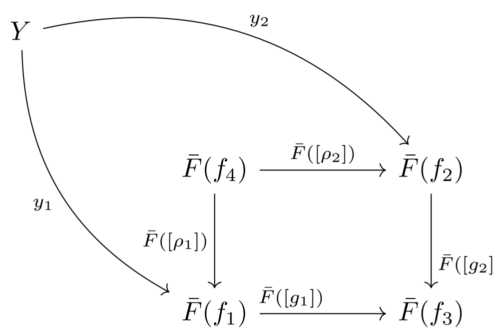
in $\mathbf{Reg}(\mathbf{X})$. Consider $F(f_i)$ for each $1 \leq i \leq 3$ and take their cover-image factorizations in $\mathbf{Y}$:



.

Note that the domain of f_4 is the pullback P of the composites g_1f_3 and g_2f_3, as constructed in proposition 32, so its image factorization looks slightly different. Now apply $\bar{F}$ to the pullback diagram in $\mathbf{Reg}(\mathbf{X})$, and we get a commuting square. Further suppose there exists some Y, y_1, y_2 such that the diagram,



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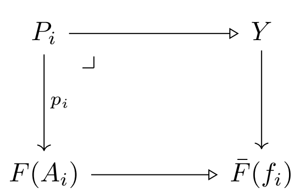
commutes. Well we can pull each y_i back along the cover of the image factorization of f_i and obtain two covers of Y since covers are stable under pullback:



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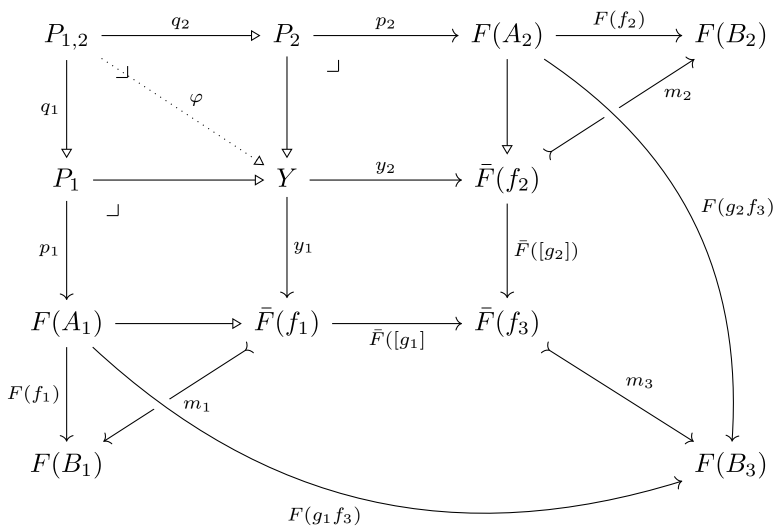
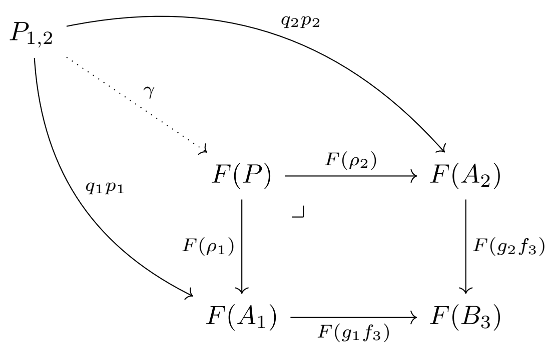
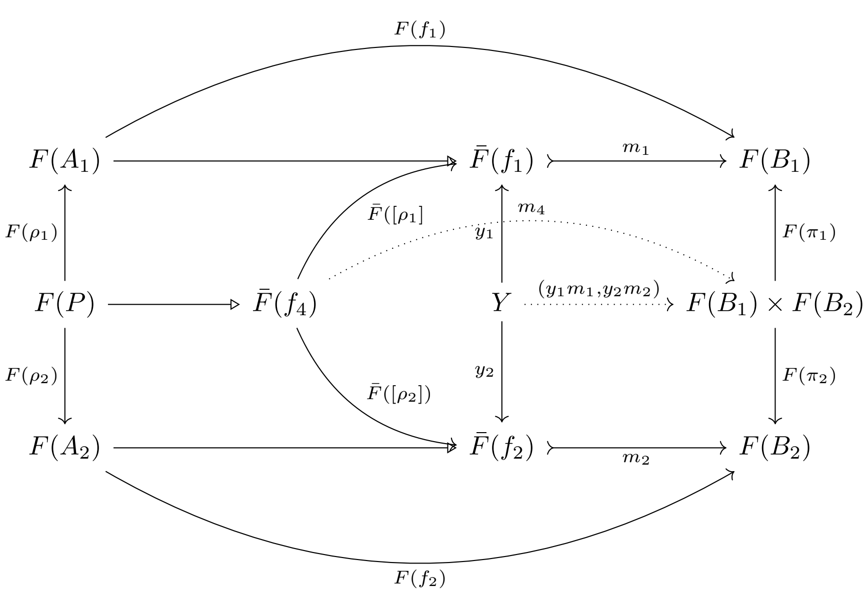
Similarly we can pull these covers back along one another to obtain covers of each of the pullbacks P_i. Moreover, we showed that covers are closed under composition so the diagonal of the last pullback is a cover. All of this can be seen by the following pullbacks:



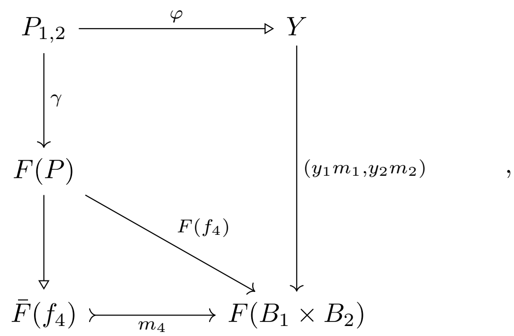
The outer square commuting above induces a unique map $\gamma : P_{1,2} \to P$ in the following pullback:



Post-composing γ with the cover from the image factorization of $\bar{F}(f_4)$ gives a map $Y \to \bar{F}(f_4)$. On the other hand, F is cartesian so it preserves binary products and post composing the y_i with the monics in the image factorization of f_i induces a unique pairing map out of Y pictured below.



Putting all our efforts together we get a commuting square,



which can be verified by computing the projections and referring back to the diagrams drawn above/ On that note, for $i = 1, 2$ we have,

$$\gamma F(f_4)F(\pi_i) = \gamma F(f_4\pi_i) = \gamma F(\rho_i f_i) = \gamma F(\rho_i)F(f_i) = q_i p_i F(f_i) = \varphi y_i m_i.$$

But covers are left orthogonal to monics by lemma 12, so there exists a unique splitting morphism, $k : Y \to \bar{F}(f_4)$ which makes both triangles in the previous square commute. Once more we can post-compose with the projects to see,

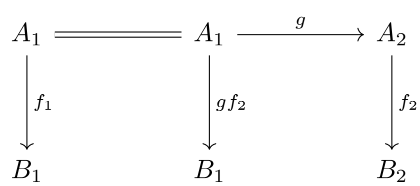
$$k\bar{F}([\rho_i])m_i = km_3\pi_i = (y_1m_1, y_2m_2)\pi_i = y_im_i$$

and since m_i is monic, we conclude that $k\bar{F}([\rho_o]) = y_i$. Since k is unique we can see that $\bar{F}$ applied to a pullback square in $\mathbf{Reg}(\mathbf{X})$ is still a pullback in $\mathbf{Y}$, which is exactly what it means for $\bar{F}$ to preserve pullbacks. Having already seen it preserves the terminal object, we've shown that $\bar{F}$ is cartesian by lemma 2.

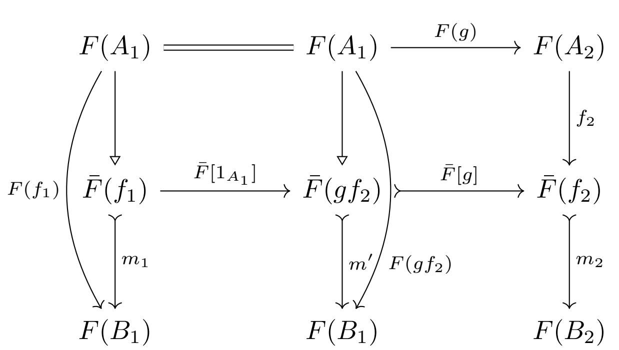
Now let $f_1 \xrightarrow{[g]} f_2$ be an arbitrary cover in $\mathbf{Reg}(\mathbf{X})$ with image factorization,

$$f_1 \xrightarrow{[1_{A_1}]} gf_2 \rightarrowtail^{[g]} f_2 \ .$$

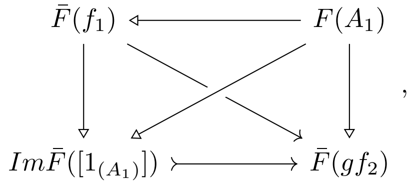
Back in $\mathbf{X}$, this looks like



Applying F and taking the image factorization (in $\mathbf{Y}$) of the vertical arrows, we can see $\bar{F}$ applied to the factorization of $[g]$ sitting in the middle:



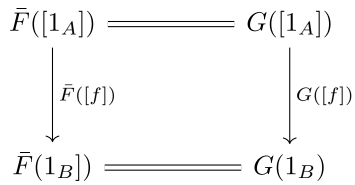
We already know $\bar{F}([g])$ is monic because $\bar{F}$ is cartesian and monomorphisms are (finite) limits, so we only need to show that $\bar{F}([1_{A_1}])$ is a cover. The top left square in the previous diagram reduces to a triangle for which two of the three morphisms are covers, and if we factorize the third morphism we get a factorization of a cover through a monic,



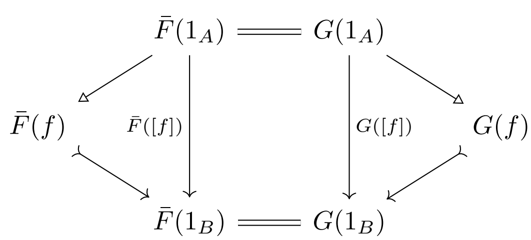
which implies the monic is an isomorphism. This implies the map we started with factors as a composite of two covers and must be a cover itself.

So far we've shown that $\bar{F}$ is a cartesian functor which preserves the cover-monic factorization system of **Reg**(**X**), which means it's a regular functor.

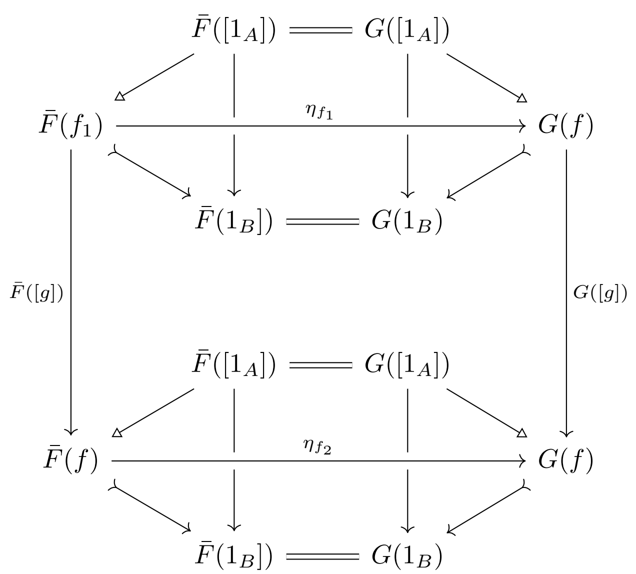
Now suppose there exists another regular functor $G : \mathbf{Reg}(\mathbf{X}) \to \mathbf{Y}$ such that $F = IG$. Then $I\bar{F} = F = IG$, which means that for any object A and morphism $f : A \to B$ in **X** we get the following equalities in **Y**:



This means they must have the same image factorization, but we can see these images are exactly $\bar{F}(f_1)$ and $G(f_1)$ by factoring $[f]$ in **Reg**(**X**) and apply $\bar{F}$ and G, which both preserve the factorizations.



Let η_f denote the isomorphism between the subobjects $\bar{F}(f)$ and $G(f)$. Then for any $[g] : f_1 \to f_2$ in **Reg**(**X**), we have a naturality square,



where the η_{f_i} are isomorphisms. Hence $\eta : \bar{F} \implies G$ is a natural isomorphism and $\bar{F}$ is unique up to natural isomorphism.
□