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The differential λ-calculus: syntax and semantics for differential geometry

by

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Abstract

The differential λ-calculus was introduced to study the resource usage of programs. This thesis marks a change in that belief system; our thesis can be summarized by the analogy

$$\lambda\text{-calculus} : \text{functions} :: \partial\lambda\text{-calculus} : \text{smooth functions}$$

To accomplish this, we will describe a precise categorical semantics for the differential λ-calculus using categories with a differential operator. We will then describe explicit models that are relevant to differential geometry, using categories like Sikorski spaces and diffeological spaces.

Preface

This thesis is the original work of the author.

This thesis represents my attempt to understand the differential λ-calculus, its models and closed structures in differential geometry. I motivated to show that the differential λ-calculus should have models in all the geometrically accepted settings that combine differential calculus and function spaces.

To accomplish this, I got the opportunity to learn chunks of differential geometry. I became fascinated by the fact that differential geometers make use of curves and (higher order) functionals everywhere, but use these spaces of functions without working generally in a closed category. It is in a time like this, that one can convince oneself, that differential geometers secretly wish that there is a closed category of manifolds.

There are concrete problems that become impossible to solve without closed structure. One of the first geometers to propose a generalization of manifolds was Chen, who wanted to iterate a path space construction to formalize how iterated integrals can be computed. These spaces, now called Chen spaces, and their closely related counterparts, diffeological spaces and Sikorski spaces have many properties that make them categorically more well behaved (they are Cartesian closed categories with all limits and colimits). Yet, differential geometers do not seem to have simply flocked to these generalized smooth spaces; the reason is that they have badly behaved limits – that is, they admit non-transverse intersections. Examples of badly behaved limits come from catastrophe theory: one has two submanifolds M, N of some manifold Z, and in Z there is an intersection of M, N where the tangent vectors at those intersections do not span Z. When taking the pullback of M, N one obtains a space B with degenerate behaviour: there can be places where the dimension of B collapses leading to catastrophic behaviour.

I find it exciting that these sorts of issues come up when trying to construct models of the differential λ-calculus. It seems perhaps important that a simple coherence required to model the differential λ-calculus in settings for differential geometry brings up the kind of issues with "bad" limits in differential geometry, which is the subject of part of chapter 6.

Chapter 2 revisits the *term logic* developed in (Blute, Cockett, and Seely, 2009) from the point of view of a representable multicategory. The view of representable multicategory I took is based on conversations and an unpublished paper with Robin Cockett on *Monoidal Turing Categories*. The separation of the product from the rules of differentiation is new. The equivalence of categories between the category of differential theories and the category of Cartesian differential categories is new. The approach to type theories I take in regarding sytax as a presentation of a Cartesian multicategory is heavily influenced and shaped by the opportunity to serve as a teaching assistant for the AARMS summer school "Higher category theory and categorical logic" which was taught by Mike Shulman and Peter LeFanu Lumsdaine.

Chapter 3 is based on reading (Bucciarelli, Ehrhard, and Manzoneto, 2010). I discussed a bit of their paper with the third named author, Giulio Manzonetto. Their paper has no completeness theorem, and Robin Cockett suggested that the chain rule might not hold for theories with function symbols. I wrote this chapter to complete the story. (Bucciarelli, Ehrhard, and Manzoneto, 2010) showed soundness for the differential λ-calculus in Cartesian closed categories for a free theory with no function symbols. The soundness theorem for the differential λ-calculus with function symbols and equations is new, and the completeness theorem is new.

Chapter 4 is based on a review for a journal version of a paper Robin Cockett and I wrote, and have submitted to MSCS on the *untyped differential λ-calculus*. The reviewer wrote that it was not obvious that the type theory I used was related to the differential λ-calculus. I constructed the interpretations to ensure that they were indeed equivalent. However, this review prompted me to learn about some of the more delicate aspects of constructing categorical models for intensional calculi and untyped calculi. Reading papers by Scott, Koymans, Lambek, Barendregt, Meyers, and others leads to a much richer and nuanced history than one might expect to find. In the end, I think the objection was based on a misunderstanding, but I think this detour was important because it lead to a bit more discernment in constructing models.

Chapter 5 is based on a talk I gave in the Peripatetic Seminar at the University of Calgary, in which I proved that synthetic differential geometry gives a model of the differential λ-calculus. Robin Cockett verbally suggested at the end of the talk, that my proof did not involve the particulars of synthetic differential geometry, and seemed to revolve around the strength. I then spent some time analyzing the strength, and proved that indeed, a property of the strength is crucial for modelling the differential λ-calculus.

Chapter 6 arose from trying to understand two seemingly separate things. First was the work of Nishimura, and second was as to whether diffeological spaces have any non-trivial tangent structure. I had never managed to adequately figure out either, until I read Leung's paper (Leung, 2017). Leung proved that tangent structures on a category $\mathbb{X}$ are equivalent to monoidal functors from Weil algebras (with tensor given by coproduct) into the functor category $[\mathbb{X}, \mathbb{X}]$ (with tensor given by functor composition). Nishimura had been asking for what he called *Weil prolongation*, and I realized that Leung's description of tangent structure could be immediately reformulated as *Weil prolongation* using actegories which give a categorically coherent version of Nishimura's notion of prolongation. (Kock, 1986) also used Weil prolongation to obtain tangent structure under the name *semidirect product of categories* instead of actegories. I met the notion of actegory while working with an MSc student of Robin Cockett, named Masuka Yeasin, on linear protocols for message passing types. Garner also used actegories in characterizing tangent categories as enriched categories (Garner, 2018). In this thesis, I try to attribute the theorems Garner proved, to Garner.

The intended audience for this thesis is people interested in the differential λ-calculus and people interested in how to obtain closed categories for differential geometry. I try to take little for granted at the level of theories, but at the level of category theory, I do take some basic material for granted.

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I would like to thank the PIMS organization for funding the peripatetic seminar which allowed visitors to interact with our group on differential and tangent structure. I would like to thank NSERC who, through my supervisor Robin Cockett, allowed me to travel and disseminate my research.

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For Mom, Dad, Grandma, Poppy, Grandma, Andrew, Jakob, BJ, Josh, and Veronika.

> In other words, that the so called involuntary circulation of your blood is one continuous process with the stars shining.
>
> If you find out that it's you who circulates your blood you will at the same moment find out that you are shining the sun.
>
> Because your physical organism is one continuous process with everything else that is going on. Just as the waves are continous with the ocean. Your body is continous with the total energy system of the cosmos and its all you.
>
> Only you're playing the game that you're only this bit of it. – Alan Watts

Table of Contents

List of Tables

Notation

In this thesis we will try to keep the notation as uncluttered as possible. We record notational conventions and symbols here.

Categories

We use the diagrammatic notation for composition of morphisms. This means that $A \xrightarrow{f} B \xrightarrow{g} C$ is written as fg. In case this is hard to parse, we will use $f;g$.

Generic categories will be named with the *bb* font, for example $\mathbb{X}$. Generic 2-categories will be named with the *scr* font, for example $\mathscr{A}$. For generic functors we use capital letters, for example $\mathbb{X} \xrightarrow{F} \mathbb{Y}$. For generic natural transformations, we will use greek letters, for example $F \xrightarrow{\alpha} G$.

In some category theory texts, one sees $\varinjlim F$ and $\varprojlim F$ to denote the limit and colimit of a functor. We will use $\mathsf{lim} F$ and $\mathsf{colim} F$. When $\mathsf{lim}_i X_i$ is a limit, we say a functor H *preserves* the limit when $H(\mathsf{lim}_i X_i) \simeq \mathsf{lim}_i H(X_i)$, mutatis mutandis for preserving a colimit. We often call a functor that preserves limits continuous, and a functor that preserves colimits cocontinuous.

List of categories

Below is a list of categories used in this thesis.

Name	Meaning
Set	Sets and set functions
CVS	Convenient vector spaces and smooth maps
SMan	Smooth manifolds and smooth maps
$\mathsf{Fröl}$	Frölicher spaces and smooth maps
$\mathsf{Fré\text{-}S}$	Fréchet spaces and smooth maps
$\mathsf{Weil\text{-}S}$	Weil spaces and smooth maps
Sik	Sikorski spaces and smooth maps
$\mathscr{C}[\mathscr{T}]$	Classifying category of the theory $\mathscr{T}$
Weil	Weil algebras and augmented algebra maps
$\mathscr{W}_1$	An important subcategory of Weil
$\mathbb{X}[A]$	The simple slice
$\mathbb{X}/A$	The slice
$\mathscr{D}$	A display system
$\mathbb{X}[\mathbb{X}]$	The total category of the simple fibration
$\mathbb{X}^{\mathscr{D}}$	The total category of the display fibration

Combinators and non-functor structures

One term that comes up often in categorical type theory is *currying* a map. This does not mean making a map spicy; instead, it means using Cartesian closed structure to obtain a map of type

$$A \longrightarrow [B, C]$$

from a map of type

$$B \times A \longrightarrow C$$

We sometimes refer to combinators. These were originally meant to be closed formula, and through a loose connection we take them to mean operators on homsets. Some common non-functorial combinators are listed below.

Name	Meaning
$D[f]$	The derivative of f (in a C.D.C.)
$\langle \_, \_ \rangle$	The pairing of maps
ev	Evaluation in a C.C.C.
$\bullet$	Application in a Turing category
$\lambda(f)$	The transpose of f in a C.C.C.
$\lambda(f)$	The canonical code of f
c_f	The code for f
$A \lhd B$ or $A \lhd_r^s B$	A is a retract of B
$\mathscr{E}^n$	The algebra of smooth germs at 0

List of functors

We use S for a comonad, and M or T for a monad. We use $L \dashv R$ to denote that L is a left adjoint to R.

Name	Meaning
$\mathscr{Y}$	The yoneda embedding
T	The tangent functor
$\otimes$	A monoidal structure
$\otimes$	A monoidal action
$\times$	The product functor
$[\_, \_]$	The internal hom
$\mathscr{C}$	Classifying category
Th	The theory of a category
$f*$	The reindexing functor (often pullback)
Σ_f	The left adjoint to $f*$
$\prod_f$	The right adjoint to $f*$

List of natural transformations

We use ϵ for the counit of a comonad or adjunction, η for the unit of a monad or adjunction, δ for the comultiplication of a comonad, and μ for the multiplication of a monad.

In a monoidal category (or monoidal category like setting) we write all associators from left to right: $(A \otimes B) \otimes C \xrightarrow{a_\otimes} A \otimes (B \otimes C)$. We use *right unitor* to mean that the unit is being cancelled on the right $U \otimes I \xrightarrow{u_R} U$ and mutatis mutandis *left unitor*.

We "curry on the left." That is in a monoidal closed category, we have

$$\mathbb{X}(A \otimes B, C) \simeq \mathbb{X}(B, [A, C])$$

Currying the left argument tends to lead to fewer applications of $c_\otimes$.

Name	Meaning
p	The tangent bundle projection
$+$	Tangent vector addition
0	The zero section of the tangent bundle
c	The canonical flip
l	The vertical lift
$c_\otimes$ and $c_\times$	The symmetry maps for $\otimes$ and $\times$
π_i	The i^{th} projection out of a product
$m_\otimes$	The natural map $FA \otimes FB \longrightarrow F(A \otimes B)$ (often the tensor is cartesian and we write $m_\times$)
$m_\top$	$I \longrightarrow FI$
θ	Strength $A \otimes TB \longrightarrow T(A \otimes B)$
ψ	Exponential strength $T[A, B] \longrightarrow [A, TB]$
cur	The internal curry $[A \times B, C] \longrightarrow [B, [A, C]]$
Δ	The copy map $A \xrightarrow{\Delta} A \times A$

In general, we will try to use $m_\otimes$ to denote the monoidal map $FA \otimes FB \longrightarrow F(A \otimes B)$. However, sometimes there will be more than one monoidal functor used in the same calculation, and when this occurs we will state more explicitly the notation we are using.

Logic and type theory

Name	Meaning
$\mathscr{T}$	An equational theory
$\frac{\partial m}{\partial x}(a) \cdot v$	The differential of a term
$m[a/x]$	The substitution operation
$m\ v$	Application in a combinatory or λ-algebra
$\lambda x.m$	λ abstraction
$m : A$	m has type A
$\Gamma \vdash m : A$	The term m derived in context Γ
$\mathsf{s}, \mathsf{k}, \mathsf{w}, \mathsf{b}, \mathsf{i}$	combinators
$\lambda^* x.m$	Derived abstraction
$\Sigma x : A, B$	The sigma type
$\prod x : A, B$	The dependent product
$t =_A s$ or $\mathsf{Eq}(t, s)$	Equality type
$\mathsf{Id}_A(t, s)$ or $A(t, s)$	Identity type

We write proofs as

$$\frac{H_1 \quad \cdots \quad H_k}{C}$$

This means that given the hypotheses $H_1, \cdots, H_k$, the conclusion C may be deduced. It is recommended to read these from bottom to top: to establish or construct C, we construct $H_1, \ldots, H_k$.

Algebra

The symbol $+$ is used for addition in a monoid, for example in a homset in a Cartesian left additive category. The symbol $\cdot$ is used for a multiplicative action; for example multiplication in a rig or algebra, or the action of a scalar on a vector.

Chapter 1

Introduction

This thesis is about understanding the theory of differentiation in settings that contain spaces of smooth functions between spaces. Historically an analytic approach was taken: various categories of topological vector spaces such as Hilbert spaces, Banach spaces, and Fréchet spaces with smooth maps were all proposed, as they do allow infinite dimensional spaces. Fréchet spaces also allow the space of smooth functions between finite dimensional vector spaces. However none of these categories has the property of being Cartesian closed which means that they are not closed to creating arbitrary function spaces.

Many non-analytical approaches have been taken to obtain Cartesian closedness in a category whose maps generalize smoothness in some way. For example, (**chen1967**) used a generalization of smooth manifold based on having a subalgebra of $C^\infty(M,\mathscr{R})$ for a manifold M in order to iterate the path space construction. Souriau proposed diffeological spaces as an alternative formulation of differentiation (**book:iz-diffeology**), and these ideas have been used to study loop spaces, quotients of manifolds, and moment maps in a setting of generalized smooth maps. Sikorski spaces or differential spaces have been used to study orbifolds, orientifolds and stratifolds (**book:diff-alg-top**). Convenient vector spaces have also been used for spaces of diffeomorphisms, mapping spaces, and path spaces (**book:kriegl-frolicher**; Kriegl and Michor, 1997). These approaches all have a common feature: they make use of a notion of differentiation by analogy with features of the category of smooth maps between $\mathscr{R}^n$ to ensure that one has a *good notion of differentiation for spaces of smooth functions*.

In this thesis, we take a different approach using categorical logic and type theory. We will make use of abstract axiomatizations of differentiation and smoothness: Cartesian differential categories (Blute, Cockett, and Seely, 2009) and tangent categories (Rosický, 1984; Cockett and Cruttwell, 2014b). Having an abstract notion of differentiation allows us to add Cartesian closure as an axiom, thus giving us an axiomatization of smooth Cartesian closed categories. Using our axiomatic framework also allows us to phrase *good notion of differentiation for spaces of smooth functions* as a coherence problem instead of making an analogy.

We split the thesis into two parts to separate two main aspects of the investigation. In the first part of the thesis we study the differential λ-calculus. The ordinary λ-calculus is regarded as the theory of functions, and the differential λ-calculus adds differentiation to the λ-calculus in a minimal way to ensure properties like the chain rule hold. Thus, we view the differential λ-calculus as being the theory of smooth functions. To make this assertion precise, we make use of tools from categorical logic that allow us to prove that the differential λ-calculus is fully expressive of our axiomatic framework; that is, any equality in a Cartesian closed differential category may be expressed as an equality in the differential λ-calculus.

Our approach to the differential λ-calculus marks a departure from the traditional view. The differential λ-calculus was introduced in (Ehrhard and Regnier, 2003) to provide the syntax for an interpretation of the λ-calculus into Köthe sequence spaces (Ehrhard, 2002), which is a setting that has a derivative. It was noticed that the derivative could be used to determine the resource commitments of functions (Pagani and Rocca, 2010). The first attempt to give a categorical semantics for the differential λ-calculus was in (Bucciarelli, Ehrhard, and Manzoneto, 2010), and the models given were the coKleisli categories of the finite multiset comonad on the category of relations, Köthe sequence spaces, and modules of finiteness space (Ehrhard, 2005). Thus, the models of the differential λ-calculus focused on resource sensitivity. They are also not easy to obtain.

The main result in the first part of the thesis is a soundness and completeness theorem which makes precise the sense in which the differential λ-calculus is the same subject as Cartesian closed differential categories.

In the second part of the thesis, we seek to understand models of the differential λ-calculus that are not focused on resource sensitivity, but instead have a more differential geometric feel. We also provide constructions that show that models of the differential λ-calculus are "everywhere."

We develop Cartesian closed tangent categories and show that a coherence allows extracting a Cartesian closed differential category, and hence a model of the differential λ-calculus. The coherence is expressed in terms of strength, and we are able to simplify the requirement needed for obtaining a Cartesian closed differential category: the coherence in the first part is quantified over all maps in the category, but the re-expression is phrased in terms of a single natural transformation.

We spend considerable time investigating the consequences of the coherence we require, as it leads to some interesting material on its own. The coherence allows for a description of vector fields as an object. It also gives a concrete description of closed tangent categories from an enriched point of view.

Finally, we investigate some general tools for constructing coherently closed tangent categories, and provide examples. This extends the class of models of the differential λ-calculus into the setting of differential geometry.

To summarize, the main contributions of this thesis are:

1. A sound and complete categorical semantics of the differential λ-calculus;

2. Simplification of the coherence required for models using strength and an elaboration on the consequences of this coherence;

3. Models of the differential λ-calculus that come from differential geometry.

Part I

The differential λ-calculus

Chapter 2

The differential calculus

In this chapter we introduce the differential calculus as an equational theory. Doing this separates out the equational theory of differentiation from the analytic aspects. Our axiomatization has seven equations that express laws of differentiation like the chain rule, the additivity of differentiation, and the independence of partial derivatives. We introduce differential calculus as a formal type theory, but the presentation given here should be familiar to classical calculus.

We also give a semantics for the differential calculus using an abstract framework for differentiation called Cartesian differential categories (Blute, Cockett, and Seely, 2009). We will prove that any equation that can be derived in differential calculus is true in any Cartesian differential category, and vice versa.

We will formalize the connection between differential calculus and Cartesian differential categories using categorical type theory. We introduce a notion of differential theory on top of differential calculus, and show that differential theories form a category. We then establish an adjoint equivalence between differential theories and Cartesian differential categories.

2.1 The categorical setting

In this subsection, we introduce the abstract setting for defining differentiation known as a Cartesian differential category. These categories generalize the category of smooth maps between finite dimensional $\mathscr{R}$-vector spaces by requiring an operator that when applied to a map, yields its total derivative. Cartesian differential categories require homsets to have an addition (although no enrichment is required), so we first introduce Cartesian left additive categories.

By chosen product functor and terminal object, we mean chosen right adjoints

$$\begin{tikzcd} \mathbb{X} \arrow[rr, bend left, "\Delta"] \arrow[rr, phantom, "\perp"] & & \mathbb{X}\times\mathbb{X} \arrow[ll, bend left, "\_\times\_"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X} \arrow[rr, bend left, "!"] \arrow[rr, phantom, "\perp"] & & 1. \arrow[ll, bend left, "1"] \end{tikzcd}$$

The pairing of maps $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$ is the unique map $C \xrightarrow{\langle f,g\rangle} A\times B$ induced by the universal property of the product.

Definition 2.1.1. *Let $\mathbb{X}$ be a Cartesian category, that is a category $\mathbb{X}$, with a chosen terminal object, and chosen product functor $\_\times\_ : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{X}$. $\mathbb{X}$ is **Cartesian left additive** when every hom-set is a commutative monoid with the property that for all f,g,h*

$$f(g+h) = fg + fh \qquad \textit{and} \qquad x0 = 0$$

and where projections π_i are additive in the sense that for any f,g, $(f+g)\pi_i = f\pi_i + g\pi_i$.

In a Cartesian left additive category, a map h such that for all f,g, we have $(f+g)h = fh+gh$ and $0h = 0$ is called **additive**. Also, a map $h : A\times B \longrightarrow C$ is **additive in its second argument** when for all x,y,z: $\langle x, y+z\rangle h = \langle x,y\rangle h + \langle x,z\rangle h$ and $\langle x,0\rangle h = 0$. Likewise, a map h is **additive in its first argument** when $\langle x+y, z\rangle h = \langle x,z\rangle h + \langle y,z\rangle h$ and $\langle 0,z\rangle h = 0$.

Example 2.1.2. The categories of abelian monoids, abelian groups, and modules with homomorphisms are individually Cartesian left additive categories. Every map is additive.

Example 2.1.3. The categories of abelian monoids, abelian groups, and modules with functions between the underlying sets are also individually Cartesian left additive categories (these are just functions and need not preserve the addition). Additition of morphisms is pointwise in the image $(f+g)(x) = f(x)+g(x)$. The additive maps are precisely homomorphisms of the additive monoid structure.

In the following lemma, we use the term monoidal structure map. This is meant to be maps like the associativity morphism, $A \times (B \times C) \longrightarrow (A \times B) \times C$ and the commutativity morphism $A \times B \longrightarrow B \times A$ and the other maps involved in a symmetric monoidal category. However, we take it just to mean composites and pairings of projections.

Lemma 2.1.4. *Let $\mathbb{X}$ be a Cartesian left additive category.*

1. *Additive maps are closed under composition.*
2. $\left\langle f, g \right\rangle + \langle h, k \rangle = \left\langle f + h, g + k \right\rangle$ *and* $0 = \langle 0, 0 \rangle$.
3. $(f \times g) + (h \times k) = (f + h) \times (g + k)$ *and* $0 = 0 \times 0$.
4. *The pairing of additive maps is additive.*
5. *The map* $\Delta := \langle 1, 1 \rangle : A \longrightarrow A \times A$ *is additive.*
6. *The product of additive maps is additive.*
7. *The monoidal structure maps are additive.*

Proof.

1. Apply additivity twice.
2. Consider the post composition of $\left\langle f, g \right\rangle + \langle h, k \rangle$ with π_0.
$$\left(\left\langle f, g \right\rangle + \langle h, k \rangle\right)\pi_0 = \left\langle f, g \right\rangle \pi_0 + \langle h, k \rangle \pi_0 = f + h$$
Similarly, $\left(\left\langle f, g \right\rangle + \langle h, k \rangle\right)\pi_1 = g + k$. But, $\left\langle f + h, g + k \right\rangle$ is unique with this property. Likewise, $0\pi_0 = 0$ and $0\pi_1 = 0$, and again $\langle 0, 0 \rangle$ is unique with this property, giving the result.
3. $$(f \times g) + (h \times k) = \left\langle \pi_0 f, \pi_1 g \right\rangle + \langle \pi_0 h, \pi_1 k \rangle = \left\langle \pi_0(f + h), \pi_1(g + k) \right\rangle = (f + h) \times (g + k)$$
Likewise, $0 = \langle 0, 0 \rangle = \langle \pi_0 0, \pi_1 0 \rangle = 0 \times 0$.

4. Let f, g be additive. Then,

$$\begin{aligned}(x+y)\left\langle f,g\right\rangle &= \left\langle (x+y)f,(x+y)g\right\rangle = \left\langle xf+yf, xg+yg\right\rangle \\ &= \left\langle xf,xg\right\rangle + \left\langle yf,yg\right\rangle = x\left\langle f,g\right\rangle + y\left\langle f,g\right\rangle\end{aligned}$$

   Also

$$0\left\langle f,g\right\rangle = \left\langle 0f,0g\right\rangle = \langle 0,0\rangle = 0$$

5. The map 1 is additive, and $\Delta = \langle 1,1\rangle$.

6. Use that f, g are additive, and that π_0, π_1 are additive, and conclude that $\pi_0 f$ and $\pi_1 g$ are additive. Thus, with the above, $f \times g = \left\langle \pi_0 f, \pi_1 g\right\rangle$ is additive.

7. A monoidal structure map is compositions and pairings of projections. The proof is then that projections are additive, and additive maps are closed to composition and pairing.

□

Definition 2.1.5. *Let $\mathbb{X}$ be a Cartesian left additive category. It is a* ***Cartesian differential category*** *(Blute, Cockett, and Seely, 2009) when there is a combinator* [1]*:*

$$\frac{A \xrightarrow{f} B}{A \times A \xrightarrow[D[f]]{} B}$$

that satisfies the following axioms:

[CD.1] $D[f+g] = D[f] + D[g]$ *and* $D[0] = 0$ *(additivity of differentiation);*

[CD.2] $\langle a+b, c\rangle D[f] = \langle a,c\rangle D[f] + \langle b,c\rangle D[f]$ *and* $\langle 0,c\rangle D[f] = 0$ *(additivity in the "direction");*

[CD.3] $D[1] = \pi_0$ *(linearity of the identity)*

[CD.4] $D[\left\langle f,g\right\rangle] = \left\langle D[f], D[g]\right\rangle$*;*

[CD.5] $D[fg] = \left\langle D[f], \pi_1 f\right\rangle D[g]$ *(chain rule);*

[1] For our current purposes, a combinator just means a function on homsets.

[CD.6] $\langle\langle a,0\rangle,\langle b,c\rangle\rangle D[D[f]] = \langle a,c\rangle D[f]$ *(linearity of the derivative in the "direction");*

[CD.7] $\langle\langle a,b\rangle,\langle c,d\rangle\rangle D[D[f]] = \langle\langle a,c\rangle,\langle b,d\rangle\rangle D[D[f]]$ *(interchange).*

One thinks of $A\times A \xrightarrow{D[f]} B$ as the total derivative derivative of f: the first A gives the directional vector and the second A gives the point that the derivative was taken at.

A map h is **linear** when $D[h] = \pi_0 h$.

Example 2.1.6. *The category of commutative monoids or R-modules for a commutative Rig R with maps homomorphisms are both Cartesian differential category in which every map is linear. The derivative is defined $D[f] := \pi_0 f$.*

Example 2.1.7 (Polynomials)**.**

Let R be a commutative rig. There is a Lawvere theory $\mathscr{P}$ of polynomials:

Obj: Natural numbers

Arr: A map $n \longrightarrow 1$ is a polynomial in n variables over R. A map $n \longrightarrow m$ is an m tuple of polynomials, each in n variables.

There is a partial derivative

$$\frac{\partial p(x_1,\ldots,x_n)}{\partial x_i}$$

that may be defined by induction on the structure of polynomials using the sum rule and the product rule and the rule for differentiating constants.

$$\frac{\partial c}{\partial x} = 0 \qquad c \text{ is a constant}$$

$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 0 & x_i \neq x_j \\ 1 & x_i = x_j \end{cases}$$

$$\frac{\partial p+q}{\partial x} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x}$$

$$\frac{\partial p*q}{\partial x} = p*\frac{\partial q}{\partial x} + \frac{\partial p}{\partial x}*q$$

The partial derivatives extend to a total derivative for maps $n \longrightarrow 1$:

$$D[p](v,a) := \sum_i \frac{\partial p(x_1,\ldots,x_n)}{\partial x_i}(a) * v_i$$

where $v = (v_1,\ldots,v_n)$ and similarly $a = (a_1,\ldots,a_n)$. Then this extends to a derivative for the whole category $\mathscr{P}$:

$$D[(p_1,\ldots,p_m)] = (D[p_1],\ldots,D[p_m])$$

In $\mathscr{P}$ the linear maps are precisely the linear polynomials

$$p(x) = a * x$$

Example 2.1.8. *The category of smooth maps between finite dimensional vector spaces is a Cartesian differential category. Here the derivative is given by the Jacobian:*

$$D[f](v,a) := J_f(a) \cdot v$$

where J_f is is the Jacobian matrix of partial derivatives of f. The linear maps are precisely the linear transformations of the underlying vector spaces.

Example 2.1.9 ((Bauer et al., 2018))**.** *A Cartesian differential category may be extracted from the category of all functors between abelian categories (denoted* AbCat*)*

There is a monad Ch *on* AbCat*. The Kleisli category of this monad is a Cartesian left additive category. Given a functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ there is something like a finite difference operator called a cross-effect; for example:*

$$F(A \oplus B) \oplus F(0) \simeq F(A) \oplus F(B) \oplus \mathsf{cr}_2(F)(A,B)$$

There are also linear approximation operations D_1 and D_1^V. D_1 is a totalization of the chain complex induced by iterating cross effects. This linearization may be done in context, and D_1^V denotes linearizing with respect to just V. The derivative is

$$D[F](V,X) := D_1 F(V) \oplus D_1^V \mathsf{cr}_2(F)(X,V)$$

The cross-effects can be linearized, giving, up to homotopy, the best linear approximation to a functor $D[F](A;V)$ which acts just like a directional derivative. And *the theorem is that the homotopy category of* $\mathsf{AbCat}_{\mathsf{Ch}}$ *is a Cartesian differential category with this derivative.*

Example 2.1.10 ((Dubuc and Kock, 1984))**.** *Let T be a unary typed algebraic theory in which the single type R is a commutative rig, and satifies the following* Fermat property*: For any map*

$$\Gamma, x : R, \Gamma' \mapsto f(x) : R$$

There is a unique map

$$\Gamma, a : R, b : R, \Gamma' \mapsto \frac{\Delta f(x)}{\Delta x}(a, b) : R$$

Such that

$$f(a+b) = f(a) + \frac{\Delta f(x)}{\Delta x}(a, b) \cdot b$$

With the Fermat property, every f has a partial derivative

$$\frac{\partial f(z)}{\partial z}(a) \cdot v := \frac{\Delta f(z)}{\Delta z}(a, 0) \cdot v$$

Then given a map $x_1, \cdots, x_n \mapsto f(x_1, \ldots, x_n) : R^n \longrightarrow R$ its total derivative at a point a in the direction v is

$$D[f](v_1, \ldots, v_n; a_1, \ldots, a_n) := \sum_{i=1}^{n} \frac{\partial f(x_i)[a_j / x_j]_{i \neq j}}{\partial x_i}(a_i) \cdot v_i$$

The total derivative of a map $R^n \xrightarrow{\langle f_1, \ldots, f_n \rangle} R^m$ is given by taking the total derivative of each of the m components.

Given a ring R with the Fermat property, (**journal:dg-arb-ring**) extend example 2.1.10 to all R-modules with a Fermat property. They also extend this by looking at models in topological spaces. This means that we can extend the smooth maps between finite dimensions to the infinite dimensional case.

Example 2.1.11. *The categories of smooth maps between Banach spaces and smooth maps between Fréchet spaces are Cartesian differential categories. The derivative is the Fréchet derivative: which gives both categories the structure required for Fermat modules. For Banach spaces when $V \xrightarrow{f} W$ is smooth then there is a unique map*

$$D[f] : U \longrightarrow \mathsf{Lin}(V, W)$$

such that
$$f(x+h) = f(x) + D[f](x)(h) + o(f,x,h)$$
where
$$o(f,x,h) = \frac{\Delta \frac{\Delta f(z)}{\Delta z}(x,w)\cdot h}{\Delta w}(0,h)\cdot h$$
For Fréchet spaces $\mathsf{Lin}(V,W)$ *is not a Fréchet space, but there is still a unique map* $D[f]: V \times U \longrightarrow W$ *that is linear in* V*, and satsifies*

$$f(x+h) = f(x) + D[f](h,x) + o(f,x,h)$$

Another example is convenient vector spaces.

Example 2.1.12 (Convenient vector spaces)**.**

To explain the category of convenient vector spaces, we must first explain Frölicher spaces (**book:kriegl-frolicher**). In this example, let $\mathscr{R}$ denote the real numbers. A **Frölicher space** is a triple (X,C,F) where

- X is a set;
- $C \subseteq \mathsf{Set}(\mathscr{R},X)$ and $F \subseteq \mathsf{Set}(X,\mathscr{R})$;
- $c \in C$ if and only if for all $f \in F$ the compoiste $\mathscr{R} \xrightarrow{c} X \xrightarrow{f} \mathscr{R}$; is a smooth function $\mathscr{R} \longrightarrow \mathscr{R}$, and similarly $f \in F$ iff for all $c \in C$, cf is smooth.

A **map of Frölicher spaces** $(X,C_X,F_X) \xrightarrow{h} (Y,C_Y,F_Y)$ is a function $X \xrightarrow{h} Y$ of the underlying sets such that for every $c \in C_X$, $ch \in C_Y$.

If V is a Fréchet space, then $(V, C^{\infty}(\mathscr{R},V), C^{\infty}(V,\mathscr{R}))$ is a Frölicher space. These examples capture the intuition of a Frölicher space: they are specified in terms of sets of smooth curves and functionals. A **Frölicher vector space** is an $\mathscr{R}$-vector space V, where V has a Frölicher space structure, and where addition and scalar multiplication are maps of Frölicher spaces.

A **convenient vector space** is a Frölicher vector space V such that the C_V and F_V are generated from the linear maps $\mathsf{Lin}(V,\mathscr{R})$, and is separated and complete. The definition of a Frölicher space sets up a Galois connection between curves and functionals, so it makes sense to talk about generation. A space is separated when for any two points, there are functionals that separate them. The completeness condition is about convergence with respect to a structure called a Mackey-Cauchy filter (see **book:kriegl-frolicher** section 2.6).

(Blute, Ehrhard, and Tasson, 2010) proved that there is a notion of derivative, and importantly that the category of convenient vector spaces and Frölicher maps form a Cartesian differential category. In 6.7.5, we give a direct description of this differential structure.

The following basic result will be used throughout this chapter.

Lemma 2.1.13. *Let $\mathbb{X}$ be a Cartesian differential category.*

1. *Linear maps are closed under composition.*
2. *Every linear map is additive.*
3. *The requirement of being Cartesian left additive is redundant in the sense that if $\mathbb{X}$ is left additive, has products, and a differential combinator, then it is Cartesian left additive.*
4. *If f is linear, then $D[fg] = (f \times f)D[g]$ and $D[hf] = D[h]f$.*
5. ***CD.4** is equivalent to $D[\pi_i] = \pi_0\pi_i$.* [2].
6. *All the monoidal structure isomorphisms are linear.*

Proof. Let $\mathbb{X}$ be a Cartesian differential category.

1. Let f, g be linear. $D[fg] = \langle D[f], \pi_1 f\rangle D[g] = \langle \pi_0 f, \pi_1 f\rangle \pi_0 g = \pi_0 f g$
2. Let h be linear.

   $$(x+y)h = \langle x+y, 1\rangle \pi_0 h = \langle x+y, 1\rangle D[h] = \langle x, 1\rangle + \langle y, 1\rangle D[h] = xh + yh$$

   and similarly, $0h = \langle 0, 1\rangle \pi_0 h = \langle 0, 1\rangle D[h] = 0$.
3. Since $D[\pi_i] = \pi_0\pi_i$ we have that projections are linear, and by the above, they preserve addition. Hence $\mathbb{X}$ is a Cartesian left additive category.

[2]Linearity of projections was originally an axiom. J.S. Lemay pointed out that $D\langle f, g\rangle = \langle Df, Dg\rangle$ was redundant, but it is actually equivalent to linearity of projection.

4. These follow directly from an application of **[CD.5]**.

5. Suppose **CD.4** holds. We will prove that projections are linear. We have $A \times B \xrightarrow{1=\langle \pi_0, \pi_1 \rangle} A \times B$, and

$$D[1_{A\times B}] = D[\langle \pi_0, \pi_1 \rangle] = \pi_0 = \pi_0 \langle \pi_0, \pi_1 \rangle = \langle \pi_0 \pi_0, \pi_0 \pi_1 \rangle$$

   on the other hand

$$D[\langle \pi_0, \pi_1 \rangle] = \langle D[\pi_0], D[\pi_1] \rangle$$

   Thus by universality $D[\pi_i] = \pi_0 \pi_i$.

   Conversely, suppose that projections are linear. We will prove that **CD.4** holds. The proof is by the universality for products. Consider that as π_i satisfies $D[\pi_i] = \pi_0 \pi_i$, we have that for any k,

$$D[k]\pi_i = \langle D[k], \pi_1 k \rangle \pi_0 \pi_i = \langle D[k], \pi_1 k \rangle D[\pi_i] = D[k\pi_i]$$

   Then

$$D[\langle f, g \rangle]\pi_0 = D[\langle f, g \rangle \pi_0] = D[f]$$

   and similarly $D[\langle f, g \rangle]\pi_1 = D[g]$. But $\langle D[f], D[g] \rangle$ is unique with this property, thus $D[\langle f, g \rangle] = \langle D[f], D[g] \rangle$.

6. Projection is assumed to be linear. The identity is also assumed to be linear, hence by the pairing of linear maps, Δ is linear. The associator and commutator are formed by the pairing of composites of projections, and hence are linear.

□

In a Cartesian differential category, there are notions of **partial derivative**. Given $A \times B \xrightarrow{f} C$:

$$D_B[f] := A \times (B \times B) \xrightarrow{\langle 0 \times \pi_0, 1 \times \pi_1 \rangle} (A \times B) \times (A \times B) \xrightarrow{D[f]} C$$

$$D_A[f] := (A \times A) \times B \xrightarrow{\langle \pi_0 \times 0, \pi_1 \times 1 \rangle} (A \times B) \times (A \times B) \xrightarrow{D[f]} C$$

are the partial derivatives with respect to B and A respectively.

A map $A \times B \xrightarrow{f} C$ is **linear in its second argument** when $D_B[f] = (1 \times \pi_0)f$, and **linear in its first argument when** $D_A[f] = (\pi_0 \times 1)f$.

The proof of the following is straightforward:

Lemma 2.1.14. *In any Cartesian differential category*

1. *A map* $A \times B \xrightarrow{f} C$ *is linear in its first argument if and only if* $\mathsf{sw} f$ *is linear in its second where* $\mathsf{sw} = \langle \pi_1, \pi_0 \rangle$.
2. *For any* f, $D[f]$ *is linear in its first argument. In fact, this is equivalent to* ***CD.6***.
3. *If* h *is linear in its second argument, then for any* g, $(g \times 1)h$ *is linear in its second argument. Likewise if* h *is linear in its first argument, then for any* g, $(1 \times g)h$ *is linear in its first argument.*
4. *If* h *is linear in an argument, then it is additive in that argument.*
5. *For* $A \times B \xrightarrow{f} C$, $D_B[f] = \mathsf{sw} D_A[\mathsf{sw} f]$, *and* $D[f] = \langle \pi_0 \times \pi_0, \pi_1 \pi_1 \rangle D_A[f] + \langle \pi_1 \pi_0, \pi_1 \times \pi_1 \rangle D_B[f]$.

2.2 Differential type theory

In classical mathematics, one often works with expressions, or terms [3], like $\sin^2(x) + xy$. Expressions can be nested, for example $\sin(\cos(x)) + xy$. In type theory, such expressions are taken to be pure syntax – that is, they have no meaning on their own. The meaning of an expression is given by assigning an actual function to an expression, and doing so gives the semantics of the expression. In order to provide such semantics for expressions consistently, we need to minimally know the type of input and output that a function corresponding to a particular expression will have. We annotate an expression to provide the kind of information we need:

$$x : \mathscr{R}, y : \mathscr{R} \vdash \sin^2(x) + xy : \mathscr{R}$$

[3] In type theory, the word term is used instead of expression.

says that any function which provides the semantics of the expression must at least take in two real inputs, and produce a real output. This can be seen as defining a function $f(x, y) := \sin^2(x) + xy : \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{R}$.

In categorical semantics, instead of providing a function that realizes a term, we provide a map in a category whose type is consistent with the type of the term; for example, map $A \times A \longrightarrow A$ as a potential model for our example above.

One then adds the ability to specify equations between terms, for example $\sin(0) = 0$. A type theory consists of rules for producing well formed terms with well defined domains and codomains, rules that determine how to compose terms, equations that relate terms, and rules that determine how to derive new equalities from established equalities.

Differential type theory adds an additional operation to terms: a derivative. Typically, terms are produced starting from a set of named, typed function symbols called a signature. However, the derivative of an expression does not arise this way. One takes the partial derivative of an expression with respect to a variable. For example we write

$$\frac{\partial \sin^2(x) + xy}{\partial x}(a) \cdot v$$

to make it clear that the meaning of the derivative depends on the variable being differentiated. The derivative of terms satisfies familiar laws like the chain rule and symmetry of partial derivatives.

In the first subsection we will introduce simple differential type theories. These theories have only simple types, that is we do not allow forming products of types. In the second subsection we introduce Cartesian differential theories by building on the differential theories subsection. Cartesian differential theories do allow forming product types. These subsections will follow the same general pattern. The final subsection provides the categorical semantics as an equivalence of categories between differential theories and Cartesian differential categories.

2.2.1 Simple differential type theory

In this subsection, we develop simple differential type theory in three movements. First, we introduce the terms of differential type theory, and the cut (or composition) relation. Second, we then introduce equations for the derivative of terms, and prove some basic results that these equations force. These equalities express what it means to be a differential Cartesian multicategory, but it is still in a sense free: the only equalities between terms are those dictating how the derivative works. Third, we provide an interpretation of terms into a Cartesian differential category. To give an interpretation of terms into a Cartesian differential category, we provide a construction that produces from a term, a map in a Cartesian differential category. We prove this interpretation is *sound* which means a few related things. Most directly, it means that any equation between terms can be realized by equality of the associated maps under interpretation. In addition, soundness means that the equations for the derivative are reasonable in the sense that they would always be true under interpretation even if they had not been axiomatized. In the sequel, we will extend soundness, and show that it gives an adjunction between the category of differential theories and the category of Cartesian differential categories.

2.2.1.1 Syntax and free differential theories

In this subsubsection, we define a differential signature, terms over a signature, and then provide equalities for composing and differentiating terms.

Definition 2.2.1. *A* ***differential signature*** $\Sigma = (\Sigma_0, \Sigma_1)$ *is a multigraph; that is, a collection* Σ_0 *of atomic types, and a collection of multiedges or function symbols together with source and target functions. The source of* $f \in \Sigma_1$ *is a list of types, and the target is a single type; we write* $f \in \Sigma(A_1, \ldots, A_n; B)$ *to denote its source list and target. A* `type` *denotes a definable type, but for now, this just means that* $A \in \Sigma_0$.

The **terms**, $\mathscr{M}(\Sigma)$ of a differential signature are found in table 2.1. In the Cut rule, we produce a term $t[s/x]$. A term in context $\Gamma, x : X, \Gamma' \vdash t : Z$ can be thought of as a map, with a specified wire (namely the x wire) or domain, and cutting s into t for x should be thought of as substitution. The Fun rule shows how to use the named function symbols from our signature to produce new terms from old, for example producing the term $\sin(x)$ from the symbol sin and the variable x. The Zero and Plus rules introduce the addition structure on terms. The Derivative rule creates a term that represents the partial derivative of a term.

The set of free variables of a term is defined inductively:

$$\frac{A \text{ type}}{\Gamma, x : A, \Gamma' \vdash x : A}\ \text{Projection} \qquad \frac{\Gamma, \Gamma' \vdash s : X \quad \Gamma, x : X, \Gamma' \vdash t : Z}{\Gamma, \Gamma' \vdash t[s/x] : Z}\ \text{Cut}$$

$$\frac{\{\Gamma \vdash t_i : A_i\}_i \quad f \in \Sigma(A_1, \ldots, A_n; B)}{\Gamma \vdash f(t_1, \ldots, t_m) : B}\ \text{Fun}$$

$$\frac{}{\Gamma \vdash 0 : B}\ \text{Zero} \qquad \frac{\Gamma \vdash t_1 : B \quad \Gamma \vdash t_2 : B}{\Gamma \vdash t_1 + t_2 : B}\ \text{Plus}$$

$$\frac{\Gamma, x : A, \Gamma' \vdash m : B \quad \Gamma, \Gamma' \vdash a : A \quad \Gamma, \Gamma' \vdash v : A}{\Gamma, \Gamma' \vdash \frac{\partial m}{\partial x}(a) \cdot v : B}\ \text{Derivative}$$

Table 2.1: Term formation rules for differential type theory

$$\mathsf{fv}(x) = \{x\} \qquad \mathsf{fv}(t[m/x]) = (\mathsf{fv}(t) - \{x\}) \cup \mathsf{fv}(m)$$
$$\mathsf{fv}(f(t_1, \ldots, t_n)) = \mathsf{fv}(t_1) \cup \cdots \cup \mathsf{fv}(t_n)$$
$$\mathsf{fv}(0) = \emptyset \qquad \mathsf{fv}(m + n) = \mathsf{fv}(m) \cup \mathsf{fv}(n)$$
$$\mathsf{fv}\left(\frac{\partial m}{\partial x}(a) \cdot v\right) = (\mathsf{fv}(m) - \{x\}) \cup \mathsf{fv}(a) \cup \mathsf{fv}(v)$$

Note that the differential binds the variable x which occurs in m. Also, note, that we use explicit cuts, which is also a binding rule. We sometimes write an equation in shorthand as $m = n$ but really mean that m, n are terms that have the same type, can be formed in the same context, so we really have $\Gamma \vdash m = n : B$. The requirement that both m, n be formed in the same context means that equations cannot introduce variable captures.

The first group of equalities put on terms regards the cut:

Definition 2.2.2 (Composition/cut equalities)**.**

[Cut.1] $x[m/x] = m \qquad y[m/x] = y$*;*

[Cut.2] $f(t_1, \ldots, t_n)[m/x] = f(t_1[m/x], \ldots, t_n[m/x])$*;*

[Cut.3] $(s + t)[m/x] = s[m/x] + t[m/x] \qquad 0[m/x] = 0$*;*

[Cut.4] $(\frac{\partial t}{\partial y}(a) \cdot v)[m/x] = \frac{\partial t[m/x]}{\partial y}(a[m/x]) \cdot v[m/x]$*. Here, to be a well formed term,* $y \notin \mathsf{fv}(m)$.

The derivative is a binding operation. To ensure the name of the bound variable is irrelevant, we require alpha-conversion as an additional equality:

$$\boldsymbol{D[\alpha]}: \quad \frac{\partial m}{\partial x}(a)\cdot v = \frac{\partial m[y/x]}{\partial y}(a)\cdot v$$

The following lemma states important properties of the cut relation that we use throughout this chapter and the next.

Lemma 2.2.3. *With **Cut.1–4** and $\boldsymbol{D[\alpha]}$:*

1. *The equalities for cut, oriented from left to right form locally a confluent and terminating (hence confluent) rewriting system modulo α-conversion, whose normal forms are cut free.*

2. *If $y \notin \mathsf{fv}(a)$, $m[t/y][a/x] = m[a/x][t[a/x]/y]$. Thus if additionally $x \notin \mathsf{fv}(m)$ then $m[t/y][a/x] = m[t[a/x]/y]$. On the other hand, if additionally $x \notin \mathsf{fv}(t)$ then $m[t/y][a/x] = m[a/x][t/y]$.*

Proof.

1. The bag ordering defined in (**journal:Dershowitz:BagTermination**) allows a proof of termination. Termination may be seen simply: the cut equalities oriented from left to right always move the $[m/x]$ into smaller terms. Local confluence is proved by the Knuth-Bendix argument: all critical pairs converge.

2. This is a straightforward proof by induction on the structure of m.

□

Remark 2.2.4. *A multicategory or coloured operad is much like a category, except that the domain of an arrow is a list of objects rather than a single object (***book:leinster-operads***). In a multicategory, a list of maps $\Gamma_i \xrightarrow{f_i} A_i$ is composed with a single map $A_1, \ldots, A_n \xrightarrow{g} B$, and this composition is associative for each of the paths through a tree of compositions.*

One way to interpret lemma 2.2.3.(2) is that the terms modulo the cut equalities form a multicategory. That $m[t/y][a/x] = m[a/x][t/y]$ as long as $x \notin \mathsf{fv}(t)$ and

$y \notin \mathsf{fv}(a)$ says that parallel composition is unambiguous. That $m[t/y][a/x] = m[t[a/x]/y]$ when $x \notin \mathsf{fv}(m)$ says that associativity holds.

A Cartesian multicategory also has weakening and contraction. Weakening in type theory can be seen by subsituting a single variable for two different variables: $m[z/x][z/y]$. Contraction can be seen because $\Gamma \vdash x : A$ does not require Γ to be the singleton $x : A$.

Thus the terms modulo the cut equalities form a Cartesian multicategory.

Next, we introduce the equations that the derivative must satisfy, and prove a few consequences of these equations.

Definition 2.2.5 (Equalities of differentiation)**.**

[Dt.1] *$m + (n + t) = (m + n) + t$, $m + n = n + m$, and $m + 0 = m$.*

[Dt.2] *$\frac{\partial m+n}{\partial x}(a) \cdot v = \frac{\partial m}{\partial x}(a) \cdot v + \frac{\partial n}{\partial x}(a) \cdot v$ and $\frac{\partial 0}{\partial x}(a) \cdot v = 0$.*

[Dt.3] *$\frac{\partial m}{\partial x}(a) \cdot (v + w) = \frac{\partial m}{\partial x}(a) \cdot v + \frac{\partial m}{\partial x}(a) \cdot w$ and $\frac{\partial m}{\partial x}(a) \cdot 0 = 0$.*

[Dt.4] *$\frac{\partial x}{\partial x}(a) \cdot v = v$ and when $x \notin \mathsf{fv}(m)$ we have $\frac{\partial m}{\partial x}(a) \cdot v = 0$.*

[Dt.5] *$\frac{\partial f[g/x]}{\partial y}(a) \cdot v = \frac{\partial f[g[a/y]/x]}{\partial y}(a) \cdot v + \frac{\partial f[a/y]}{\partial x}(g[a/y]) \cdot (\frac{\partial g}{\partial y}(a) \cdot v)$. Here $x \notin \mathsf{fv}(a)$.*

[Dt.6] *$\frac{\partial \frac{\partial m}{\partial x}(a) \cdot y}{\partial y}(b) \cdot v = \frac{\partial m}{\partial x}(a) \cdot v$. Here $y \notin \mathsf{fv}(m, a)$.*

[Dt.7] *$\frac{\partial \frac{\partial m}{\partial x}(a) \cdot v}{\partial y}(b) \cdot w = \frac{\partial \frac{\partial m}{\partial y}(b) \cdot w}{\partial x}(a) \cdot v$. Here $y \notin \mathsf{fv}(a, v)$ and $x \notin \mathsf{fv}(b, w)$.*

These basic equalities may be extended to be a congruence on terms by the following definition.

Definition 2.2.6. *A **theorem** or **derivable equality** is an equation that can be derived by closing the **Cut** and **Dt** equalities with respect to being an equivalence relation*

$$\frac{\Gamma \vdash m : B}{\Gamma \vdash m = m : B} \qquad \frac{\Gamma \vdash m = n : B}{\Gamma \vdash n = m : B} \qquad \frac{\Gamma \vdash m = n : B \quad \Gamma \vdash n = t : B}{\Gamma \vdash m = t : B}$$

and with respect to congruence

$$\frac{\Gamma, x : A, \Gamma' \vdash m = m' : B \quad \Gamma, \Gamma' \vdash a, v : A}{\Gamma, \Gamma' \vdash \frac{\partial m}{\partial x}(a) \cdot v = \frac{\partial m'}{\partial x}(a) \cdot v} \qquad \frac{\Gamma, \Gamma' \vdash m = m' : A \quad \Gamma, x : A, \Gamma' \vdash t = t' : B}{\Gamma, \Gamma' \vdash t[m/x] = t'[m'/x] : B}$$

Lemma 2.2.7. *In differential type theory:*

1. *When $y \notin \mathsf{fv}(f)$,*
$$\frac{\partial f[g/x]}{\partial y}(a) \cdot v = \frac{\partial f}{\partial x}(g[a/y]) \cdot \left(\frac{\partial g}{\partial y}(a) \cdot v\right)$$
Note, this was the original form of the chain rule for the theory presented by (Blute, Cockett, and Seely, 2009).

2. *Differential α-conversion is actually derivable from the chain rule: $\frac{\partial m}{\partial x}(a) \cdot v = \frac{\partial m[y/x]}{\partial y}(a) \cdot v$ holds.*

3. *Derivable equality is a congruence on terms.*

4. *When $y_i \notin \mathsf{fv}(g_j)$ for all i, j*
$$\frac{\partial f[g_i/y_i]_{i=1}^n}{\partial x}(a) \cdot v = \frac{\partial f[g_i[a/x]/y_i]_{i=1}^n}{\partial x}(a) \cdot v + \sum_{i=1}^n \frac{\partial f[g_j/y_j]_{i \neq j}[a/x]}{\partial y_i}(g_i[a/x]) \cdot \frac{\partial g_i}{\partial x}(a) \cdot v$$
In particular, when $x \notin \mathsf{fv}(f)$ and $y_i \notin \mathsf{fv}(g_j)$ for all i, j
$$\frac{\partial f[g_i/y_i]_{i=1}^n}{\partial x}(a) \cdot v = \sum_{i=1}^n \frac{\partial f[g_j/y_j]_{j \neq i}[a/x]}{\partial y_i}(g_i[a/x]) \cdot \left(\frac{\partial g_i}{\partial x}(a) \cdot v\right)$$

5. *When $y \notin \mathsf{fv}(b, w)$*
$$\begin{aligned}\frac{\partial \frac{\partial t}{\partial y}(a) \cdot v}{\partial x}(b) \cdot w = &\frac{\partial \frac{\partial t}{\partial x}(b) \cdot w}{\partial y}(a[b/x]) \cdot (v[b/x]) \\ &+ \frac{\partial \frac{\partial t[b/x]}{\partial y}(z_1) \cdot (v[b/x])}{\partial z_1}(a[b/x]) \cdot \left(\frac{\partial a}{\partial x}(b) \cdot w\right) \\ &+ \frac{\partial t[b/x]}{\partial y}(a[b/x]) \cdot \left(\frac{\partial v}{\partial x}(b) \cdot w\right)\end{aligned}$$
In particular, when $x \notin \mathsf{fv}(a)$ and $y \notin \mathsf{fv}(b, w)$ then
$$\frac{\partial \frac{\partial t}{\partial y}(a) \cdot v}{\partial x}(b) \cdot w = \frac{\partial \frac{\partial t}{\partial x}(b) \cdot w}{\partial y}(a) \cdot (v[b/x]) + \frac{\partial t[b/x]}{\partial y}(a) \cdot \left(\frac{\partial v}{\partial x}(b) \cdot w\right)$$

Proof.

1. As $y \notin \mathsf{fv}(f)$, then $y \notin f[g[a/y]/x]$, so the derivative of this term vanishes by **Dt.4.2**:

$$\frac{\partial f[g/x]}{\partial y}(a)\cdot v = \frac{\partial f[g[a/y]/x]}{\partial y}(a)\cdot v + \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot(\frac{\partial g}{\partial y}(a)\cdot v)$$
$$= \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot(\frac{\partial g}{\partial y}(a)\cdot v) = \frac{\partial f}{\partial x}(g[a/y])\cdot(\frac{\partial g}{\partial y}(a)\cdot v)$$

2. $$\frac{\partial m[y/x]}{\partial y}(a)\cdot v = \frac{\partial m}{\partial x}(a)\cdot \frac{\partial y}{\partial y}(a)\cdot v = \frac{\partial m}{\partial x}(a)\cdot v$$

3. This proof is a straightforward proof by induction on the structure of terms, and relies mostly on the equalities for cut to move equalities into and out of subterms.

4. The proof is by induction on n. The base case is **Dt.5**. For the inductive case:

$$\begin{aligned}
&\frac{\partial f[g_i/y_i]}{\partial x}(a)\cdot v = \frac{\partial f[g_1/y_1][g_i/y_i]_{i=2}^{n}}{\partial x}(a)\cdot v \\
&= \frac{\partial f[g_1[a/x]/y_1][g_i[a/x]/y_i]_{i=2}^{n}}{\partial x}(a)\cdot v + \frac{\partial f[g_i/y_i]_{i\neq 1}}{\partial y_1}(g_1[a/x])\cdot \frac{\partial g_1}{\partial x}(a)\cdot v \qquad \textbf{Dt.5} \\
&= \frac{\partial f[g_1[a/x]/y_1][g_i[a/x]/y_i]_{i=2}^{n}}{\partial x}(a)\cdot v + \sum_{i=2}^{n}\frac{\partial f[g_j/y_j]_{i\neq j}^{n\geq 2}[a/x]}{\partial y_i}(g_i[a/x])\cdot \frac{\partial g_i}{\partial x}(a)\cdot v \\
&\quad + \frac{\partial f[g_i/y_i]_{i\neq 1}}{\partial y_1}(g_1[a/x])\cdot \frac{\partial g_1}{\partial x}(a)\cdot v \qquad \text{Ind. Hyp.} \\
&= \frac{\partial f[g_i[a/x]/y_i]_{i=1}^{n}}{\partial x}(a)\cdot v + \sum_{i=1}^{n}\frac{\partial f[g_j/y_j]_{i\neq j}[a/x]}{\partial y_i}(g_i[a/x])\cdot \frac{\partial g_i}{\partial x}(a)\cdot v
\end{aligned}$$

5. The claim follows from a couple of uses of **Dt.5** followed by **Dt.6**. Consider:

$$
\begin{aligned}
&\frac{\partial \frac{\partial t}{\partial y}(a)\cdot v}{\partial x}(b)\cdot w = \frac{\partial \frac{\partial t}{\partial y}(z_1)\cdot z_2[a/z_1, v/z_2]}{\partial x}(b)\cdot w \\
&= \frac{\partial \frac{\partial t}{\partial y}(z_1)\cdot (v[b/x])[a/z_1]}{\partial x}(b)\cdot w + \frac{\partial t[b/x]}{\partial y}(a[b/x])\cdot\left(\frac{\partial v}{\partial x}(b)\cdot w\right) \\
&= \frac{\partial \frac{\partial t}{\partial y}(a[b/x])\cdot (v[b/x])}{\partial x}(b)\cdot w + \frac{\partial \frac{\partial t[b/x]}{\partial y}(z_1)\cdot (v[b/x])}{\partial z_1}(a[b/x])\cdot\left(\frac{\partial a}{\partial x}(b)\cdot w\right) \\
&+ \frac{\partial t[b/x]}{\partial y}(a[b/x])\cdot\left(\frac{\partial v}{\partial x}(b)\cdot w\right) \\
&= \frac{\partial \frac{\partial t}{\partial x}(b)\cdot w}{\partial y}(a[b/x])\cdot (v[b/x]) + \frac{\partial \frac{\partial t[b/x]}{\partial y}(z_1)\cdot (v[b/x])}{\partial z_1}(a[b/x])\cdot\left(\frac{\partial a}{\partial x}(b)\cdot w\right) \\
&+ \frac{\partial t[b/x]}{\partial y}(a[b/x])\cdot\left(\frac{\partial v}{\partial x}(b)\cdot w\right)
\end{aligned}
$$

□

2.2.1.2 Interpretation of a differential theory

In this subsubsection, we show that the terms of a differential theory may be interpreted as maps in Cartesian differential category in way that substitution is interpreted by composition and the differentiation of terms is interpreted by the differential operator of a Cartesian differential category.

The following definitions of specification and theory are extensions of definitions 3.2.1 and 3.2.5 (resp.) given in (Jacobs, 1999). A **differential specification** is $\mathscr{T} = (\Sigma_0, \Sigma_1, E)$ where E is a set of equations between terms of $\mathscr{M}(\Sigma)$ of the same type and formed in the same context; we write $m =_\Gamma n : A$ to denote an equation in E. We then expand the notion of theorem to include the rule:

$$\frac{m =_\Gamma n : B \in E}{\Gamma \vdash m = n : B}$$

A **differential theory** is a differential specification that is closed to derivable equality, in the sense that E contains every derivable equality from $\mathscr{T}$. Every specification generates a minimal theory.

An **interpretation** for a differential specification $\mathcal{T}$ into a Cartesian differential category $\mathbb{X}$ consists of an assignment of types to objects of $\mathbb{X}$, denoted $[\![A]\!]$, together with an assignment of $f \in \Sigma(A_1,\dots,A_n;B)$ to a map

$$(\cdots([\![A_1]\!] \times [\![A_2]\!])\cdots \times [\![A_n]\!] \xrightarrow{[\![f]\!]} [\![B]\!]$$

We extend the interpretation to contexts:

$$[\![]\!] := 1 \qquad [\![x : A]\!] := [\![A]\!] \qquad [\![\Gamma, x : A]\!] := [\![\Gamma]\!] \times [\![A]\!].$$

Finally, extend the interpretation to terms:

Proj:
- $[\![x : A \vdash x : A]\!] := 1_{[\![A]\!]}$;
- $[\![\Gamma, x : A \vdash x : A]\!] := [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{\pi_1} [\![A]\!]$;
- $[\![\Gamma, y : B \vdash x : A]\!] := [\![\Gamma]\!] \times [\![B]\!] \xrightarrow{\pi_0} [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash x : A]\!]} [\![A]\!]$.

Cut: $[\![\Gamma \vdash t[m/x] : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle 1, [\![\Gamma \vdash m : A]\!]\rangle} [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{[\![\Gamma, x : A \vdash t : B]\!]} [\![B]\!]$.

Fun: Let $f \in \Sigma(A_1,\dots,A_n;B)$. Then

$$[\![\Gamma \vdash f(t_1,\dots,t_n) : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash t_i : A_i]\!]\rangle_{i=1}^n} (\cdots([\![A_1]\!] \times [\![A_2]\!])\cdots \times [\![A_n]\!] \xrightarrow{[\![f]\!]} [\![B]\!].$$

CMon: $[\![\Gamma \vdash m + n : B]\!] := [\![\Gamma \vdash m : B]\!] + [\![\Gamma \vdash n : B]\!]$ and $[\![\Gamma \vdash 0 : B]\!] := 0$.

Diff: The interpretation of the derivative, $\left[\!\!\left[\Gamma \vdash \frac{\partial m}{\partial x}(a) \cdot v\right]\!\!\right]$, is

$$[\![\Gamma]\!] \xrightarrow{\langle\langle 0, [\![\Gamma \vdash v : A]\!]\rangle, \langle 1, [\![\Gamma \vdash a : A]\!]\rangle\rangle} ([\![\Gamma]\!] \times [\![A]\!]) \times ([\![\Gamma]\!] \times [\![A]\!]) \xrightarrow{D[[\![\Gamma, x : A \vdash m : B]\!]]} [\![B]\!]$$

Note that derivative of a term is interpreted into the partial derivative in $\mathbb{X}$

$$\begin{aligned}
&\langle\langle 0, [\![v]\!]\rangle, \langle 1, [\![a]\!]\rangle\rangle D[[\![\Gamma, x : \Gamma \times A \vdash m : B]\!]] \\
&= \langle\langle 0, \langle [\![v]\!], [\![a]\!]\rangle \pi_0\rangle, \langle 1, \langle [\![v]\!], [\![a]\!]\rangle \pi_1\rangle\rangle D[[\![\Gamma, x : \Gamma \times A \vdash m : B]\!]] \\
&= \langle 1, \langle [\![v]\!], [\![a]\!]\rangle\rangle \langle 0 \times \pi_0, 1 \times \pi_1\rangle D[[\![\Gamma, x : \Gamma \times A \vdash m : B]\!]] \\
&= \langle 1, \langle [\![v]\!], [\![a]\!]\rangle\rangle D_{[\![x:A]\!]}[[\![\Gamma, x : \Gamma \times A \vdash m : B]\!]]
\end{aligned}$$

In what follows, we will sometimes drop the types and the context when they are not relevant to the calculation.

Lemma 2.2.8. *We have*

- *For any m if $x \notin \mathsf{fv}(m)$ then $[\![\Gamma, x \vdash m]\!] = \pi_0 [\![\Gamma \vdash m]\!]$.*
- *For any Γ, x, y, m, $[\![\Gamma, x, y \vdash m]\!] = c_2 [\![\Gamma, y, x \vdash m]\!]$ where c_2 is the involution $(A \times B) \times C \xrightarrow{\langle\langle \pi_0\pi_0, \pi_1\rangle, \pi_0\pi_1\rangle} (A \times C) \times B$.*

The following theorem establishes the soundness of the interpretation.

Theorem 2.2.9. *Let $\mathscr{T} = (\Sigma_0, \Sigma_1, E)$ be any differential specification. If every equality $t_1 =_\Gamma t_2 : A \in E$ has $[\![\Gamma \vdash t_1 : A]\!] = [\![\Gamma \vdash t_2 : A]\!]$, then every theorem of $\mathscr{T}$ holds under interpretation.*

Since every equality in E holds under interpretation by assumption, we must show that all the derivable equalities hold under interpretation. This is equivalent to proving that every equality in the theory generated by $(\Sigma_0, \Sigma_1, \emptyset)$ holds under interpretation.

To prove this, we must show that every generating equality gives an equation in $\mathbb{X}$, and that $\mathbb{X}$ is closed under the derivation of theorems.

Proof. We start with the cut equations: **[Cut.1-3]** are immediate. For example,

$$[\![\Gamma \vdash x[m/x]]\!] = \left\langle 1, [\![\Gamma \vdash m]\!] \right\rangle [\![\Gamma, x \vdash x]\!] = \left\langle 1, [\![\Gamma \vdash m]\!] \right\rangle \pi_1 = [\![\Gamma \vdash m]\!]$$

The proof for **[Cut.4]** is a bit longer, so we give its proof here:

Let $w := \left\langle \left\langle 0, [\![\Gamma \vdash v[m/x]]\!] \right\rangle, \left\langle 1, [\![\Gamma \vdash a[m/x]]\!] \right\rangle \right\rangle$. Then note that

$$\begin{aligned} & wD[[\![\Gamma, y \vdash m]\!]] = wD[\pi_0 [\![\Gamma \vdash m]\!]] = w(\pi_0 \times \pi_0) D[[\![\Gamma \vdash m]\!]] \\ & = \langle 0, 1 \rangle D[[\![\Gamma \vdash m]\!]] = 0 \end{aligned}$$

Also note that the natural isomorphism $c_2 : (A \times B) \times C \longrightarrow (A \times C) \times B$ used in lemma 2.2.8 is linear and gives,

$$D[[\![\Gamma, y, x \vdash t]\!]] = D[c_2 [\![\Gamma, x, y \vdash t]\!]] = (c_2 \times c_2) D[[\![\Gamma, x, y \vdash t]\!]]$$

Then,

$$\left[\!\!\left[\Gamma \vdash \frac{\partial t[m/x]}{\partial y}(a[m/x])\cdot v[m/x]\right]\!\!\right] = wD[\langle 1, [\![\Gamma, y \vdash m]\!]\rangle [\![\Gamma, y, x \vdash t]\!]]$$
$$= w\left\langle D[\langle 1, [\![\Gamma, y \vdash m]\!]\rangle], \pi_1\langle 1, [\![\Gamma, y \vdash m]\!]\rangle\right\rangle D[[\![\Gamma, y, x \vdash t]\!]]$$
$$= w\left\langle\langle \pi_0, D[[\![\Gamma, y \vdash m]\!]]\rangle, \pi_1\langle 1, [\![\Gamma, y \vdash m]\!]\rangle\right\rangle D[[\![\Gamma, y, x \vdash t]\!]]$$
$$= \left\langle\langle\langle 0, [\![\Gamma \vdash v[m/x]]\!]\rangle, 0\rangle, \langle\langle 1, [\![\Gamma \vdash a[m/x]]\!]\rangle, [\![\Gamma \vdash m]\!]\rangle\right\rangle (c_2 \times c_2) D[[\![\Gamma, x, y \vdash t]\!]]$$
$$= \left\langle\langle\langle 0, 0\rangle, [\![\Gamma \vdash v[m/x]]\!]\rangle, \langle\langle 1, [\![\Gamma \vdash m]\!]\rangle, [\![\Gamma \vdash a[m/x]]\!]\rangle\right\rangle D[[\![\Gamma, x, y \vdash t]\!]]$$
$$= \left\langle\langle 0, [\![v[m/x]]\!]\rangle, \langle\langle 1, [\![\Gamma \vdash m]\!]\rangle, [\![\Gamma \vdash a[m/x]]\!]\rangle\right\rangle D[[\![\Gamma, x, y \vdash t]\!]]$$
$$= \langle 1, [\![\Gamma \vdash m]\!]\rangle \left\langle\langle 0, [\![\Gamma, x \vdash v]\!]\rangle, \langle 1, [\![\Gamma, x \vdash a]\!]\rangle\right\rangle D[[\![\Gamma, x, y \vdash t]\!]]$$
$$= \langle 1, [\![\Gamma \vdash m]\!]\rangle \left[\!\!\left[\Gamma, x \vdash \frac{\partial t}{\partial y}(a)\cdot v\right]\!\!\right]$$
$$= \left[\!\!\left[\Gamma \vdash \left(\frac{\partial t}{\partial y}(a)\cdot v\right)[m/x]\right]\!\!\right]$$

Now for the differential identities:

[Dt.1] This is immediate.

[Dt.2] This is also immediate.

[Dt.3] Consider,

$$\frac{\partial m}{\partial x}(a)\cdot(v_1 + v_2) = \left\langle\langle 0, [\![v_1 + v_2]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![m]\!]]$$
$$= \left\langle\langle 0, [\![v_1]\!] + [\![v_2]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![m]\!]] = \left\langle\langle 0, [\![v_1]\!]\rangle + \langle 0, [\![v_2]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![m]\!]]$$
$$= \left\langle\langle 0, [\![v_1]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![m]\!]] + \left\langle\langle 0, [\![v_2]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![m]\!]]$$
$$= \left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v_1\right]\!\!\right] + \left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v_2\right]\!\!\right] = \left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v_1 + \frac{\partial m}{\partial x}(a)\cdot v_2\right]\!\!\right]$$

The 0 case is similar.

[Dt.4] Consider,

$$\left[\!\!\left[\Gamma \vdash \frac{\partial x}{\partial x}(a)\cdot v\right]\!\!\right] = \left\langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle D[[\![\Gamma, x \vdash x]\!]]$$
$$= \left\langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![a]\!]\rangle\right\rangle \pi_0\pi_1 = [\![\Gamma \vdash v]\!]$$

And suppose $x \notin m$, then

$$\left[\!\!\left[\Gamma \vdash \frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right] = \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle D[[\![\Gamma, x \vdash m]\!]]$$
$$= \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle(\pi_0 \times \pi_0)D[[\![\Gamma \vdash m]\!]] = \langle 0, 1\rangle D[[\![\Gamma \vdash m]\!]] = 0$$

[Dt.5] To make the calculation digestible, break it up into a few steps. Let $w = \left\langle\left\langle 0, [\![v]\!]\right\rangle, \left\langle 1, [\![a]\!]\right\rangle\right\rangle$, and let $w_2 = D[[\![\Gamma, y \vdash g[a/y]]\!]]$. Now, we expand w_2. Note that $y \notin \mathsf{fv}(g[a/y])$.

$$w_2 = D[[\![\Gamma, y \vdash g[a/y]]\!]] = D[\pi_0[\![\Gamma \vdash g[a/y]]\!]] = (\pi_0 \times \pi_0)D[\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle [\![\Gamma, y \vdash g]\!]]$$
$$= (\pi_0 \times \pi_0)\left\langle\left\langle pi_0, D[[\![\Gamma \vdash a]\!]]\right\rangle, \pi_1\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\gamma, y \vdash g]\!]] \qquad \textbf{CD.4}$$
$$= \left\langle\left\langle \pi_0\pi_0, (\pi_0 \times \pi_0)D[[\![\Gamma \vdash a]\!]]\right\rangle, \pi_1\pi_0\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, y \vdash g]\!]]$$

Now, we expand $w\,w_2$ using $0 = \langle 0, 0\rangle$ followed by **CD.2**.

$$w\,w_2 = \left\langle\left\langle 0, \langle 0, 1\rangle\, D[[\![\Gamma \vdash a]\!]]\right\rangle w\pi_1\pi_0\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, y \vdash g]\!]]$$
$$= \left\langle 0, w\pi_1\pi_0\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, y \vdash g]\!]] = 0$$

Thus,

$$\left[\!\!\left[\frac{\partial f[g[a/y]/x]}{\partial y}(a)\cdot v\right]\!\!\right] = wD[[\![\Gamma, y \vdash f[g[a/y]/x]]\!]]$$
$$= wD[\left\langle 1, [\![\Gamma, y \vdash g[a/y]]\!]\right\rangle [\![\Gamma, y, x \vdash f]\!]]$$
$$= w\left\langle\langle \pi_0, D[w_2]\rangle, \pi_1\left\langle 1, \pi_0[\![\Gamma \vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma, y, x \vdash f]\!]]$$
$$= \left\langle\left\langle\left\langle 0, [\![\Gamma \vdash v]\!]\right\rangle, 0\right\rangle, \left\langle\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle, [\![\Gamma \vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma, y, x \vdash f]\!]]$$

We call the equality established immediately above (1)

Next, we expand $D[[\![\Gamma, x \vdash f[a/y]]\!]]$:

$$D[[\![\Gamma, x \vdash f[a/y]]\!]] = D[\left\langle 1, [\![\Gamma, x \vdash a]\!]\right\rangle [\![\Gamma, x, y \vdash f]\!]]$$
$$= \left\langle\left\langle \pi_0, D[\pi_0[\![\Gamma \vdash a]\!]]\right\rangle, \pi_1\left\langle 1, \pi_0[\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, x, y \vdash f]\!]]$$
$$= \left\langle\left\langle \pi_0, (\pi_0 \times \pi_0)D[[\![\Gamma \vdash a]\!]]\right\rangle, \pi_1\left\langle 1, \pi_0[\![\Gamma \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, x, y \vdash f]\!]]$$

Thus,

$$\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right)\right]\!\!\right]\\
&=\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,\left\langle 1,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma,x\vdash f[a/y]]\!]]\\
&=\left\langle\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,\langle 0,1\rangle\, D[[\![\Gamma\vdash a]\!]]\right\rangle,\left\langle 1,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\left\langle 1,\pi_0[\![\Gamma\vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma,x,y\vdash f]\!]]\\
&=\left\langle\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,0\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash g[a/y]]\!]\right\rangle,[\![\Gamma\vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma,x,y\vdash f]\!]]\\
&=\left\langle\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,0\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash g[a/y]]\!]\right\rangle,[\![\Gamma\vdash a]\!]\right\rangle\right\rangle (c_2\times c_2) D[[\![\Gamma,y,x\vdash f]\!]]\\
&=\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma,y,x\vdash f]\!]]
\end{aligned}$$

We call the equality established immediately above (2).

Now, note that $\left\langle 0,\left[\!\!\left[\vdash \frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle + \left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,0\right\rangle = \left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle$,
and use this with **CD.2**:

$$
\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right)+\frac{\partial f[g[a/y]/x]}{\partial y}(a)\cdot v\right]\!\!\right]\\
&=\left[\!\!\left[\Gamma \vdash \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right)\right]\!\!\right]+\left[\!\!\left[\frac{\partial f[g[a/y]/x]}{\partial y}(a)\cdot v\right]\!\!\right]\\
&=\left\langle\left\langle 0,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma,y,x\vdash f]\!]]\\
&+\left\langle\left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,0\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma,y,x\vdash f]\!]] \qquad (1,2)\\
&=\left\langle\left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,\left[\!\!\left[\Gamma\vdash\frac{\partial g}{\partial y}(a)\cdot v\right]\!\!\right]\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle,[\![\Gamma\vdash g[a/y]]\!]\right\rangle\right\rangle D[[\![\Gamma,y,x\vdash f]\!]]\\
&=\langle\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle 1,[\![\Gamma\vdash a]\!]\rangle\rangle D[[\![\Gamma,y\vdash g]\!]]\rangle,\langle\langle 1,[\![\Gamma\vdash a]\!]\rangle,[\![\Gamma\vdash g[a/y]]\!]\rangle\rangle D[[\![\Gamma,y,x\vdash f]\!]]\\
&=\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle 1,[\![\Gamma\vdash a]\!]\rangle\rangle\langle\langle\pi_0,D[[\![\Gamma,y\vdash g]\!]]\rangle,\pi_1\langle 1,[\![\Gamma,y\vdash g]\!]\rangle\rangle D[[\![\Gamma,y,x\vdash f]\!]]\\
&=\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle 1,[\![\Gamma\vdash a]\!]\rangle\rangle\langle D[\langle 1,[\![\Gamma,y\vdash g]\!]\rangle],\pi_1\langle 1,[\![\Gamma,y\vdash g]\!]\rangle\rangle D[[\![\Gamma,y,x\vdash f]\!]]\\
&=\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle 1,[\![\Gamma\vdash a]\!]\rangle\rangle D[\langle 1,[\![\Gamma,y\vdash g]\!]\rangle[\![\Gamma,y,x\vdash f]\!]]\\
&=\langle\langle 0,[\![\Gamma\vdash v]\!]\rangle,\langle 1,[\![\Gamma\vdash a]\!]\rangle\rangle D[[\![\Gamma,y\vdash f[g/x]]\!]]\\
&=\left[\!\!\left[\Gamma\vdash\frac{\partial f[g/x]}{\partial y}(a)\cdot v\right]\!\!\right]
\end{aligned}
$$

[Dt.6] Again, break the computation down to make it simpler. To start consider,

$$
\begin{aligned}
&D\left[\left[\!\!\left[\Gamma,y\vdash\frac{\partial m}{\partial x}(a)\cdot y\right]\!\!\right]\right]=D[\left\langle\left\langle 0,[\![\Gamma,y\vdash y]\!]\right\rangle,\left\langle 1,[\![a]\!]\right\rangle\right\rangle D[[\![\Gamma,y,x\vdash m]\!]]]\\
&=D[\left\langle\left\langle 0,[\![\Gamma,y\vdash y]\!]\right\rangle,\left\langle 1,[\![a]\!]\right\rangle\right\rangle(c_2\times c_2)D[[\![\Gamma,x,y\vdash m]\!]]]\\
&=D[\left\langle\left\langle\langle 0,\pi_1\rangle,0\right\rangle,\left\langle\left\langle\pi_0,\pi_0[\![\Gamma\vdash a]\!]\right\rangle,\pi_1\right\rangle\right\rangle(\pi_0\times\pi_0)D[[\![\Gamma,x\vdash m]\!]]]\\
&=D[\left\langle\langle 0,\pi_1\rangle,\left\langle\pi_0,\pi_0[\![\Gamma\vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma,x\vdash m]\!]]]\\
&=\left\langle D[\left\langle\langle 0,\pi_1\rangle,\left\langle\pi_0,\pi_0[\![\Gamma\vdash a]\!]\right\rangle\right\rangle],\pi(\left\langle\langle 0,\pi_1\rangle,\left\langle\pi_0,\pi_0[\![\Gamma\vdash a]\!]\right\rangle\right\rangle)\right\rangle D[D[[\![\Gamma,x\vdash m]\!]]]\\
&=\left\langle\left\langle\langle 0,\pi_0,\pi_1\rangle,\left\langle\pi_0\pi_0,(\pi_0\times\pi_0)D[[\![\Gamma\vdash a]\!]]\right\rangle\right\rangle,\pi_1\left\langle\langle 0,\pi_1\rangle,\left\langle\pi_0,\pi_0[\![\Gamma\vdash a]\!]\right\rangle\right\rangle\right\rangle D[D[[\![\Gamma,x\vdash m]\!]]]
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left[\!\!\left[\frac{\partial \frac{\partial m}{\partial x}(a)\cdot y}{\partial y}(b)\cdot v\right]\!\!\right] \\
&=\left\langle\left\langle 0,[\![\Gamma,\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma\vdash b]\!]\right\rangle\right\rangle D[\left[\!\!\left[\Gamma,y\vdash\frac{\partial m}{\partial x}(a)\cdot y\right]\!\!\right]] \\
&=\left\langle\left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,0\right\rangle,\left\langle\left\langle 0,[\![\Gamma\vdash b]\!]\right\rangle,\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle\right\rangle\right\rangle D[D[[\![\Gamma,x\vdash m]\!]]] \\
&=\left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma\vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma,x\vdash m]\!]] \\
&=\left[\!\!\left[\Gamma\vdash\frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right]
\end{aligned}
$$

[Dt.7] First, note that

$$D[\left\langle\left\langle 0,[\![\Gamma,y\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma,y\vdash a]\!]\right\rangle\right\rangle]=\left\langle\left\langle 0,(\pi_0\times\pi_0)D[[\![\Gamma\vdash v]\!]]\right\rangle,\left\langle\pi_0,(\pi_0\times\pi_0)D[[\![\Gamma\vdash a]\!]]\right\rangle\right\rangle$$

Then

$$
\begin{aligned}
&\left\langle\left\langle 0,[\![\Gamma\vdash w]\!]\right\rangle,\left\langle 1,[\![\Gamma\vdash b]\!]\right\rangle\right\rangle D[\left\langle\left\langle 0,[\![\Gamma,y\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma,y\vdash a]\!]\right\rangle\right\rangle] \\
&=\left\langle\left\langle 0,\langle 0,1\rangle\, D[[\![\Gamma\vdash v]\!]]\right\rangle,\left\langle\left\langle 0,[\![\Gamma\vdash w]\!]\right\rangle,\langle 0,1\rangle\, D[[\![\Gamma\vdash a]\!]]\right\rangle\right\rangle \\
&=\left\langle\langle 0,0\rangle,\left\langle\left\langle 0,[\![\Gamma\vdash w]\!]\right\rangle,0\right\rangle\right\rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\langle\left\langle 0,[\![\Gamma\vdash w]\!]\right\rangle,\left\langle 1,[\![\Gamma\vdash b]\!]\right\rangle\right\rangle\left\langle D[\left\langle\left\langle 0,[\![\Gamma,y\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma,y\vdash a]\!]\right\rangle\right\rangle],\pi_1\left\langle\left\langle 0,[\![\Gamma,y\vdash v]\!]\right\rangle,\left\langle 1,[\![\Gamma,y\vdash a]\!]\right\rangle\right\rangle\right\rangle \\
&=\left\langle\left\langle\langle 0,0\rangle,\left\langle\left\langle 0,[\![\Gamma\vdash w]\!]\right\rangle,0\right\rangle\right\rangle,\left\langle\left\langle 0,[\![\Gamma\vdash v]\!]\right\rangle,\left\langle\left\langle 1,[\![\Gamma\vdash b]\!]\right\rangle,[\![\Gamma\vdash a]\!]\right\rangle\right\rangle\right\rangle
\end{aligned}
$$

We use the equality immediately above in the third step of the following calculuation.

$$\left[\!\!\left[\Gamma \vdash \frac{\partial \frac{\partial m}{\partial x}(a)\cdot v}{\partial y}(b)\cdot w\right]\!\!\right]$$

$$= \left\langle\left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle, \left\langle 1, [\![\Gamma \vdash b]\!]\right\rangle\right\rangle D[\left\langle\left\langle 0, [\![\Gamma, y \vdash v]\!]\right\rangle, \left\langle 1, [\![\Gamma, y \vdash a]\!]\right\rangle\right\rangle D[[\![\Gamma, y, x \vdash m]\!]]]$$

$$= \left\langle\left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle, \left\langle 1, [\![\Gamma \vdash b]\!]\right\rangle\right\rangle$$
$$\left\langle D[\left\langle\left\langle 0, [\![\Gamma, y \vdash v]\!]\right\rangle, \left\langle 1, [\![\Gamma, y \vdash a]\!]\right\rangle\right\rangle], \pi_1 \left\langle\left\langle 0, [\![\Gamma, y \vdash v]\!]\right\rangle, \left\langle 1, [\![\Gamma, y \vdash a]\!]\right\rangle\right\rangle\right\rangle D[D[[\![\Gamma, y, x \vdash m]\!]]]$$

$$= \left\langle\left\langle \langle 0,0\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle, 0\right\rangle\right\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash v]\!]\right\rangle, \left\langle\left\langle 1, [\![\Gamma \vdash b]\!]\right\rangle, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle\right\rangle D[D[[\![\Gamma, y, x \vdash m]\!]]]$$

$$= \left\langle\left\langle \langle 0,0\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle, 0\right\rangle\right\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash v]\!]\right\rangle, \left\langle\left\langle 1, [\![\Gamma \vdash b]\!]\right\rangle, [\![\Gamma \vdash a]\!]\right\rangle\right\rangle\right\rangle$$
$$((c_2 \times c_2) \times (c_2 \times c_2)) D[D[[\![\Gamma, y, x \vdash m]\!]]]$$

$$= \left\langle\left\langle \langle 0,0\rangle, \left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle\right\rangle, \left\langle\left\langle\left\langle 0, [\![\Gamma \vdash v]\!]\right\rangle, 0\right\rangle, \left\langle\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle, [\![\Gamma \vdash b]\!]\right\rangle\right\rangle\right\rangle$$
$$D[D[[\![\Gamma, y, x \vdash m]\!]]] \qquad (\text{as } 0 = \langle 0,0\rangle)$$

$$= \left\langle\left\langle \langle 0,0\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash v]\!]\right\rangle, 0\right\rangle\right\rangle, \left\langle\left\langle 0, [\![\Gamma \vdash w]\!]\right\rangle, \left\langle\left\langle 1, [\![\Gamma \vdash a]\!]\right\rangle, [\![\Gamma \vdash b]\!]\right\rangle\right\rangle\right\rangle$$
$$D[D[[\![\Gamma, y, x \vdash m]\!]]] \qquad \textbf{CD.6}$$

$$= \left[\!\!\left[\Gamma \vdash \frac{\partial \frac{\partial m}{\partial y}(b)\cdot w}{\partial x}(a)\cdot v\right]\!\!\right]$$

Thus, all the generating equalities hold. To show that all derivable theorems give equalities in $\mathbb{X}$, we show that the interpretation is closed to derivable equality. The equivalence relation requirements: reflexivity, symmetry, and transitivity hold because equality in a category is an equivalence relation.

For congruence, suppose $[\![\Gamma, x \vdash m]\!] = [\![\Gamma, x \vdash m']\!]$. We must show that $\left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right] = \left[\!\!\left[\frac{\partial m'}{\partial x}(a)\cdot v\right]\!\!\right]$ for all a, v, yet this is immediate as $D[[\![\Gamma, x \vdash m]\!]] = D[[\![\Gamma, x \vdash m']\!]]$. Similarly, suppose that $[\![\Gamma \vdash m]\!] = [\![\Gamma \vdash m']\!]$ and $[\![\Gamma, x \vdash t]\!] = [\![\Gamma, x \vdash t']\!]$. It is then immediate that $[\![\Gamma \vdash t[m/x]]\!] = [\![\Gamma \vdash t'[m'/x]]\!]$.

This completes the proof. $\square$

2.2.2 Cartesian differential type theory

In the previous subsection we introduced differential theory, and in this subsection we add products to a differential theory. This involves both adding the product of objects to our theory, and also adding the pairing of terms. We make the same moves, but we can move a bit faster since most of the interpretation was defined in the previous section.

We use a standard notion of classifying category: we formally add rules to the logic that allow for product formation and elimination. This allows replacing a context $x_1 : A_1, \ldots, x_n : A_n$ with a single product type $z : A_1 \times \cdots \times A_n$. The terms that can be formed with a single type on the left of turnstile always form a category, and the extension of the product rules, ensures that this category is always a Cartesian category.

For a differential specification $\mathcal{T}$, we denote the extension of it to a representable Cartesian multicategory by $\mathcal{M}_\times(\Sigma)$.

We extend the notion of definable type, recall that atomic types A have $A\ \mathsf{ty}$, and now

$$\frac{}{1\ \mathsf{ty}} \qquad \text{and} \qquad \frac{A\ \mathsf{ty} \quad B\ \mathsf{ty}}{A \times B\ \mathsf{ty}}$$

We then add the representation terms:

$$\frac{\Gamma, \Gamma' \vdash m : C}{\Gamma, z : 1, \Gamma' \vdash z \lhd ().m : C} \qquad \text{and} \qquad \frac{\Gamma, x : A, y : B, \Gamma' \vdash m : C}{\Gamma, z : A \times B, \Gamma' \vdash z \lhd (x, y).m : C}$$

Above the line, we have two variables, $x : A, y : B$, and to form a map out of the product $A \times B$, we need a new variable $z : A \times B$. The notation $z \lhd (x, y).m$ really means that z is the variable representing (x, y). We will show below that $z \lhd (x, y).m = m[\pi_0(z)/x, \pi_1(z)/y]$.

We also have product formation rules:

$$\frac{}{\Gamma \vdash () : 1} \qquad \text{and} \qquad \frac{\Gamma \vdash m : A \quad \Gamma \vdash n : B}{\Gamma \vdash (m, n) : A \times B}$$

And projection operations:

$$\frac{\Gamma \vdash m : A \times B}{\Gamma \vdash \pi_0(m) : A} \qquad \text{and} \qquad \frac{\Gamma \vdash m : A \times B}{\Gamma \vdash \pi_1(m) : B}$$

Next, extend the notion of derivable equality to include projection computation, surjective pairing axioms, a congruence for equality, and the projection linearity axiom:

$$\frac{}{\Gamma \vdash \pi_0(m, n) = m}\ \pi_0\text{-elim} \qquad \frac{}{\Gamma \vdash \pi_1(m, n) = n}\ \pi_1\text{-elim}$$

$$\frac{\Gamma \vdash m : 1}{\Gamma \vdash m = () : 1}\ \text{surj-pair} \qquad \frac{\Gamma \vdash m : A \times B}{\Gamma \vdash m = (\pi_0(m), \pi_1(n)))}\ \text{surj-pair}$$

$$\frac{\Gamma, x : A, y : B, \Gamma' \vdash m = n}{\Gamma, z : A \times B, \Gamma' \vdash z \lhd (x, y).m = z \lhd (x, y).n : C}\ \text{rep-cong}$$

And the following equations that force projection to be linear.

$\boldsymbol{\pi_0}$**-linear:** $\dfrac{\partial z \lhd (x, y).x}{\partial z}(a) \cdot v = \pi_0(v)$ $\qquad$ $\boldsymbol{\pi_1}$**-linear:** $\dfrac{\partial z \lhd (x, y).y}{\partial z}(a) \cdot v = \pi_1(v)$

Also we must extend the equalities for cut to the representation of $\times$:

[R-Cut.1] $(z \lhd ().m)[()/z] = m$ and $(z \lhd (x, y).t)[m/z] = t[\pi_0(m)/x][\pi_1(m)/y]$;

[R-Cut.2] $()[t/x] = ()$ and $(m, n)[t/x] = (m[t/x], n[t/x])$.

Remark 2.2.10. *We have added ability to form an explicit projection $z : A \times B \vdash z \lhd (x, y).x : A$. The linearity of projection does not seem to be provable, and hence the linearity of projections must be assumed*[4]*.*

We call this system Cartesian differential type theory. A *Cartesian* differential specification is just a differential specification but where the notion of theorem is expanded as above; likewise a *Cartesian* differential theory is a Cartesian differential specification that is closed to provable equality.

We have the following structural lemma for these terms.

Lemma 2.2.11. *In Cartesian differential type theory we have*

1. $z \lhd (x, y).m = m[\pi_0(z)/x, \pi_1(z)/y]$ *and* $z \lhd ().m = m$*. Thus* $(t, s) \lhd (x, y).m = m[t/x][s/y]$.
2. $(x, y) \lhd (x, y).m = m$ *and* $() \lhd ().m = m$.
3. $z \lhd (x, y).m[(x, y)/z] = m$ *and* $z \lhd ().m[()/z] = m$.

[4]JS Lemay pointed out that for a Cartesian left additive category, $D[f] := 0$ satisfies all the axioms except **CDC.3**. It could still however be the case that $D[1] = \pi_0$ is enough to force the rest of **CDC.3**, but neither a proof nor counter example is currently known. If all of **CDC.3** follows from $D[1] = \pi_0$, then the requirement postulated here is redundant.

4. *$\pi_0(m) = (z \lhd (x, y).x)[m/z]$ and $\pi_1(m) = (z \lhd (x, y).y)[m/z]$.*

5. *The representation moves in and out of all term formations. That is:*

   - $z \lhd (x, y).f(t_1, \ldots, t_n) = f(z \lhd (x, y).t_1, \ldots, z \lhd (x, y).t_n)$;
   - $z \lhd (x, y).(m, n) = (z \lhd (x, y).m, z \lhd (x, y).n)$ *and* $z \lhd (x, y).() = ()$;
   - $z \lhd (x, y).(m + n) = (z \lhd (x, y).m) + (z \lhd (x, y).n)$ *and* $z \lhd (x, y).0 = 0$;
   - $z \lhd (x, y).\frac{\partial m}{\partial r}(a) \cdot v = \frac{\partial z \lhd (x,y).m}{\partial r}(a) \cdot v$ *whenever* $x, y \notin \mathsf{fv}(a, v)$.

6. *Equality is a congruence on terms, and moreover, $(m, n) = (m', n')$ if and only if $m = m'$ and $n = n'$.*

7. *If $\Gamma, z : A \times B, \Gamma' \vdash m : C$ then there is a term m' such that $m = z \lhd (x, y).m'$.*

Proof.

1. $z \lhd (x, y).m = (z \lhd (x, y).m)[z/z] = m[\pi_0(z)/x][\pi_1(z)/y]$. For $z \lhd ().m = m$, note that the m on the right hand side is in the weakened context containing $z : 1$.

2. We do the case for binary pairs, as the case for nullary pairs is similar. The case for binary pairs follows immediately from the above:

$$(x, y) \lhd (x, y).m = m[x/x][y/y] = m$$

3. We do the case for binary products, as the case for nullary products is similar.

$$\begin{aligned}
&z \lhd (x, y).m[(x, y)/z] \\
&= m[(x, y)/z][\pi_0(z)/x, \pi_1(z)/y] \\
&= m[(\pi_0(z), \pi_1(z))/z] = m[z/z] \qquad \text{surj. pair} \\
&= m
\end{aligned}$$

4. $(z \lhd (x, y).x)[m/z] = x[\pi_0(m)/x][\pi_1(m)/y] = \pi_0(m)$ and similarly for $\pi_1(m)$.

5. This follows immediately from *1*.

6. For projection, suppose $m = m'$:

$$\pi_0(m) = (z \lhd (x, y).x)[m/z] = (z \lhd (x, y).x)[m'/z] = \pi_0(m')$$

   Similarly $\pi_1(m) = \pi_1(m')$.

   For the empty tuple, there is nothing to do as $m : 1$ implies $m = ()$. For a nonempty tuple, suppose $m = m'$ and $n = n'$:

$$(m, n) = (x, y)[m/x, n/y] = (x, y)[m'/x, n'/y] = (m', n')$$

   Finally assume $t = t'$ and $m = m'$ then

$$t \lhd (x, y).m = z \lhd (x, y).m[t/z] = z \lhd (x, y).m'[t'/z] = t' \lhd (x, y).m'$$

   Moreover, if $(m, n) = (m', n')$ then $m = \pi_0(m, n) = \pi_0(m', n') = m'$ and similarly $n = n'$.

7. We sketch the proof. If $\Gamma, z : A \times B, \Gamma' \vdash m : C$, then $z \in \mathsf{fv}(m)$. In particular, m cannot take the form $\frac{\partial m'}{\partial z}(a) \cdot v$, $w = (z, r).m'$, nor $n[m'/z]$. Thus we can apply part *3*, to pull the $z \lhd (x, y)$. to the outside of the term, giving the term m' as the leftover part of the term.

□

Corollary 2.2.12. *There is an equivalence of proofs*

$$\Gamma, x : A, y : B, \Gamma' \vdash m : C \qquad \textit{and} \qquad \Gamma, z : A \times B, \Gamma' \vdash m' : C$$

Proof. This is points *2* and *3* of lemma 2.2.11. □

For the interested reader, the above corollary can be stated categorically by saying that $\mathscr{M}_\times(\Sigma)$ under the cut and representation equalities form a representable multicategory in the sense of (**journal:hermida-representable**).

One might think that an immediate consequence of point *6* of lemma 2.2.11 is that the theorems of a Cartesian differential specification are completely generated by the tuple closure of the theorems of the underlying differential specification; i.e. every theorem is of the form $(t_1, \cdots, t_n) = (t'_1, \cdots, t'_n)$ where each $t_i = t'_i$ is provable in the operadic differential theory. However, this may not be so, as the equation for linearity of projection is believed to have no proof in the operadic theory.

However, what is true is:

Remark 2.2.13. *Any differential theory $\mathcal{T}$ generates a Cartesian differential theory, by closing the theorems with respect to derivable equality.*

We have added some new term formation rules and equalities, so we must extend the interpretation and again show that soundness holds.

To extend the interpretation on types

$$[\![1]\!] := 1 \qquad \text{and} \qquad [\![A \times B]\!] := [\![A]\!] \times [\![B]\!]$$

and on contexts

$$[\![\Gamma, z : 1]\!] := [\![\Gamma]\!] \times 1 \qquad \text{and} \qquad [\![\Gamma, z : A \times B]\!] := [\![\Gamma]\!] \times ([\![A]\!] \times [\![B]\!])$$

and finally on terms:

Representation:

It is important to note that the isomorphisms used in the following defintions are linear maps.

- $[\![\Gamma, z : 1, \Gamma' \vdash z \lhd ().m : C]\!] := [\![\Gamma, z : 1, \Gamma']\!] \xrightarrow[i_1]{\simeq} [\![\Gamma, \Gamma']\!] \xrightarrow{[\![\Gamma, \Gamma' \vdash m : C]\!]} [\![C]\!]$
- $[\![\Gamma, z : A \times B, \Gamma' \vdash z \lhd (x, y).m : C]\!]$ is

$$[\![\Gamma, z : A \times B, \Gamma']\!] \xrightarrow[i_2]{\simeq} [\![\Gamma, x : A, y : B, \Gamma']\!] \xrightarrow{[\![\Gamma, x : A, y : B, \Gamma' \vdash m : C]\!]} [\![C]\!]$$

Tuple:

$$[\![\Gamma \vdash () : 1]\!] := [\![\Gamma]\!] \xrightarrow{!} 1 \qquad [\![\Gamma \vdash (m, n) : A \times B]\!] := [\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash m]\!], [\![\Gamma \vdash n]\!] \rangle} [\![A \times B]\!]$$

Projection terms:

$$[\![\Gamma \vdash \pi_0(m)]\!] := [\![\Gamma]\!] \xrightarrow{[\![m]\!]} [\![A \times B]\!] \xrightarrow{\pi_0} [\![A]\!] \qquad [\![\Gamma \vdash \pi_1(m)]\!] := [\![\Gamma]\!] \xrightarrow{[\![m]\!]} [\![A \times B]\!] \xrightarrow{\pi_1} [\![B]\!]$$

Theorem 2.2.14. *Let $\mathscr{T} = (\Sigma_0, \Sigma_1, E)$ be any Cartesian differential specification. If every equality $t_1 =_\Gamma t_2 : A \in E$ has $[\![\Gamma \vdash t_1 : A]\!] = [\![\Gamma \vdash t_2 : A]\!]$, then every theorem of $\mathscr{T}$ holds under interpretation. That is the interpretation is sound.*

The proof follows the same pattern as 2.2.9: we prove that every equality provable in $(\Sigma_0, \Sigma_1, \emptyset)$ holds under interpretation.

Proof. The projection computation equalities, surjective pairing, **R-Cut.1,2**, and the congruence that when $[\![m]\!] = [\![n]\!]$ we have $[\![z \lhd (x, y).m]\!] = [\![z \lhd (x, y).n]\!]$ are standard. This leaves the linearity of the projections.

First, note that the isomorphism $i_2 : [\![z : A \times B]\!] \longrightarrow [\![x : A, y : B]\!]$ is $1_{[\![A \times B]\!]}$.

$$\begin{aligned}
&D[[\![\Gamma, z : A \times B \vdash z \lhd (x, y).x]\!]] = D[\pi_1 [\![z : A \times B \vdash z \lhd (x, y).x]\!]] \\
&= (\pi_1 \times \pi_1) D[[\![z : A \times B \vdash z \lhd (x, y).x]\!]] = (\pi_1 \times \pi_1) D[[\![x : A, y : B \vdash x]\!]] \\
&= (\pi_1 \times \pi_1)(\pi_0 \times \pi_0) D[[\![x \vdash x]\!]] = (\pi_1 \times \pi_1)(\pi_0 \times \pi_0)\pi_0
\end{aligned}$$

A similar calculation shows $D[[\![\Gamma, z : A \times B \vdash z \lhd (x, y).y]\!]] = (\pi_1 \times \pi_1)(\pi_0 \times \pi_0)\pi_1$.

This implies

$$\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \frac{\partial z \lhd (x, y).x}{\partial z}(a) \cdot v\right]\!\!\right] = \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle D[[\![\Gamma, z \vdash z \lhd (x, y).x]\!]] \\
&= \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle (\pi_1 \times \pi_1)(\pi_0 \times \pi_0)\pi_0 \\
&= [\![\Gamma \vdash v : A]\!]\pi_0 = [\![\Gamma \vdash \pi_0(v)]\!]
\end{aligned}$$

And similarly,

$$\left[\!\!\left[\Gamma \vdash \frac{\partial z \lhd (x, y).y}{\partial z}(a) \cdot v\right]\!\!\right] = [\![\pi_1(v)]\!]$$

□

2.2.3 Differential categories-type correspondence

In the previous two subsections of section 2.2, we defined differential and Cartesian differential type theories, and showed that they admit a sound interpretation into a Cartesian differential category. In this subsection, we define a category of differential theories, and show it is equivalent to the category of Cartesian differential categories.

The left adjoint in the equivalence is the classifying category functor. The classifying category functor assigns to each theory a Cartesian differential category. The right adjoint is the theory functor, and it assigns to a Cartesian differential category, a differential theory. That the unit of the adjunction is an isomorphism means that the theory of the classifying category is equivalent to the starting theory, which means that one can prove a theorem by proving it in all models hence the models are complete. That the counit of the adjunction is an isomophism says that every Cartesian differential category arises as the classifying category of some differential theory, hence the theories are fully expressive.

In (Blute, Cockett, and Seely, 2009), the type theory of a Cartesian differential category was defined directly on top of the type theory for a Cartesian representable multicategory. In this section we considered starting with a simpler equational type theory with a differentiation, and defined the extension to representable products. Lemma 2.2.7 gives their **[Dt.5]**, and lemma 2.2.15 gives their **[Dt.3]** and **[Dt.4]**; thus, the type theory presented here proves every theorem in the type theory of (Blute, Cockett, and Seely, 2009). As the classifying category of their differential theory is a Cartesian differential category, the soundness theorem says that every theorem in a Cartesian differential theory as presented here holds in their theory. This gives an equivalence between the two ways of presenting a Cartesian differential category.

The classifying category $\underline{C}[\mathscr{T}]$ of a Cartesian differential theory $\mathscr{T}$ is defined as:

Obj: Types of $\mathscr{T}$

Arr: A map $A \xrightarrow{m} B$ is an equivalence class of terms of the form $p : A \vdash m : B$ under the equivalence relation given by provable equality, and $p : A$ is a singleton context.

Id: $p : A \vdash p : A$

Comp: $(p \vdash m : B)(q : B \vdash n : C) := p \vdash n[m/q] : C.$

Diff:

$$\frac{p : A \vdash m : B}{D[m] := va : A \times A \vdash va \triangleleft (v,a).\frac{\partial m}{\partial p}(a) \cdot v : B}$$

The following is what is needed to prove that the classifying category of a Cartesian differential theory is a Cartesian differential category.

Lemma 2.2.15. *In Cartesian differential type theory we have:*

1. *If* $\Gamma, z : 1, \Gamma' \vdash m : B$, *then* $\frac{\partial m}{\partial z}(a) \cdot v = 0$;
2. $\pi_0(\frac{\partial m}{\partial r}(a) \cdot v) = \frac{\partial \pi_0(m)}{\partial r}(a) \cdot v$ *and* $\pi_1(\frac{\partial m}{\partial r}(a) \cdot v) = \frac{\partial \pi_1(m)}{\partial r}(a) \cdot v$;
3. $\frac{\partial ()}{\partial r}(a) \cdot v = 0 = ()$ *and* $\frac{\partial (m,n)}{\partial r}(a) \cdot v = (\frac{\partial m}{\partial r}(a) \cdot v, \frac{\partial n}{\partial r}(a) \cdot v)$;
4. *Suppose* $\Gamma, z : A \times B, \Gamma' \vdash m' : C$. *Then* $m' = z \lhd (x, y).m$. *Then*
$$\frac{\partial m'}{\partial z}(a) \cdot v = \frac{\partial m[\pi_1(a)/y]}{\partial x}(\pi_0(a)) \cdot \pi_0(v) + \frac{\partial m[\pi_0(a)/x]}{\partial y}(\pi_1(a)) \cdot \pi_1(v)$$
5. *Suppose* $\Gamma, z : A \times B, \Gamma' \vdash m' : C$, *so again* $m' = z \lhd (x, y).m$. *Then*
$$\frac{\partial m'}{\partial z}(a,b)\cdot(v,0) = \frac{\partial m[b/y]}{\partial x}(a)\cdot v \quad \textit{and} \quad \frac{\partial m'}{\partial z}(a,b)\cdot(0,w) = \frac{\partial m[a/x]}{\partial y}(b)\cdot w$$

Intuitively *1* says that the derivative of a term with respect to x that is constant in x, is 0.

Proof.

1. By term formation rules, $z \notin \mathsf{fv}(m)$ hence $\frac{\partial m}{\partial z}(a) \cdot v = 0$.
2. For $\pi_0(m)$:
$$\begin{aligned} &\frac{\partial \pi_0(m)}{\partial r}(a) \cdot v = \frac{\partial (z \lhd (x,y).x)[m/z]}{\partial r}(a) \cdot v \\ &= \frac{\partial z \lhd (x,y).x}{\partial z}(m[a/r]) \cdot \left(\frac{\partial m}{\partial r}(a) \cdot v\right) \\ &= \pi_0\left(\frac{\partial m}{\partial r}(a) \cdot v\right) \end{aligned}$$

   For $\pi_1(m)$ is similar.

3. By surjective pairing if $m : 1$ then $m = ()$. Hence $() = 0$, and so

$$\frac{\partial ()}{\partial r}(a)\cdot v = \frac{\partial 0}{\partial r}(a)\cdot v = 0$$

For the tuple case:

$$\begin{aligned}\frac{\partial (m,n)}{\partial r}(a)\cdot v &= \left(\pi_0\left(\frac{\partial (m,n)}{\partial r}(a)\cdot v\right), \pi_1\left(\frac{\partial (m,n)}{\partial r}(a)\cdot v\right)\right)\\ &= \left(\frac{\partial \pi_0(m,n)}{\partial r}(a)\cdot v, \frac{\partial \pi_1(m,n)}{\partial r}(a)\cdot v\right) = \left(\frac{\partial m}{\partial r}(a)\cdot v, \frac{\partial n}{\partial r}(a)\cdot v\right)\end{aligned}$$

4. First note that

$$\frac{\partial \pi_0(z)}{\partial z}(a)\cdot v = \pi_0\left(\frac{\partial z}{\partial z}(a)\cdot v\right) = \pi_0(v)$$

and similarly that $\frac{\partial \pi_1(z)}{\partial z}(a)\cdot v = \pi_1(v)$. Then consider:

$$\begin{aligned}&\frac{\partial m'}{\partial z}(a)\cdot v = \frac{\partial z \lhd (x,y).m}{\partial z}(a)\cdot v = \frac{\partial m[\pi_0(z)/x][\pi_1(z)/y]}{\partial z}(a)\cdot v\\ &= \frac{\partial m[\pi_0(z)/x][\pi_1(z)[a/z]/y]}{\partial z}(a)\cdot v + \frac{\partial m[\pi_0(z)/x][a/z]}{\partial y}(\pi_1(a))\cdot\left(\frac{\partial \pi_1(z)}{\partial z}(a)\cdot v\right)\\ &= \frac{\partial m[\pi_1(a)/y][\pi_0(z)/x]}{\partial z}(a)\cdot v + \frac{\partial m[\pi_0(a)/x]}{\partial y}(\pi_1(a))\cdot \pi_1(v)\\ &= \frac{\partial m[\pi_1(a)/y][\pi_0(a)/x]}{\partial z}(a)\cdot v + \frac{\partial m[\pi_1(a)/y]}{\partial x}(\pi_0(a))\cdot \pi_0(v)\\ &\quad + \frac{\partial m[\pi_0(a)/x]}{\partial y}(\pi_1(a))\cdot \pi_1(v)\\ &= \frac{\partial m[\pi_1(a)/y]}{\partial x}(\pi_0(a))\cdot \pi_0(v) + \frac{\partial m[\pi_0(a)/x]}{\partial y}(\pi_1(a))\cdot \pi_1(v)\end{aligned}$$

5. Immediate from the above.

□

From lemma 2.2.15.(5) we have also that when $\Gamma, z : A\times B, \Gamma' \vdash m' : C$ so that $m' = z \lhd (x,y).m$ then

$$\frac{\partial m'}{\partial z}(a,b)\cdot(v,w) = \frac{\partial m[b/y]}{\partial x}(a)\cdot v + \frac{\partial m[a/x]}{\partial y}(b)\cdot w$$

Proposition 2.2.16. *For any Cartesian differential theory $\mathscr{T}$, $\underline{C}[\mathscr{T}]$ is a Cartesian differential category.*

A signature (Σ_0, Σ_1) such as above, is the same thing as a multigraph. The adjunction between multigraphs and representable cartesian multicategories puts a monad on the category of multigraphs, and the Kleisli category of this monad is the category of signatures and signature morphisms (see Jacobs, 1999). Thus a morphism of signatures $\Sigma \longrightarrow \Sigma'$ is an assignment of types $\Sigma_0 \xrightarrow{F} \Sigma_0'$ and for each function symbol $f : A_1, \ldots, A_n \longrightarrow B$ of Σ_1 a term $x_1 : F(A_1), \ldots, x_n : F(A_n) \vdash F(f) : F(B)$. The interpretation $[\![\_]\!]$ of Σ into $\underline{C}[\mathscr{T}]$ yields an extension of signature morphism to all terms, $\hat{F}$. A morphism of signatures $\Sigma \longrightarrow \Sigma'$ that underlie theories $\mathscr{T}, \mathscr{T}'$ is a morphism of theories when for every theorem $t_1 = t_2$ of $\mathscr{T}$, $\hat{F}(t_1) = \hat{F}(t_2)$ is a theorem of $\mathscr{T}'$. Now, since we are in a Kleisli category, composition and identity are from the Kleisli structure, and give the correct notion of theory equivalence: $\mathscr{T}$ is equivalent to $\mathscr{T}'$ when they are equivalent as representable cartesian multicategories. Using the equivalence between representable cartesian multicategories and categories with products, we get that $\mathscr{T}$ and $\mathscr{T}'$ are equivalent when $\underline{C}[\mathscr{T}] \simeq \underline{C}[\mathscr{T}']$ as categories with products, and that this holds when they prove the same theorems.

Thus the category $\mathsf{Diff\text{-}Th}$ whose objects are differential theories, and whose morphisms are theory morphisms has the property that differential theories $\mathscr{T}, \mathscr{T}'$ are equivalent when $\underline{C}[\mathscr{T}] \simeq \underline{C}[\mathscr{T}']$.

Let $\mathbb{X}$ be a Cartesian differential category with a chosen product functor. Define $\mathsf{Th}(\mathbb{X})$ to be the specification with Σ_0 the set of objects of $\mathbb{X}$, and Σ_1 to have a function symbol $f : A_1, \ldots, A_n \longrightarrow B$ for every map $(\cdots(A_1 \times A_2)\cdots \times A_n) \xrightarrow{f} B$ of $\mathbb{X}$. We have an interpretation of (Σ_0, Σ_1) in $\mathbb{X}$ where $[\![f]\!] := f$ and $[\![A]\!] := A$. Then $\mathsf{Th}(\mathbb{X})$ has as its sets of equations $E := \{t_1 = t_2 | t_1, t_2 \in \mathscr{M}(\mathsf{Th}(\mathbb{X})), [\![t_1]\!] = [\![t_2]\!]\}$.

We immediately have that $\mathsf{Th}(\mathbb{X})$ is a differential specification – however it is also a differential theory.

Proposition 2.2.17. *For any Cartesian differential category $\mathbb{X}$, $\mathsf{Th}(\mathbb{X})$ is a differential theory.*

Proof. The proof actually follows from soundness: theorem 2.2.14. By definition we have that $t_1 = t_2 \in E$ implies $[\![t_1]\!] = [\![t_2]\!]$. Then by corollary 2.2.14, every theorem of $\mathsf{Th}(\mathbb{X})$ holds under interpretation. But this means that for every theorem $m_1 = m_2$ of $\mathsf{Th}(\mathbb{X})$ we have $[\![m_1]\!] = [\![m_2]\!]$, but then by definition we have $m_1 = m_2 \in E$. Thus, $\mathsf{Th}(\mathbb{X})$ is a differential theory. $\square$

Now, suppose $\mathscr{T}$ is any differential theory, and $\mathbb{X}$ is a Cartesian differential category. Further suppose that there is a theory morphism $\mathscr{T} \xrightarrow{M} \mathsf{Th}(\mathbb{X})$. This means, that for each symbol of $\mathscr{T}$ there is a map of $\mathbb{X}$. This then gives an interpretation of $\mathscr{T}$ into $\mathbb{X}$. Since M is a theory morphism, for each theorem $t_1 = t_2$ of $\mathscr{T}$, we have $\hat{M}(t_1) = \hat{M}(t_2)$, but by definition of $\hat{M}$ this just means $[\![t_1]\!] = [\![t_2]\!]$. Then by corollary 2.2.14, every theorem is satisfied by the interpretation. Now, the interpretation $[\![\_]\!]$ restricted to the classifying category gives a functor

$$\underline{C}[\mathscr{T}] \xrightarrow{[\![]\!]} \mathbb{X}$$

This is a functor by definition $[\![(p : A \vdash m : B)(q : B \vdash n : C)]\!] = [\![p : A \vdash n[m/q] : C]\!] = [\![m]\!][\![n]\!]$. This is well defined by soundness: for each theorem $t_1 = t_2$ we have $[\![t_1]\!] = [\![t_2]\!]$. We provide the calculation that the derivative is preserved on the nose:

$$\begin{aligned}
&[\![D[p : A \vdash m : B]]\!] \\
&= \left[\!\!\left[(v,a) \vdash \frac{\partial m}{\partial p}(a) \cdot v : B\right]\!\!\right] \\
&= \big\langle \big\langle 0, [\![(v,a) \vdash v]\!]\big\rangle, \big\langle 1, [\![(v,a) \vdash a]\!]\big\rangle\big\rangle D[[\![((v,a),p) \vdash m]\!]] \\
&= \big\langle \big\langle 0, \pi_0[\![v \vdash v]\!]\big\rangle, \big\langle 1, \pi_1[\![a \vdash a]\!]\big\rangle\big\rangle D[\pi_1[\![p \vdash m]\!]] \\
&= \big\langle \big\langle 0, \pi_0[\![v \vdash v]\!]\big\rangle, \big\langle 1, \pi_1[\![a \vdash a]\!]\big\rangle\big\rangle (\pi_1 \times \pi_1) D[[\![p \vdash m]\!]] \\
&= \big\langle \pi_0[\![v \vdash v]\!], \pi_1[\![a \vdash a]\!]\big\rangle D[[\![p \vdash m]\!]] \\
&= D[[\![p \vdash m]\!]]
\end{aligned}$$

Completeness says that if an equality is provable in all models, then it is provable in the theory. $\underline{C}[\mathscr{T}]$ is a model whose theory $\mathsf{Th}(\underline{C}[\mathscr{T}])$ is equivalent to $\mathscr{T}$; in other words a theorem is provable in $\mathscr{T}$ iff it holds under interpretation into $\underline{C}[\mathscr{T}]$ – giving completeness. A direct proof of this is straightforward.

Theorem 2.2.18.

1. *Soundness: There is an adjunction:*

$$\begin{tikzcd} \textit{Diff-Th} \arrow[r, bend left, "\underline{C}"] \arrow[r, phantom, "\perp"] & \textit{CDC} \arrow[l, bend left, "\mathsf{Th}"] \end{tikzcd}$$

2. *Completeness: The unit of the adjunction has* $\mathscr{T} \simeq \mathsf{Th}(\underline{C}[\mathscr{T}])$.

3. *Expressivenes: The counit of the adjunction has* $\underline{C}[\mathsf{Th}(\mathbb{X})] \simeq \mathbb{X}$.

Proof.

1. Use the characterization of adjunctions by the universal property of the unit. We have object construction $\underline{C}$ and a functor Th for which

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\eta"] \arrow[dr, "M"'] & \mathsf{Th}(\underline{C}[\mathscr{T}]) \arrow[d, "\mathsf{Th}(\llbracket\_\rrbracket)"] \\ & \mathsf{Th}(\mathbb{X}) \end{tikzcd}$$

   Supposing M is a theory morphism, we get the functor $\underline{C}[\mathscr{T}] \xrightarrow{\llbracket\_\rrbracket} \mathbb{X}$ by simply restricting the interpretation to sequents with one type on the left. η is the map which assigns the symbol $f : A_1,\dots,A_n \longrightarrow B$ the term $x_1 : A_1,\dots,x_n : A_n \vdash f(x_1,\dots,x_n) : B$. The triangle clearly commutes. Uniqueness is immediate – the construction of $\llbracket\_\rrbracket$ is forced by the requirements for functors to strictly preserve products, sums, and differentials.

2. To show that $\mathscr{T} \simeq \mathsf{Th}(\underline{C}[\mathscr{T}])$ we must show that $\underline{C}[\mathscr{T}] \simeq \underline{C}[\mathsf{Th}(\underline{C}[\mathscr{T}])]$. When we form $\mathsf{Th}(\underline{C}[\mathscr{T}])$, we obtain a new function symbol $\underline{t}$ for each formable term t of $\mathscr{T}$. If t was formed with variables $x_1,\dots,x_n$, then $\mathsf{Th}(\underline{C}[\mathscr{T}])$ will come with an equation $\underline{t}(x_1,\dots,x_n) = t$. In particular, any equation of $\mathscr{T}$ holds in $\mathsf{Th}(\underline{C}[\mathscr{T}])$ thus there is a fully-faithful functor $\underline{C}[\mathscr{T}] \longrightarrow \underline{C}[\mathsf{Th}(\underline{C}[\mathscr{T}])]$.

   When we form $\underline{C}[\mathscr{T}]$ we explicitly form products of objects, and when we take $\mathsf{Th}(\underline{C}[\mathscr{T}])$ we now have named atomic product types in Σ_0. We write them as $\underline{A \times B}$. There are equations that express that this is the product of

A, B, so that $\underline{A \times B} \simeq A \times B$, and hence the functor $\underline{C}[\mathscr{T}] \longrightarrow \underline{C}[\mathsf{Th}(\underline{C}[\mathscr{T}])]$ is essentially surjective on objects.

3. The counit of the adjunction $\underline{C}[\mathsf{Th}(\mathbb{X})] \xrightarrow{\epsilon} \mathbb{X}$ is just the restriction of the the interpretation of $\mathsf{Th}(\mathbb{X})$ into $\mathbb{X}$. But the interpretation of $\mathsf{Th}(\mathbb{X})$ into $\mathbb{X}$ sends f to $[\![f]\!] = f$ and A to $[\![A]\!] = A$. Thus it is full and faithful. Moreover, it is actually an isomorphism on objects: $\mathsf{Th}(\mathbb{X})$ has a type for each object of $\mathbb{X}$, and this gives the equivalence $\underline{C}[\mathsf{Th}(\mathbb{X})] \simeq \mathbb{X}$.

$\square$

Chapter 3

higher-order differential calculus

In this chapter, we extend the results of chapter 2 to a higher-order setting; [1] i.e., a setting with function spaces, also called a Cartesian closed setting.

Classically, having differentiation in a setting where the domain of a function is the vector space of smooth functions between $\mathscr{R}$-vector spaces is both important and non-trivial to obtain. Taking the derivative of functions whose type is for example $\mathscr{R}^{\mathscr{R}} \longrightarrow \mathscr{R}$ is at the heart of the calculus of variations. However, neither of the categories of smooth maps between Banach spaces nor Fréchet spaces is Cartesian closed. The failure of a category of smooth functions between vector spaces to be Cartesian closed led to the development of convenient vector spaces (**book:kriegl-frolicher**; Kriegl and Michor, 1997) which are Cartesian closed, and as we saw in chapter 2 are a Cartesian differential category.

In this chapter we extend our abstract framework for differentiation to handle Cartesian closed categories of smooth maps. We also extend our differential calculus with rules and equalities that allow for computing the derivative of higher-order functions. To do this, we combine the differential calculus with the λ-calculus which is the established theory of functions. We mirror the structure of chapter 2, and we provide a categorical semantics similar to the equivalence of chapter 2.

[1]Sometimes in differential geometry, the word 'higher-order' is used to refer to differential equations with higher-derivatives. At the present we are using higher-order in a computer science sense, which means a function that can operate on and return functions.

3.1 The categorical setting

In this section we describe the abstract setting for differentiation with function spaces. There are actually two notions of "function space" in a category: intensional and extensional function spaces. Extensional function spaces have the additional property that functions that are equal on all inputs are equal functions. Extensional function spaces are found in a Cartesian closed category, and intensional function spaces are found in a λ_β-category.

On the side of syntax, extensional theories are those that have the η-conversion rule for the λ-calculus. For categorical models of the ordinary λ-calculus, we will make use of a notion called a λ_β-category. A slightly weaker structure than λ_β-categories was investigated in (Hayashi, 1985) (here the surjective pairing axiom is also weakened, so that one does not have genuine products), and a setting very similar to λ_β categories was described in (Martini, 1992).

We first describe λ_β-categories, and then we add addition and then differentiation.

Definition 3.1.1. *Let $\mathbb{X}$ have products and a terminal object. $\mathbb{X}$ is a λ_β-category when there is a semifunctor*

$$\mathbb{X}^{op} \times \mathbb{X} \xrightarrow{[\_,\_]} \mathbb{X}$$

and maps

$$\mathbb{X}(A \times B, C) \xrightarrow{\lambda} \mathbb{X}(A, [B, C]) \xrightarrow{\psi} \mathbb{X}(A \times B, C)$$

such that $\lambda\psi = 1$ (so that λ is a section with ψ a retraction) and that ψ, λ are both natural in A.

Naturality in A for ψ means that $(g \times 1)\psi(h) = \psi(gh)$. Naturality in A for λ means that $g\lambda(f) = \lambda((g \times 1)f)$. For ψ in particular we have $\psi(g) = (g \times 1)\psi(1)$, and if we define $\mathsf{ev} := \psi(1)$ we have

$$f = \psi(\lambda(f)) = (\lambda(f) \times 1)\psi(1) = (\lambda(f) \times 1)\mathsf{ev}$$

Proposition 3.1.2. *Let $\mathbb{X}$ have products.*

1. *$\mathbb{X}$ is a λ_β-category if and only if for each B, C there is an object $[B, C]$ and an arrow $[B, C] \times B \xrightarrow{\mathsf{ev}} C$ such that for any $A \times B \xrightarrow{f} C$ there is a map $A \xrightarrow{\lambda(f)} [B, C]$ such that $(\lambda(f) \times 1)\mathsf{ev} = f$ and $g\lambda(f) = \lambda((g \times 1)f)$.*

2. *In a λ_β-category both ψ, λ are natural in C.*

3. *When $\mathbb{X}$ is a λ_β-category, then the following are equivalent:*

   a) *$\psi\lambda = 1$;*

   b) *$\mathbb{X}$ is cartesian closed;*

   c) *$[\_, \_]$ is a functor;*

   d) *$\lambda(\mathsf{ev}) = 1$.*

Proof.

1. The forward direction of the proof was described preceding this lemma. For the converse:

$$\frac{\dfrac{C \xrightarrow{f} C'}{[B, C] \times B \xrightarrow{\mathsf{ev}} C \xrightarrow{f} C'}}{[B, C] \xrightarrow[{[B, f] := \lambda(\mathsf{ev} f)}]{} [B, C']} \qquad \frac{\dfrac{B' \xrightarrow{g} B}{[B, C] \times B' \xrightarrow{1 \times g} [B, C] \times B \xrightarrow{\mathsf{ev}} C}}{[B, C] \xrightarrow[{[g, C] := \lambda((1 \times g)\mathsf{ev})}]{} [B', C]}$$

   These both preserve composition, but the identity is sent to an idempotent. The map $\lambda(\_)$ is provided by assumption, and the naturality in A is $g\lambda(f) = \lambda((g \times 1)f)$. Define $\psi(g) := (g \times 1)\mathsf{ev}$. Then naturality is just

$$(g \times 1)\psi(h) = (g \times 1)(h \times 1)\mathsf{ev} = \psi(gh)$$

2. For the naturality of λ in C, we must prove $\lambda(gh) = \lambda(g)[B, h]$:

$$\lambda(g)\lambda(\mathsf{ev}h) = \lambda((\lambda(g)\ x 1)\mathsf{ev}h) = \lambda(gh)$$

   For the naturality of ψ in C, we must prove $\psi(g[B, h]) = \psi(g)h$:

$$\psi(g[B, h]) = (g\lambda(\mathsf{ev}h) \times 1)\mathsf{ev} = (g \times 1)\mathsf{ev}h = \psi(g)h$$

3. That a implies b and b implies c is immediate.

   That c implies d: Assume $[\_,\_]$ is a functor, so that it is a functor in both arguments, and hence $[B,1]=1$. But $[B,1]:=\lambda(\mathsf{ev})$, giving the result.

   That d implies a: Suppose $\lambda(\mathsf{ev})=1$. Consider:

$$\lambda(\psi(g))=\lambda((g\times 1)\mathsf{ev})=g\lambda(\mathsf{ev})=g.$$

□

In contrast with a λ_β-category, we will sometimes call a Cartesian closed category a **λ-category**.

Definition 3.1.3. *Let $\mathbb{X}$ be a Cartesian left additive category that is also a λ_β-category. It is called an **additive λ_β-category** when* ev *is additive in its first argument and $\lambda(f+g)=\lambda(f)+\lambda(g)$ and $\lambda(0)=0$. If $\mathbb{X}$ is additionally closed, then it is called an **additive λ-category** or **cartesian closed left additive category**.*

For the type theory correspondence, note that we will work up to chosen product and internal hom semifunctors.

Lemma 3.1.4. *In an additive λ-category the requirement that* ev *be additive in its first argument is redundant.*

Proof. In an additive λ-category, $\lambda(\_)$ has an inverse. Then consider the following calculation:

$$\begin{aligned}\langle m+n,t\rangle\mathsf{ev} &= \langle\lambda(\lambda^{-1}(m))+\lambda(\lambda^{-1}(n)),t\rangle\mathsf{ev}\\ &= \langle 1,t\rangle(\lambda^{-1}(m)+\lambda^{-1}(n))=\langle m,t\rangle\mathsf{ev}+\langle n,t\rangle\mathsf{ev}\end{aligned}$$

The case for 0 is similar. □

However, this condition seems to be required for additive λ_β-categories.

Suppose $\mathbb{X}$ is Cartesian closed and Cartesian differential. Consider,

$$\cfrac{\cfrac{\cfrac{A \times B \xrightarrow{f} C}{A \xrightarrow{\lambda(f)} [B,C]}}{A \times A \xrightarrow{D[\lambda(f)]} [B,C]}}{(A \times A) \times B \xrightarrow[\lambda^{-1}(D[\lambda(f)])]{} C}$$

The resulting map $\lambda^{-1}(D[\lambda(f)])$ seems like a candidate for partial derivative $D_A[f]$. In fact, we require that it is in the following definition.

Definition 3.1.5. *Let $\mathbb{X}$ be a λ_β-category that is also a Cartesian differential category. $\mathbb{X}$ is a **differential λ_β-category** when* ev *is linear in its first argument and $\lambda(D_A[f]) = D[\lambda(f)]$ for any $A \times B \xrightarrow{f} C$. When $\mathbb{X}$ is additionally closed, it is a **differential λ-category** or **cartesian closed differential category**.*

To spell out these conditions in more detail we have:

$$\langle \pi_0 \times 0, \pi_1 \times 1 \rangle D[\mathsf{ev}] = (\pi_0 \times 1)\mathsf{ev} \qquad D[\lambda(f)] = \lambda(\langle \pi_0 \times 0, \pi_1 \times 1 \rangle D[f])$$

Example 3.1.6. *The category of convenient vector spaces is an abstract coKleisli category in the sense of (***journal:BCS-Storage***). For any convenient vector space V, there is a free convenient vector space $!V$. In (Kriegl and Michor, 1997), the following adjunction was provided (theorem 23.6):*

$$\mathsf{Smooth}(V, W) \simeq \mathsf{Lin}(!V, W)$$

The category of convenient vector spaces and linear maps has products, a tensor product, is monoidal closed with respect to the tensor product, and the coKleisli category of this comonad is the category of convenient vector spaces and smooth maps. Thus the category of convenient vector spaces and smooth maps is Cartesian closed. It is not hard to show that the required coherence holds, and we will give a direct argument for this coherence in subsection 6.7.5.

Lemma 3.1.7. *In a differential λ-category, the requirement that* ev *be linear in its first argument is redundant.*

$$\frac{\Gamma, x : A, \Gamma' \vdash m : B}{\Gamma, \Gamma' \vdash \lambda x.m : A \longrightarrow B}\ \text{Abs} \qquad \frac{\Gamma \vdash m : A \longrightarrow B \quad \Gamma \vdash n : A}{\Gamma \vdash m\, n : B}\ \text{App}$$

Table 3.1: Term formation rules for differential λ type theory

Proof. Recall that $\lambda^{-1}(1) = \mathsf{ev}$. Then,

$$\begin{aligned}(\pi_0 \times 1)\mathsf{ev} &= (D[1] \times 1)\mathsf{ev} = (D[\lambda(\mathsf{ev})] \times 1)\mathsf{ev} \\ &= (\lambda(\langle \pi_0 \times 0, \pi_1 \times 1\rangle D[\mathsf{ev}]) \times 1)\mathsf{ev} = \langle \pi_0 \times 0, \pi_1 \times 1\rangle D[\mathsf{ev}]\end{aligned}$$

as required. □

The extra coherence, however, still seems to be necessary for differential λ_β-categories.

3.2 higher-order differential type theory

In section 2.2, we introduced type theory for the differential calculus. In this section, we extend this type theory with higher-order terms, also called λ-terms.

3.2.1 Simple higher-order differential type theory

A **signature for a differential λ-theory** is a pair $\Sigma = (\Sigma_0, \Sigma_1)$. Σ_0 is a collection of atomic types, as before except that we have types:

$$\mathsf{type} := A \in \Sigma_0 \mid \mathsf{ty} \longrightarrow \mathsf{ty}$$

The set of differential λ-terms is defined by the inference rules in table 2.1, together with the two additional rules in table 3.1. The term $\lambda x.m$ is a function term; it may be thought of as an unnamed function that uses the variable x. The term $m\,n$ is application. From the types, $m : A \longrightarrow B$ and $n : A$, writing $m\,n$ is thought of as evaluation $m(n)$.

The cut identities are expanded to include the following.

Definition 3.2.1 (Cut/Composition Equalities)**.**

[CD-Sub.1] $(\lambda x.m)[n/y] = \lambda x.(m[n/y])$*;*

[CD-Sub.2] $(m\,n)[t/x] = (m[t/x])(n[t/x])$.

The term $(\lambda x.m)$ represents an unnamed function that uses x, and $(\lambda x.m)\,n$ is supposed to be thought of as evaluating that unnamed function at the argument n. The β equality makes this precise, because it says that $(\lambda x.m)\,n$ is $m[n/x]$ which is m with every x substituted by n. For example

$$(\lambda x.\sin^2(x) + x\,y)\,\pi =_\beta \sin^2(\pi) + \pi \cdot y$$

Definition 3.2.2 (λ-calculus identities)**.** *The λ-calculus equalities:*

***α*:** $\lambda x.m = \lambda y.(m[y/x])$.

***β*:** $(\lambda x.m)n = m[n/x]$.

Definition 3.2.3 (Differentiation of higher-order functions)**.** *The following equalities specify the interaction between differentiation and higher-order terms:*

[CCDT.1] $\lambda y.\frac{\partial m}{\partial x}(a)\cdot v = \frac{\partial \lambda y.m}{\partial x}(a)\cdot v$ *note that* $y \notin \mathsf{fv}(a, v)$.

[CCDT.2] $\frac{\partial y\,z}{\partial y}(a)\cdot v = v\,z$ *when* $y \neq z$.

The notion of **theorem** or **derivable equality** is expanded to include the ξ rule:

$$\frac{\Gamma, x : A \vdash m = n : B}{\Gamma \vdash \lambda x.m = \lambda x.n : A \longrightarrow B}\ \xi$$

We may choose to include or ignore the $\boldsymbol{\eta}$-rule as a theorem:

$$\frac{\Gamma \vdash m : A \longrightarrow B}{\Gamma \vdash m = \lambda x.m\,x : A \longrightarrow B}\ \boldsymbol{\eta}$$

We call a type theory with the $\boldsymbol{\eta}$-rule **extensional**.

The structual lemma holds for this type theory.

Lemma 3.2.4. *The equations for cut form a terminating and locally confluent rewrite system whose normal forms are cut free.*

sketch. The proof of this lemma is the same as lemma 2.2.3. □

The equality congruence lemma also holds.

Lemma 3.2.5. *Equality is a congruence on terms.*

Proof. We prove that equality is closed under term formation. There are only two rules. That $m = n$ implies $\lambda x.m = \lambda x.n$ is the ξ rule. For application

$$m\,n = (x\,y)[m/x, n/y] = (x\,y)[m'/x, n'/y] = m'\,n'$$

$\square$

In (Ehrhard and Regnier, 2003) the chain rule was dropped, and instead the $\boldsymbol{D}$**[App]** equality was used:

$$\boldsymbol{D}[\mathbf{App}]: \qquad \frac{\partial\, m\, n}{\partial x}(a)\cdot v = \left(\frac{\partial m}{\partial x}(a)\cdot v\right) n[a/x] + \frac{\partial (m[a/x])\, z}{\partial z}(n[a/x])\cdot\left(\frac{\partial n}{\partial x}(a)\cdot v\right)$$

Note, that we are not *assuming* that $\boldsymbol{D}$**[App]** holds; it is provable in this setting.

We also have the following lemma about derivable equality:

Lemma 3.2.6. *In a differential type theory, the following equations always hold:*

1. *The η equality holds if and only if $m\,x = n\,x$ implies $m = n$.*
2. $\lambda x.(f+g) = (\lambda x.f) + (\lambda x.g)$ *and* $\lambda x.0 = 0$.
3. *When $x \notin \mathsf{fv}(m)$, $\frac{\partial\, x\, m}{\partial x}(a)\cdot v = v\, m$.*
4. $(m+n)t = m\,t + n\,t$ *and* $0\,t = 0$.
5. *Consider closed differential type theory, but without* ***[CCDT.2]****. Then the following are equivalent:*
   - $\lambda x.v\,x = \frac{\partial \lambda x.y\,x}{\partial y}(a)\cdot v$;
   - ***[CCDT.2]****;*
   - *For a fresh z, $D[\mathsf{App}]$ holds:* $\frac{\partial\, m\, n}{\partial x}(a)\cdot v = \left(\frac{\partial m}{\partial x}(a)\cdot v\right) n[a/x] + \frac{\partial (m[a/x])\, z}{\partial z}(n[a/x])\cdot \left(\frac{\partial n}{\partial x}(a)\cdot v\right)$.
6. *With the η-rule,* ***[CCDT.2]*** *is redundant.*

7. $\boldsymbol{D}[\boldsymbol{\beta}]$ *holds:* $\frac{\partial(\lambda x.m)z}{\partial z}(a)\cdot v = \frac{\partial m}{\partial x}(a)\cdot v$ *(where* $z \notin \mathsf{fv}(m,a,v)$*).*

Proof. Consider the following:

1. This is standard.

2.
$$\begin{aligned}\lambda x.(f+g) &= \lambda x.\left(\frac{\partial y}{\partial y}(a)\cdot(f+g)\right) = \frac{\partial \lambda x.y}{\partial y}(a)\cdot(f+g)\\ &= \frac{\partial \lambda x.y}{\partial y}(a)\cdot f + \frac{\partial \lambda x.y}{\partial y}(a)\cdot g = \lambda x.\frac{\partial y}{\partial y}(a)\cdot f + \lambda x.\frac{\partial y}{\partial y}(a)\cdot g\\ &= \lambda x.f + \lambda x.g\end{aligned}$$

   Similarly
$$\lambda x.0 = \lambda x.\frac{\partial y}{\partial y}(a)\cdot 0 = \frac{\partial \lambda x.y}{\partial y}(a)\cdot 0 = 0$$

3.
$$\frac{\partial\, y\, m}{\partial y}(a)\cdot v = \left(\frac{\partial\, y\, z}{\partial y}(a)\cdot v\right)[m/z] = (v\, z)[m/z] = v\, m$$

4.
$$(m+n)t = \frac{\partial\, x\, t}{\partial x}(a)\cdot(m+n) = \frac{\partial\, x\, t}{\partial x}(a)\cdot m + \frac{\partial\, x\, t}{\partial x}(a)\cdot n = m\, t + n\, t$$

   and
$$0\, t = \frac{\partial\, x\, t}{\partial x}(a)\cdot 0 = 0$$

5. That $\lambda z.v\, z = \frac{\partial \lambda z.y\, z}{\partial y}(a)\cdot v$ implies **[CCDT.2]**:
$$\frac{\partial \lambda z.y\, z}{\partial y}(a)\cdot v = \lambda z.\frac{\partial\, y\, z}{\partial y}(a)\cdot v = \lambda z.v\, z.$$

   That **[CCDT.2]** implies **D[App]**, the main step is lemma 2.2.7.4, followed by **[CCDT.2]**.
$$\begin{aligned}&\frac{\partial\, m\, n}{\partial z}(a)\cdot v = \frac{\partial (x\, y)[m/x, n/y]}{\partial z}(a)\cdot v\\ &= \frac{\partial\, m[a/z]y}{\partial y}(n[a/z])\cdot\left(\frac{\partial n}{\partial z}(a)\cdot v\right) + \frac{\partial\, x\, (n[a/z])}{\partial x}(m[a/z])\cdot\left(\frac{\partial m}{\partial z}(a)\cdot v\right)\\ &= \frac{\partial\, m[a/z]y}{\partial y}(n[a/z])\cdot\left(\frac{\partial n}{\partial z}(a)\cdot v\right) + \left(\frac{\partial m}{\partial z}(a)\cdot v\right)n[a/z]\end{aligned}$$

Finally, that **D[App]** implies $\lambda z.v\,z = \frac{\partial \lambda z.y\,z}{\partial y}(a)\cdot v$:

$$\begin{aligned}
&\frac{\partial \lambda z.y\,z}{\partial y}(a)\cdot v \\
&= \lambda z.\frac{\partial y\,z}{\partial y}(a)\cdot v \qquad \textbf{[CCDT.1]} \\
&= \lambda z.\left(\left(\frac{\partial y}{\partial y}(a)\cdot v\right)z + \frac{\partial a\,z_0}{\partial z_0}(z)\cdot\left(\frac{\partial z}{\partial y}(a)\cdot v\right)\right) \qquad \boldsymbol{D}[\mathsf{App}] \\
&= \lambda z.\left(v\,z + \frac{\partial a\,z_0}{\partial z_0}(z)\cdot 0\right) \qquad \textbf{[DT.4.2]} \\
&= \lambda z.v\,z
\end{aligned}$$

6. Suppose η holds, but not **[CCDT.2]**. We show **[CCDT.2]** holds.

$$\begin{aligned}
v\,z &= \left(\frac{\partial y}{\partial y}(a)\cdot v\right)z = \left(\frac{\partial \lambda x.y\,x}{\partial y}(a)\cdot v\right)z \\
&= \left(\lambda x.\frac{\partial y\,x}{\partial y}(a)\cdot v\right)z = \frac{\partial y\,z}{\partial y}(a)\cdot v
\end{aligned}$$

7. This is just the chain rule (**[DT.5]**):

$$\frac{\partial(\lambda x.m)z}{\partial z}(a)\cdot v = \frac{\partial m[z/x]}{\partial z}(a)\cdot v = \frac{\partial m}{\partial x}(a)\cdot\left(\frac{\partial z}{\partial z}(a)\cdot v\right) = \frac{\partial m}{\partial x}(a)\cdot v$$

$\square$

More surprisingly as long as we have α-conversion for differentiation, the chain rule follows from **D[App]**

Lemma 3.2.7. *In differential λ type theory, the equation*

$$\frac{\partial m}{\partial x}(a)\cdot v = \frac{\partial m[y/x]}{\partial y}(a)\cdot v$$

*together with **D[App]** implies **[DT.5]** (see section 2.2)*[2].

[2] If the differential λ-calculus of (Ehrhard and Regnier, 2003) is augmented with function symbols, an equality like α-conversion must be added to ensure the chain rule holds. It is not strictly an α-equality because their derivative does not bind x.

Proof. Consider,

$$
\begin{aligned}
&\frac{\partial f[g/x]}{\partial y}(a)\cdot v = \frac{\partial(\lambda x.f)g}{\partial y}(a)\cdot v \\
&= \left(\frac{\partial \lambda x.f}{\partial y}(a)\cdot v\right) g[a/y] + \frac{\partial(\lambda x.(f[a/y]))z}{\partial z}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right) \\
&= \left(\lambda x.\frac{\partial f}{\partial y}(a)\cdot v\right) g[a/y] + \frac{\partial f[a/y][z/x]}{\partial z}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right) \\
&= \frac{\partial f[g[a/y]/x]}{\partial y}(a)\cdot v + \frac{\partial f[a/y]}{\partial x}(g[a/y])\cdot\left(\frac{\partial g}{\partial y}(a)\cdot v\right)
\end{aligned}
$$

□

A **differential λ-specification** $\mathscr{T} = (\Sigma_0, \Sigma_1, E)$ is given by a signature and a set of equations E between differential λ-terms. As before, we extend the notion of theorem of a theory to include

$$\frac{\Gamma \vdash m = n : A}{m =_\Gamma n : A \in E}$$

A **differential λ-theory** is a differential λ-specification $\mathscr{T}$ such that the equations are closed to derivable equality. By **extensional specification** we mean that η-equality is a theorem. A theory is then extensional when $m = \lambda x.m\ x$ is in E.

The notion of interpretation structure is similar to before, except now we interpret into a differential λ_β-category. Given the assignment of atomic types, we extend the interpretation to all types by $[\![A \longrightarrow B]\!] := \big[[\![A]\!], [\![B]\!]\big]$. The interpretation then specifies an assignment of function symbols that is consistent with the types. The interpretation is extended to contexts in the same way as before. Finally, to extend to all terms, we only need to say how to extend for the two new term formation rules.

- $[\![\Gamma \vdash \lambda x.m : A \longrightarrow B]\!] := \lambda([\![\Gamma, x \vdash m : B]\!])$ is the map that makes the following commute.

$$
\begin{tikzcd}
\big[[\![A]\!], [\![B]\!]\big] \times [\![A]\!] \arrow[r, "\mathsf{ev}"] & B \\
[\![\Gamma]\!] \times [\![A]\!] \arrow[u, "\lambda([\![\Gamma, x:A \vdash m:B]\!])\times 1"] \arrow[ur, "[\![\Gamma, x:A \vdash m:B]\!]"']
\end{tikzcd}
$$

Note this map is guaranteed to exist because the interpretation is into a differential λ_β category. It is unique when the interpretation is into a differential λ-category (and hence models the η equality).

- $[\![\Gamma \vdash m\, n : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash m : A \to B]\!], [\![\Gamma \vdash n : A]\!]\rangle} \big[[\![A]\!], [\![B]\!]\big] \times [\![A]\!] \xrightarrow{\mathsf{ev}} [\![B]\!]$

Theorem 3.2.8. *Let $\mathcal{T} = (\Sigma_0, \Sigma_1, E)$ be any differential λ-specification. If every equality $t_1 =_\Gamma t_2 : A \in E$ has $[\![\Gamma \vdash t_1 : A]\!] = [\![\Gamma \vdash t_2 : A]\!]$, then every theorem of $\mathcal{T}$ holds under interpretation. If $\mathcal{T}$ is an extensional λ-specification, and $\mathbb{X}$ is a differential λ-category, then every theorem of $\mathcal{T}$ holds under interpretation. That is, the interpretation of differential λ-type theory into a differential λ_β-category is sound, and the interpretation of an extensional theory into a differential λ-category is sound.*

To make the calculations take less space, we will omit types and contexts if they are not relevant to the computation.

Proof. For **[CDC-Sub.1]** use lemma 2.2.8:

$$
\begin{aligned}
&[\![\Gamma \vdash \lambda x.(m[n/y])]\!] = \lambda([\![\Gamma, x \vdash m[n/y]]\!]) = \lambda(\langle 1_{[\![\Gamma]\!]\times[\![A]\!]}, [\![\Gamma, x \vdash n]\!]\rangle [\![\Gamma, x, y \vdash m]\!]) \\
&= \lambda(\langle 1, [\![\Gamma, x \vdash n]\!]\rangle c_2 [\![\Gamma, y, x \vdash m]\!]) = \lambda\big((\langle 1, [\![\Gamma \vdash n]\!]\rangle \times 1)[\![\Gamma, y, x \vdash m]\!]\big) \\
&= \langle 1, [\![\Gamma \vdash n]\!]\rangle \lambda([\![\Gamma, y \vdash \lambda x.m]\!]) = [\![\Gamma \vdash (\lambda x.m)[n/y]]\!]
\end{aligned}
$$

[CDC-Sub.2] is immediate.

For **[CCDT.1]** First consider the LHS:

$$
\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \lambda y.\frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right] = \lambda\left(\left[\!\!\left[\Gamma, y \vdash \frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right]\right) \\
&= \lambda\big(\langle\langle 0, [\![\Gamma, y \vdash v]\!]\rangle, \langle 1, [\![\Gamma, y \vdash a]\!]\rangle\rangle D[[\![\Gamma, y, x \vdash m]\!]]\big)
\end{aligned}
$$

On the RHS:

$$
\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \frac{\partial \lambda y.m}{\partial x}(a)\cdot v\right]\!\!\right] = \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle D[[\![\Gamma, x \vdash \lambda y.m]\!]] \\
&= \lambda\big((\langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle \times 1)\langle \pi_0 \times 0, \pi_1 \times 1\rangle D[[\![\Gamma, x, y \vdash m]\!]]\big)
\end{aligned}
$$

As $\lambda()$ is a function, it suffices to compare what is under the λ. Then,

$$
\begin{aligned}
&\langle\langle 0, [\![\Gamma, y \vdash v]\!]\rangle, \langle 1, [\![\Gamma, y \vdash a]\!]\rangle\rangle D[[\![\Gamma, y, x \vdash m]\!]] \\
&= \langle\langle 0, \pi_0[\![\Gamma \vdash v]\!]\rangle, \langle 1, \pi_0[\![\Gamma \vdash a]\!]\rangle\rangle(c_2 \times c_2) D[[\![\Gamma, x, y \vdash m]\!]] \\
&= \langle\langle\langle 0, \pi_0[\![\Gamma \vdash v]\!]\rangle, \pi_1\rangle, \langle\langle \pi_0, \pi_0[\![\Gamma \vdash a]\!]\rangle, \pi_1\rangle\rangle D[[\![\Gamma, x, y \vdash m]\!]] \\
&= \big(\langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle \times 1\big)\langle \pi_0 \times 0, \pi_1 \times 1\rangle D[[\![\Gamma, x, y \vdash m]\!]]
\end{aligned}
$$

as desired.

For **[CCDT.2]**: Note that $[\![\Gamma \vdash z]\!]$ is a projection when z is a variable, and hence is linear, and preserves 0.

$$
\begin{aligned}
&\left[\!\!\left[\Gamma \vdash \frac{\partial\, y\, z}{\partial\, y}(a) \cdot v\right]\!\!\right] = \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle D[[\![\Gamma, y \vdash y\, z]\!]] \\
&= \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle D[\langle [\![\Gamma, y \vdash y]\!], [\![\Gamma, y \vdash z]\!]\rangle \mathsf{ev}] \\
&= \langle\langle 0, [\![\Gamma \vdash v]\!]\rangle, \langle 1, [\![\Gamma \vdash a]\!]\rangle\rangle(\langle \pi_1, \pi_0[\![\Gamma \vdash z]\!]\rangle \times \langle \pi_1, \pi_0[\![\Gamma \vdash z]\!]\rangle) D[\mathsf{ev}] \\
&= \langle\langle [\![\Gamma \vdash v]\!], 0\rangle, \langle [\![\Gamma \vdash a]\!], [\![\Gamma \vdash z]\!]\rangle\rangle D[\mathsf{ev}] \\
&= \langle\langle [\![\Gamma \vdash v]\!], [\![\Gamma \vdash a]\!]\rangle, [\![\Gamma \vdash z]\!]\rangle \langle \pi_0 \times 0, \pi_1 \times 1\rangle D[\mathsf{ev}] \\
&= \langle\langle [\![\Gamma \vdash v]\!], [\![\Gamma \vdash a]\!]\rangle, [\![\Gamma \vdash z]\!]\rangle (\pi_0 \times 1)\mathsf{ev} \\
&= \langle [\![\Gamma \vdash v]\!], [\![\Gamma \vdash z]\!]\rangle \mathsf{ev} = [\![\Gamma \vdash v\, z]\!]
\end{aligned}
$$

That β and ξ hold are routine. □

3.2.2 higher-order differential categories-type theory correspondence

We do not need to develop the Cartesian differential λ-type theory because Cartesian differential λ-type theory is the union of the term and type formation rules as well as equations of section 2.2.2 with simple higher-order differential type theory.

For a Cartesian differential λ-theory $\mathcal{T}$, the classifying category $\underline{C}[\mathcal{T}]$ is defined in section 2.2.3. For a differential λ-theory, $\underline{C}[\mathcal{T}]$ is a λ_β-category, and the λ_β-structure is:

$$\frac{z : A \times B \vdash f : C}{x : A \vdash \lambda y.f[(x,y)/z] : B \longrightarrow C}\ \lambda(\_) \qquad \frac{}{f_x : (B \longrightarrow C) \times B \vdash f_x \lhd (f,x).f\ x : C}\ \mathsf{ev}$$

Lemma 3.2.9. *When $\mathscr{T}$ is a differential λ-theory, $\underline{C}[\mathscr{T}]$ is a λ_β-category.*

Proof. By proposition 3.1.2, it suffices to show that $(\lambda(f) \times 1)\mathsf{ev} = f$ and $g\lambda(f) = \lambda((g \times 1)f)$.

Suppose we have $z : A \times B \vdash f : C$. Then we have $\lambda(f) := x : A \vdash \lambda y.f[(x,y)/z] : B \longrightarrow C$, and $\lambda(f) \times 1 := z : A \times B \vdash (\lambda y.f[(\pi_0(z), y)/z], \pi_1(z))$. Thus using surjective pairing:

$$(\lambda(f) \times 1)\mathsf{ev} = z : A \times B \vdash (\lambda y.f[(\pi_0(z), y)/z])\pi_1(z) = f[(\pi_0(z), \pi_1(z))/z] = f[z/z] = f$$

Next, suppose $t : D \vdash g : A$ and $z : A \times B \vdash f : C$. Then consider the formation of $g\lambda(f)$:

$$\frac{t : D \vdash g : A \qquad \dfrac{z : A \times B \vdash f : C}{x : A \vdash \lambda y.f[(x,y)/z] : B \longrightarrow C}}{t : D \vdash \lambda y.f[(g,y)/z] : B \longrightarrow C}$$

On the other hand consider the formation $\lambda((g \times 1)f)$:

$$\dfrac{\dfrac{q : D \times B \vdash (g[\pi_0(q)/t], \pi_1(q)) : A \times B \qquad z : A \times B \vdash f : C}{q : D \times B \vdash f[(g[\pi_0(q)/t], \pi_1(q))/z]}}{t : D \vdash \lambda y.f[(g[\pi_0(q)/t], \pi_1(q))/z][(t,y)/q] B \longrightarrow C}$$

Then

$$\begin{aligned}
&\lambda y.f[(g[\pi_0(q)/t], \pi_1(q))/z][(t,y)/q] \\
&= \lambda y.f[(g[\pi_0(t,y)/t], \pi_1(t,y))/z] \\
&= \lambda y.f[(g[t/t], y)/z] \\
&= \lambda y.f[(g,y)/z]
\end{aligned}$$

Thus, $\lambda((g \times 1)f) = g\lambda(f)$. This completes the proof that $\underline{C}[\mathscr{T}]$ is a λ_β-category. □

Proposition 3.2.10. *For any Cartesian differential theory $\mathscr{T}$, $\underline{C}[\mathscr{T}]$ is a differential λ_β-category.*

In the following proof, we use the following notational convention: we write for example $va \lhd (v, a).m$ where va is a separate, new variable that splits as (v, a), and we use va as opposed to some arbitrary variable name like t to keep the number of names easier to manage.

Proof. It suffices to show that ev is linear in its first argument and that $D[\lambda(f)] = \lambda(\langle \pi_0 \times 0, \pi_1 \times 1\rangle D[f])$. That ev is linear in its first argument follows from **[CCDT.2]**.

For the other requirement, suppose $z : A \times B \vdash f' = z \lhd (x, y).f : C$, so that $\lambda(f) = x : A \vdash \lambda y.f : B \longrightarrow C$, and $D[\lambda(f)] = v - a \lhd (v, a)\frac{\partial \lambda y.f}{\partial x}(a) \cdot v$.

On the other hand, $D[f] = v\,w\,x\,y \lhd (v\,w, x\,y).\frac{\partial f'}{\partial z}(x\,y) \cdot v\,w$ and $\langle \pi_0 \times 0, \pi_1 \times 1\rangle = v\,a\,y \vdash v\,a\,y \lhd (v\,a, y).v\,a \lhd (v, a).((v, 0), (a, y))$. Thus,

$$\begin{aligned}
\langle \pi_0 \times 0, \pi_1 \times 1\rangle D[f] &= v\,a\,y \vdash v\,a\,y \lhd (v\,a, y).v\,a \lhd (v, a).\frac{\partial z \lhd (x, y).f}{\partial z}(a, y) \cdot v, 0 \\
&= v\,a\,y \vdash v\,a\,y \lhd (v\,a, y).v\,a \lhd (v, a).\frac{\partial f}{\partial x}(a) \cdot v \qquad \text{lemma 2.2.15.5}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lambda(\langle \pi_0 \times 0, \pi_1 \times 1\rangle\, D[f]) &= v\,a \vdash v\,a \lhd (v, a).\lambda y.\frac{\partial f}{\partial x}(a) \cdot v \\
&= v\,a \vdash v\,a \lhd (v, a).\frac{\partial \lambda y.f}{\partial x}(a) \cdot v \qquad \textbf{[CCDT.1]}
\end{aligned}$$

as required. □

The notion of theory of a differential λ_β-category is the same as for a Cartesian differential category. We have the following lemma

Lemma 3.2.11.

1. *For any Cartesian differential theory $\mathcal{T}$, $\underline{C}[\mathcal{T}]$ is a differential λ-category (i.e. it is closed) if and only if $\mathcal{T}$ is an extensional theory.*

2. *The interpretation of an extensional theory $\mathcal{T}$ into a differential λ-category is sound.*

3. *The theory of a differential λ_β-category is always a Cartesian differential λ-theory.*

4. *The theory of a differential λ-category is always an extensional Cartesian differential λ-theory.*

Then for the same general reasons as before we get the categorical type theory correspondence.

Theorem 3.2.12.

- *There is an adjunction:*

$$\begin{tikzcd} \textit{Diff-}\lambda\textit{-Th} \arrow[rr, bend left, "\underline{C}"] \arrow[rr, phantom, "\perp"] & & \textit{Differential-}\lambda_\beta\textit{-cat} \arrow[ll, bend left, "\mathsf{Th}"] \end{tikzcd}$$

- *Completeness: The unit of the adjunction has* $\mathscr{T} \simeq \mathsf{Th}(\underline{C}[\mathscr{T}])$.

- *Internal logic: The counit of the adjunction has* $\underline{C}[\mathsf{Th}(\mathbb{X})] \simeq \mathbb{X}$.

Also we get as an immediate consequence:

Corollary 3.2.13. *When $\mathbb{X}$ is a differential λ_β-category then it is an additive λ_β-category.*

Proof. Consider lemma 3.2.6 points *2* and *4*. Under interpretation into $\mathbb{X}$, they give $\lambda(f+g) = \lambda(f) + \lambda(g)$ and $\lambda(0) = 0$ and the additivity of ev in the function argument respectively. By soundness and completeness these equalities for all maps in $\mathbb{X}$, giving the coherence for being an additive λ_β-category. □

Chapter 4

An explicit equivalence between syntaxes

We have shown that the type theory for higher order differential calculus has a sound and complete semantics into differential λ_β-categories in chapter 3. In this chapter, we show that if the differential λ-calculus of (Ehrhard and Regnier, 2003) has the differential extensionality equation added, then the equational theory of the differential λ-calculus is equivalent to the equational theory of the higher order differential calculus of chapter 3 provided that $\Sigma = \emptyset$ and $E = \emptyset$.

We present a simplified syntax of the original differential λ-calculus of (Ehrhard and Regnier, 2003) that was used in (Bucciarelli, Ehrhard, and Manzoneto, 2010):

$$\Lambda^d := 0 \,|\, \Lambda^s + \Lambda^d \,|\, \Lambda^s \qquad \Lambda^s := x \,|\, \lambda x.\Lambda^s \,|\, \Lambda^s\,\Lambda^d \,|\, D\Lambda^s \cdot \Lambda^s$$

These terms are taken modulo the identities for a commutative monoid and the permutation identity $D(Dm \cdot u) \cdot v = D(Dm \cdot v) \cdot u$. There are two main equations:

$$(\boldsymbol{\beta}): \quad (\lambda x.s)t = s[t/x] \qquad\qquad (\boldsymbol{D\beta}): \quad D(\lambda x.s) \cdot t = \lambda x.\frac{\partial s}{\partial x} \cdot t$$

where substitution denotes normal substitution and $\frac{\partial s}{\partial x} \cdot t$ is the differential substitution, which we define by induction on syntax:

DSub.1 $\frac{\partial y}{\partial x} \cdot T = \begin{cases} T & x = y \\ 0 & \text{else} \end{cases}$;

DSub.2 $\frac{\partial \lambda y.s}{\partial x} \cdot T = \lambda y.\frac{\partial s}{\partial x} \cdot T$;

DSub.3 $\frac{\partial s\, u}{\partial x} \cdot t = (\frac{\partial s}{\partial x} \cdot t)u + (Ds \cdot (\frac{\partial u}{\partial x} \cdot t))u$;

DSub.4 $\frac{\partial Ds \cdot u}{\partial x} \cdot t = D(\frac{\partial s}{\partial x} \cdot t) \cdot u + Ds \cdot (\frac{\partial u}{\partial x} \cdot t)$;

DSub.5 $\frac{\partial s+u}{\partial x} \cdot t = \frac{\partial s}{\partial x} \cdot t + \frac{\partial u}{\partial x} \cdot t$ and $\frac{\partial 0}{\partial x} \cdot t = 0$.

Note, $\frac{\partial s}{\partial x} \cdot t$ is *not a variable binding term formation rule*, it is an operation on terms, and x may generally be free in $\frac{\partial s}{\partial x} \cdot t$.

Thus the full equational theory of Λ^d is given by

- the equations for giving the syntactic $+, 0$ the structure of a monoid on terms;
- the permutation axiom $D(Dm \cdot v) \cdot u = D(Dm \cdot u) \cdot v$);
- the equations $\boldsymbol{\beta}$ and the equation $\boldsymbol{D\beta}$;
- $(\boldsymbol{\eta_\partial})$: $\lambda z.(Dm \cdot v)z = Dm \cdot v$ (this is differential extensionality).

We now give explicit translations between these two syntaxes and show simulation equivalence when η is not present. The case for when η is present follows immediately.

We now denote the syntax for the differential λ-calculus proposed in this thesis by Λ^D to distinguish it from the Ehrhard-Regnier syntax Λ^d.

First, define $\Lambda^d \xrightarrow{[\![]\!]^d_D} \Lambda^D$ by induction on terms

- $[\![0]\!]^d_D = 0$
- $[\![m+n]\!]^d_D = [\![m]\!]^d_D + [\![n]\!]^d_D$
- $[\![x]\!]^d_D = x$
- $[\![\lambda x.m]\!]^d_D = \lambda x.[\![m]\!]^d_D$

- $[\![m\, n]\!]_D^d = [\![m]\!]_D^d\, [\![n]\!]_D^d$
- $[\![D m \cdot v]\!]_D^d = \lambda x. \frac{\partial [\![m]\!]_D^d z}{\partial z}(x) \cdot [\![v]\!]_D^d$

First we show that $[\![\_]\!]_D^d$ preserves the operations of substitution and differential substitution. We will supress the super and subscripts in the following lemma.

Lemma 4.0.1 (Substitution lemma)**.** *The translation has the following properties*

1. $[\![m[n/x]]\!] = [\![m]\!][[\![n]\!]/x]$

2. $\left[\!\!\left[\frac{\partial m}{\partial x} \cdot v\right]\!\!\right] = \frac{\partial [\![m]\!]}{\partial x}(x) \cdot [\![v]\!]$

Proof.

1. The proof of 1 is by a straightforward induction on m.

2. The proof of 2 is by induction on m.

   Consider the case when $m = m\, n$. Then $\frac{\partial m\, n}{\partial x} \cdot v$ is $(\frac{\partial m}{\partial x} \cdot v)n + (D m \cdot (\frac{\partial n}{\partial x} \cdot v))n$ by the definition of differential substitution. Then,

$$
\begin{aligned}
&\left[\!\!\left[\frac{\partial m\, n}{\partial x} \cdot v\right]\!\!\right] \\
&= \left[\!\!\left[(\frac{\partial m}{\partial x} \cdot v)n + (D m \cdot (\frac{\partial n}{\partial x} \cdot v))n\right]\!\!\right] \\
&= \left(\frac{\partial [\![m]\!]}{\partial x}(x) \cdot [\![v]\!]\right)[\![n]\!] + \left(\lambda y. \frac{\partial [\![m]\!]\, z}{\partial z}(y) \cdot \left(\frac{\partial [\![n]\!]}{\partial x}(x) \cdot [\![v]\!]\right)\right)[\![n]\!] \\
&= \left(\frac{\partial [\![m]\!]}{\partial x}(x) \cdot [\![v]\!]\right)[\![n]\!] + \left(\frac{\partial [\![m]\!]\, z}{\partial z}([\![n]\!]) \cdot \left(\frac{\partial [\![n]\!]}{\partial x}(x) \cdot [\![v]\!]\right)\right) \\
&= \frac{\partial [\![m]\!]\, [\![n]\!]}{\partial x}(x) \cdot [\![v]\!] = \frac{\partial [\![m\, n]\!]}{\partial x}(x) \cdot [\![v]\!] \qquad 3.2.6.5
\end{aligned}
$$

   Consider the case when $m = D s \cdot u$. In this case we have that by definition of the differential substitution operation, $\frac{\partial D s \cdot u}{\partial x} \cdot t$ is $D(\frac{\partial s}{\partial x} \cdot t) \cdot u + D s \cdot (\frac{\partial u}{\partial x} \cdot t)$. Then,

$$
\begin{aligned}
&\frac{\partial [\![Ds\cdot u]\!]}{\partial x}(x)\cdot [\![t]\!]\\
&= \frac{\partial \lambda y_0.\left(\frac{\partial [\![s]\!]\,y}{\partial y}(y_0)\cdot [\![u]\!]\right)}{\partial x}(x)\cdot [\![t]\!]\\
&= \lambda y_0.\frac{\partial \frac{\partial [\![s]\!]\,y}{\partial y}(y_0)\cdot [\![u]\!]}{\partial x}(x)\cdot [\![t]\!] \qquad \textbf{[CCDT.1]}\\
&= \lambda y_0.\frac{\partial \frac{\partial z_0\,y}{\partial y}(y_0)\cdot z_1[[\![s]\!],[\![u]\!]/z_0,z_1]}{\partial x}(x)\cdot [\![t]\!]\\
&= \lambda y_0.\frac{\partial \frac{\partial z_0\,y}{\partial y}(y_0)\cdot [\![u]\!]}{\partial z_0}([\![s]\!])\cdot\left(\frac{\partial [\![s]\!]}{\partial x}(x)\cdot [\![t]\!]\right)\\
&+ \lambda y_0.\frac{\partial \frac{\partial [\![s]\!]\,y}{\partial y}(y_0)\cdot z_1}{\partial z_1}([\![u]\!])\cdot\left(\frac{\partial [\![u]\!]}{\partial x}(x)\cdot [\![t]\!]\right) \qquad 2.2.7.4\\
&= \lambda y_0.\frac{\partial \frac{\partial z_0\,y}{\partial z_0}([\![u]\!])\cdot\left(\frac{\partial [\![s]\!]}{\partial x}(x)\cdot [\![t]\!]\right)}{\partial y}(y_0)\cdot [\![u]\!]\\
&+ \lambda y_0.\frac{\partial [\![s]\!]\,y}{\partial y}(y_0)\cdot\left(\frac{\partial [\![u]\!]}{\partial x}(x)\cdot [\![t]\!]\right) \qquad \textbf{DT.7}\text{ and }\textbf{DT.6}\\
&= \lambda y_0.\frac{\partial\left(\frac{\partial [\![s]\!]}{\partial x}(x)\cdot [\![t]\!]\right)y}{\partial y}(y_0)\cdot [\![u]\!]\\
&+ \lambda y_0.\frac{\partial [\![s]\!]\,y}{\partial y}(y_0)\cdot\left(\frac{\partial [\![u]\!]}{\partial x}(x)\cdot [\![t]\!]\right) \qquad \textbf{[CCDT.2]}\\
&= \left[\!\!\left[D(\frac{\partial s}{\partial x}\cdot t)\cdot u\right]\!\!\right] + \left[\!\!\left[Ds\cdot(\frac{\partial u}{\partial x}\cdot t)\right]\!\!\right]\\
&= \left[\!\!\left[D(\frac{\partial s}{\partial x}\cdot t)\cdot u + Ds\cdot(\frac{\partial u}{\partial x}\cdot x)\right]\!\!\right] = \left[\!\!\left[\frac{\partial Ds\cdot u}{\partial x}\cdot t\right]\!\!\right]
\end{aligned}
$$

The base cases and other cases are straightforward.

□

As a corollary, observe that the above means that for each equation $l = r$ of **DSub**, we have $[\![l]\!] = [\![r]\!]$.

Before continuing, we need the following lemma:

Lemma 4.0.2 (Manzonetto)**.** *In Λ^D, suppose $(p,q):\Gamma\times A\vdash f:B$, $p:\Gamma\vdash g:A$, and $p:\Gamma\vdash h:A$. Then,*

$$\frac{\partial \frac{\partial f(p,q)}{\partial q}(q)\cdot h(p)}{\partial q}(q)\cdot g(p)=\frac{\partial \frac{\partial f(p,q)}{\partial q}(q)\cdot g(p)}{\partial q}(q)\cdot h(p)$$

where $t(r)$ denotes that r is free in t.

This lemma was stated by (Manzonetto, 2012) as lemma 4.6 using the $\star$ operator. There it was stated as $(f\star g)\star h=(f\star h)\star g$. However, a more direct and general proof may be given:

Lemma 4.0.3. *In any differential theory $\mathscr{T}$, when $z_1\notin \mathsf{fv}(m,v,a,w)$,*

$$\frac{\partial \frac{\partial m}{\partial x}(z_1)\cdot v}{\partial z_1}(a)\cdot w=\frac{\partial \frac{\partial m}{\partial x}(z_1)\cdot w}{\partial z_1}(a)\cdot v$$

Proof. The proof is from (Cockett and Seely, 2011).

First, we prove that $\frac{\partial m}{\partial x}(a)\cdot v=\frac{\partial m[a+y/x]}{\partial y}(0)\cdot v$:

$$\begin{aligned}\frac{\partial m[a+y/x]}{\partial y}(0)\cdot v&=\frac{\partial m}{\partial x}((a+y)[0/y])\cdot\frac{\partial a+y}{\partial y}(0)\cdot v\\&=\frac{\partial m}{\partial x}(a)\cdot\left(\frac{\partial a}{\partial y}(0)\cdot v+\frac{\partial y}{\partial y}(0)\cdot v\right)=\frac{\partial m}{\partial x}(a)\cdot v\end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial \frac{\partial m}{\partial x}(z_1)\cdot v}{\partial z_1}(a)\cdot w&=\frac{\partial \frac{\partial m[z_1+y/x]}{\partial y}(0)\cdot v}{\partial z_1}(a)\cdot w=\frac{\partial \frac{\partial m[z_1+y/x]}{\partial y}(0)\cdot v[a+y_2/z_1]}{\partial y_2}(0)\cdot w\\&=\frac{\partial \frac{\partial m[a+y_2+y/x]}{\partial y}(0)\cdot v}{\partial y_2}(0)\cdot w=\frac{\partial \frac{\partial m[a+y_2+y/x]}{\partial y}(0)\cdot w}{\partial y_2}(0)\cdot v=\frac{\partial \frac{\partial m}{\partial x}(z_1)\cdot w}{\partial z_1}(a)\cdot v\end{aligned}$$

□

Then,

Proposition 4.0.4. *The interpretation $[\![\_]\!]_D^d$ is sound: if $m=n$ in Λ^d, then $[\![m]\!]_D^d=[\![n]\!]_D^d$ in Λ^D.*

Proof. The equations for commutative monoids are preserved under interpretation, and the calculations for this are straightforward. This leaves the permutation identity, β, $D\beta$, and η_∂. We start with β. We will omit the scripts on the interpretation brackets for this proof.

$$
\begin{aligned}
&[\![(\lambda x.m)n]\!] \\
&= [\![\lambda x.m]\!][\![n]\!] \\
&= (\lambda x.[\![m]\!])[\![n]\!] \\
&= [\![m]\!][[\![n]\!]/x] \\
&= [\![m[n/x]]\!] \qquad \text{lemma 4.0.1}
\end{aligned}
$$

Next $\boldsymbol{D\beta}$:

$$
\begin{aligned}
&\left[\!\!\left[\lambda y.\frac{\partial s}{\partial y}\cdot t\right]\!\!\right] \\
&= \lambda y.\frac{\partial [\![s]\!]}{\partial y}(y)\cdot [\![t]\!] \qquad \text{lemma 4.0.1} \\
&= \lambda x.\frac{\partial [\![s]\!]}{\partial y}(x)\cdot [\![t]\!] \qquad \boldsymbol{\alpha} \\
&= \lambda x.\frac{\partial [\![s]\!]}{\partial y}(x)\cdot \left(\frac{\partial z}{\partial z}(x)\cdot [\![t]\!]\right) \\
&= \lambda x.\frac{\partial [\![s]\!][z/y]}{\partial z}(x)\cdot [\![t]\!] \qquad 2.2.7.1 \\
&= \lambda x.\frac{\partial [\![\lambda y.s]\!]\, z}{\partial z}(x)\cdot [\![t]\!] \\
&= [\![D(\lambda y.s)\cdot t]\!]
\end{aligned}
$$

Next, the permutation identity:

$$
\begin{aligned}
&[\![D(Dm\cdot v)\cdot u]\!] \\
&= \lambda x.\frac{\partial\left(\lambda y.\frac{\partial [\![m]\!] z_0}{\partial z_0}(y)\cdot [\![v]\!]\right) z_1}{\partial z_1}(x)\cdot [\![u]\!] \\
&= \lambda x.\frac{\partial\left(\frac{\partial [\![m]\!] z_0}{\partial z_0}(z_1)\cdot [\![v]\!]\right)}{\partial z_1}(x)\cdot [\![u]\!] \\
&= \lambda x.\frac{\partial\left(\frac{\partial [\![m]\!] z_0}{\partial z_0}(z_1)\cdot [\![u]\!]\right)}{\partial z_1}(x)\cdot [\![v]\!] \qquad \text{lemma 4.0.2} \\
&= [\![D(Dm\cdot u)\cdot v)]\!]
\end{aligned}
$$

Finally, we show that η_∂ holds:

$$
\begin{aligned}
&[\![\lambda x.(Dm\cdot v)x]\!] \\
&= \lambda x.[\![Dm\cdot v]\!]x \\
&= \lambda x.(\lambda y.\frac{\partial [\![m]\!] z}{\partial z}(y)\cdot [\![v]\!])x \\
&= \lambda x.\frac{\partial [\![m]\!] z}{\partial z}(x)\cdot [\![v]\!] \\
&= [\![Dm\cdot v]\!]
\end{aligned}
$$

□

Define $\Lambda^D \xrightarrow{[\![]\!]_d^D} \Lambda^d$ by induction on terms:

- $[\![0]\!]_d^D = 0$
- $[\![m+n]\!]_d^D = [\![m]\!]_d^D + [\![n]\!]_d^D$
- $[\![x]\!]_d^D = x$
- $[\![\lambda x.m]\!]_d^D = \lambda x.[\![m]\!]_d^D$
- $[\![m\,n]\!]_d^D = [\![m]\!]_d^D\,[\![n]\!]_d^D$
- $\left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right]_d^D = \left(\lambda x.\frac{\partial [\![m]\!]_d^D}{\partial x}\cdot [\![v]\!]_d^D\right)[\![a]\!]_d^D$

Lemma 4.0.5 (Substitution lemma)**.** *The translation has the substitution property*

$$[\![n[m/x]]\!]_d^D = [\![n]\!]_d^D[[\![m]\!]_d^D/x]$$

Proof. The proof of the substitution lemma is by induction on the structure of n. □

As before, we will drop the super and subscripts while working with only this translation.

Also note the translation of differentials yields:

$$\begin{aligned}
&\left[\!\!\left[\frac{\partial m}{\partial x}(a)\cdot v\right]\!\!\right] \\
&= \left(\lambda x.\frac{\partial [\![m]\!]}{\partial x}\cdot [\![v]\!]\right)[\![a]\!] \\
&= \left(\frac{\partial [\![m]\!]}{\partial x}\cdot [\![v]\!]\right)[[\![a]\!]/x]
\end{aligned}$$

We will need the following lemma from (Ehrhard and Regnier, 2003):

Lemma 4.0.6 (Ehrhard-Regnier)**.** *The following holds for Λ^d:*

1. *If $x \notin m$ then $\frac{\partial m}{\partial x}\cdot v = 0$;*

2. $$\frac{\partial m}{\partial x}\cdot \sum_j v_j = \sum_j \frac{\partial m}{\partial x}\cdot v_j$$

3. *If $y \notin \mathsf{fv}(u)$, then*

$$\frac{\partial \frac{\partial t}{\partial y}\cdot v}{\partial x}\cdot u = \frac{\partial \frac{\partial t}{\partial x}\cdot u}{\partial y}\cdot v + \frac{\partial t}{\partial y}\cdot\left(\frac{\partial v}{\partial x}\cdot u\right)$$

In particular, if additionally, $x \notin \mathsf{fv}(v)$, then

$$\frac{\partial \frac{\partial t}{\partial y}\cdot v}{\partial x}\cdot u = \frac{\partial \frac{\partial t}{\partial x}\cdot u}{\partial y}\cdot v$$

4. *When* $x \notin \mathsf{fv}(v)$*:*

$$\left(\frac{\partial t}{\partial x} \cdot u\right)[v/y] = \frac{\partial t[v/y]}{\partial x} \cdot u[v/y]$$

5. *When* $x \neq y$*,* $y \notin \mathsf{fv}(u, v)$*, then*

$$\frac{\partial t[v/y]}{\partial x} \cdot u = \left(\frac{\partial t}{\partial x} \cdot u\right)[v/y] + \left(\frac{\partial t}{\partial y} \cdot \left(\frac{\partial v}{\partial x} \cdot u\right)\right)[v/y]$$

In particular, when $x \notin \mathsf{fv}(v_i)$ *for* $i = 1$ *to* n*, the above inductively gives*

$$\frac{\partial t[v_i/y_i]}{\partial x} \cdot u = \left(\frac{\partial t}{\partial x} \cdot u\right)[v_i/y_i]$$

6. *When* $x \notin \mathsf{fv}(m)$ *and* $y_i \notin \mathsf{fv}(v_j)$ *for* $i, j = 1$ *to* n*, we have*

$$\frac{\partial m[v_i/y_i]}{\partial x} \cdot u = \left(\sum_{i=1}^{n} \frac{\partial m}{\partial y_i} \cdot \left(\frac{\partial v_i}{\partial x} \cdot u\right)\right)[v_i/y_i]$$

Proof. For proofs of the above see (Ehrhard and Regnier, 2003): $1, 2$ are (Ehrhard and Regnier, 2003) lemma 3, 3 is (Ehrhard and Regnier, 2003) lemma 4, 4 is (Ehrhard and Regnier, 2003) lemma 6, and 5 is (Ehrhard and Regnier, 2003) lemma 5. This leaves only 6 for us; the proof of 6 is by induction on the structure of n. The base case is trivial. Then,

$$\begin{aligned}
&\frac{\partial m[v_i/y_i]}{\partial x} \cdot u \\
&= \frac{\partial m[v_i/y_i][v_n/y_n]}{\partial x} \cdot u \\
&= \left(\frac{\partial m[v_i/y_i]_{i<n}}{\partial x} \cdot u\right)[v_n/y_n] + \frac{\partial m[v_i/y_i]_{i<n}}{\partial y_n} \cdot \left(\frac{\partial v_n}{\partial x} \cdot u\right)[v_n/y_n] && \text{lemma 4.0.6.5} \\
&= \left(\frac{\partial m[v_i/y_i]_{i<n}}{\partial x} \cdot u\right)[v_n/y_n] + \frac{\partial m}{\partial y_n} \cdot \left(\frac{\partial v_n}{\partial x} \cdot u\right)[v_i/y_i]_{i=1}^{n} && \text{lemma 4.0.6.5} \\
&= \left(\sum_{i=1}^{n-1} \frac{\partial m}{\partial y_i} \cdot \left(\frac{\partial v_i}{\partial x} \cdot u\right)\right)[v_i/y_i]_{i<n}[v_n/y_n] + \frac{\partial m}{\partial y_n} \cdot \left(\frac{\partial v_n}{\partial x} \cdot u\right)[v_i/y_i]_{i=1}^{n} && \text{ind hyp} \\
&= \left(\sum_{i=1}^{n} \frac{\partial m}{\partial y_i} \cdot \left(\frac{\partial v_i}{\partial x} \cdot u\right)\right)[v_i/y_i]_{i=1}^{n}
\end{aligned}$$

as required. □

Proposition 4.0.7. *The interpretation* $\Lambda^D \xrightarrow{[\![\,]\!]^D_d} \Lambda^d$ *is sound: if* $m = n$ *in* Λ^D*, then* $[\![m]\!]^D_d = [\![n]\!]^D_d$ *in* Λ^d.

Proof. The commutative monoid equations **DT.1** are immediate. That β and α hold is also immediate because the substitution lemma holds.

We now proceed to show that **Dt.2–7** and **CCDT.1–2** hold under translation.

[DT.2] This is obvious.

[DT.3] This is given by lemma 4.0.6.2

[DT.4] For the variable case:

$$\begin{aligned}&\left[\!\!\left[\frac{\partial x}{\partial x}(a)\cdot v\right]\!\!\right]\\ &=\left(\frac{\partial x}{\partial x}\cdot[\![v]\!]\right)[[\![a]\!]/x]\\ &=[\![v]\!][[\![a]\!]/x]=[\![v]\!]\end{aligned}$$

For the 0 case use 4.0.6.1.

[DT.5] This is given by 4.0.6.6 and a few routine manipulations.

[DT.6] By hypothesis we have $y \notin \mathsf{fv}(m)$. Then,

$$\begin{aligned}&\left[\!\!\left[\frac{\partial \frac{\partial n}{\partial x}(a)\cdot y}{\partial y}(b)\cdot v\right]\!\!\right]\\ &=\left(\frac{\partial \frac{\partial [\![m]\!]}{\partial x}\cdot y}{\partial y}\cdot[\![v]\!]\right)[[\![a]\!]/x,[\![b]\!]/y]\\ &=\left(\frac{\partial \frac{\partial [\![m]\!]}{\partial y}\cdot[\![v]\!]}{\partial x}\cdot y+\frac{\partial [\![m]\!]}{\partial x}\cdot\frac{\partial y}{\partial y}\cdot[\![v]\!]\right)[[\![a]\!]/x,[\![b]\!]/y] \quad 4.0.6.3\\ &=\left(\frac{\partial [\![m]\!]}{\partial x}\cdot[\![v]\!]\right)[[\![a]\!]/x]\end{aligned}$$

[DT.7] This is given essentially by the "in particular" part of 4.0.6.3.

[CCDT.1] We have

$$\begin{aligned}
&\left[\!\!\left[\frac{\partial \lambda y.m}{\partial p}(a)\cdot v\right]\!\!\right] \\
&= \left(\frac{\partial \lambda y.[\![m]\!]}{\partial x_i}\cdot [\![v_i]\!]\right)[\,[\![a_j]\!]/x_j] \\
&= \lambda y.\left(\left(\frac{\partial \lambda y.[\![m]\!]}{\partial x_i}\cdot [\![v_i]\!]\right)[\,[\![a_j]\!]/x_j]\right) \\
&= \left[\!\!\left[\lambda y.\frac{\partial m}{\partial p}(a)\cdot v\right]\!\!\right]
\end{aligned}$$

[CCDT.2] First note that in Λ^d:

$$\frac{\partial y\, z}{\partial y}\cdot v = \left(\frac{\partial y}{\partial y}\cdot v\right)z + (D\,y\cdot(\frac{\partial z}{\partial y}(v)\cdot))z = \left(\frac{\partial y}{\partial y}\cdot v\right)z = v\,z$$

As $D\,x\cdot 0 = 0$. Then

$$\left[\!\!\left[\frac{\partial \lambda z.y\,z}{\partial y}(a)\cdot v\right]\!\!\right] = \left(\frac{\partial \lambda z.y\,z}{\partial y}\cdot v\right)[a/y] = \lambda z.\left(\frac{\partial y\,z}{\partial y}\cdot v\right)[a/y] = \lambda z.v\,z$$

as required.

This completes the proof. □

Therefore we have sound translations between Λ^d and Λ^D. Finally, we show that these translations are inverse.

Proposition 4.0.8. *For all $m\in\Lambda^D$ we have $m = \left[\!\!\left[[\![m]\!]_d^D\right]\!\!\right]_D^d$.*

Proof. The proof is by induction on the structure of m.

Variable $\left[\!\!\left[[\![x]\!]_d^D\right]\!\!\right]_D^d = [\![x]\!]_D^d = x$

Monoid For 0, $\left[\!\!\left[[\![0]\!]_d^D\right]\!\!\right]_D^d = [\![0]\!]_D^d = 0$. For sums

$$\left[\!\!\left[[\![m+n]\!]_d^D\right]\!\!\right]_D^d = \left[\!\!\left[[\![m]\!]_d^D\right]\!\!\right]_D^d + \left[\!\!\left[[\![n]\!]_d^D\right]\!\!\right]_D^d = m+n$$

by the inductive hypothesis.

Application Similarly, $\left[\!\left[[\![m\,n]\!]_d^D \right]\!\right]_D^d = \left[\!\left[[\![m]\!]_d^D \right]\!\right]_D^d \left[\!\left[[\![n]\!]_d^D \right]\!\right]_D^d = m\,n.$

Abstraction Similarly, $\left[\!\left[[\![\lambda.m]\!]_d^D \right]\!\right]_D^d = \lambda x. \left[\!\left[[\![m]\!]_d^D \right]\!\right]_D^d = \lambda x.m.$

Differential We have

$$
\begin{aligned}
&\left[\!\left[\left[\!\left[\frac{\partial m}{\partial x}(a) \cdot v \right]\!\right]_d^D \right]\!\right]_D^d \\
&= \left[\!\left[\left(\frac{\partial [\![m]\!]_d^D}{\partial x} \cdot [\![v]\!]_d^D \right) [[\![a]\!]_d^D] \right]\!\right]_D^d \\
&= \frac{\partial \left[\!\left[[\![m]\!]_d^D \right]\!\right]_D^d}{\partial x}(x) \cdot \left[\!\left[[\![v]\!]_d^D \right]\!\right]_D^d \left[\left[\!\left[[\![a]\!]_d^D \right]\!\right]_D^d \right] \\
&= \frac{\partial m}{\partial x}(a) \cdot v \qquad \text{Ind. hyp.}
\end{aligned}
$$

This completes the proof. □

Similarly, we have

Proposition 4.0.9. *For all $m \in \Lambda^d$ we have $m = \left[\!\left[[\![m]\!]_D^d \right]\!\right]_d^D$.*

Proof. The proof is by induction on the structure of m. Similarly, to the above, the cases for variable, sums of terms, application, and abstraction are easy. We cover here case $m = D\,m \cdot v$.

$$
\begin{aligned}
&\left[\!\!\left[[\![Dm \cdot v]\!]^d_D \right]\!\!\right]^D_d \\
&= \left[\!\!\left[\lambda y. \frac{\partial\, [\![m^d_D]\!]\, z}{\partial z}(y) \cdot [\![v]\!]^d_D \right]\!\!\right]^D_d \\
&= \lambda y. \left(\frac{\partial \left[\!\!\left[[\![m]\!]^d_D \right]\!\!\right]^D_d z}{\partial z} \cdot \left[\!\!\left[[\![v]\!]^d_D \right]\!\!\right]^D_d \right)[y/z] \\
&= \lambda y. \left(\frac{\partial m\, z}{\partial z} \cdot v \right)[y/z] \qquad \text{Ind. Hyp.} \\
&= \lambda y. \left(\left(\frac{\partial m}{\partial z} \cdot v \right) z + (Dm \cdot v)\, z \right)[y/z] \\
&= \lambda y. (Dm \cdot v)\, y \\
&= Dm \cdot v \qquad \text{differential extensionality } \eta_\partial
\end{aligned}
$$

$\square$

Therefore we have:

Corollary 4.0.10. *The theory* Λ^d *is equivalent to the theory* Λ^D.

Part II

Differential Geometry

In part I of the thesis, we investigated the categorical semantics for the differential λ-calculus; the main objective was to produce soundness, completeness and expressiveness results for the differential λ-calculus. In doing this, we were able to give an explicit equational theory for the differential λ-calculus, and show that differential λ_β-categories provide an expressive, sound, and complete semantics.

A particularly intriguing model of the differential λ-calculus was mentioned 3.1.6 in the previous section: convenient vector spaces (Blute, Ehrhard, and Tasson, 2010). Kock showed that the category of convenient vector spaces and smooth maps embeds into the Cahiers topos (Kock, 1986). The Cahiers topos was introduced by Dubuc in (Dubuc, 1979) and initiated the study of well adapted models of synthetic differential geometry. The convenient vector spaces actually embed into the $\mathscr{R}$ vector spaces and smooth maps, and so their inherited tangent bundle trivializes $T(V) \simeq V \times V$.

The above insight led to asking the question: do the Kock-Lawvere vector spaces of a smooth topos (i.e. a model of SDG) always admit a sound interpretation of the differential λ-calculus? The answer is a definite yes, but the way we approached the answer, led to a better understanding of the models of the differential λ-calculus and of Cartesian closed tangent categories.

Tangent categories, introduced by (Rosický, 1984) to understand abstractly the tangent functor in synthetic differential geometry as well as the Lie algebra of a tangent functor. Tangent categories were rediscovered in (Cockett and Cruttwell, 2014b) where the manifold completion of (Grandis, 1990) was applied to a differential restriction category (Cockett, Cruttwell, and Gallagher, 2011) to understand the conditions that needed to hold of an abstract tangent functor. Remarkably, (Rosický, 1984) and (Cockett and Cruttwell, 2014b) arrived at essentially the same conditions. The main difference is that (Rosický, 1984) assumed that the addition of tangent vectors had negatives, where (Cockett and Cruttwell, 2014b) does not assume negatives. The lack of negatives in (Cockett and Cruttwell, 2014b) is crucial for applications to the differential λ-calculus, as the differential λ-calculus does not have negatives.

Synthetic differential geometry (SDG) yields a tangent category with finite limits and where the tangent bundle is representable; $TM \simeq [D, M]$. D is then forced to have certain properties making it into an object of "disembodied tangent vectors.” While SDG is a radical reformulation of differential geometry, Dubuc’s paper (Dubuc, 1979) shows that it is possible to study classical differential geometry using the synthetic method. This is because the category of smooth manifolds and smooth maps embeds into a well adapted model of SDG (for example the Cahiers topos), and importantly the embedding preserves transverse limits.

In the proof that the $\mathscr{R}$ vector spaces of a model of SDG are always a model of the differential λ-calculus, one finds that the proof does not deal with the peculiarities of SDG and revolves around an isomorphism

$$T[A,B] \simeq [A,TB]$$

It turns out that this isomorphism may be seen to arise from a strength $A \times T(B) \xrightarrow{\theta} T(A \times B)$. We axiomatize a coherently closed tangent category to be a cartesian closed tangent category in which the map $T[A,B] \longrightarrow [A,TB]$, which arises from the strength $A \times TB \longrightarrow T(A \times B)$, is an isomorphism. In any tangent category there is a notion of *differential object* (see e.g. (Cockett and Cruttwell, 2014b)) that plays the role of vector space. Indeed in SDG the differential objects are precisely $\mathscr{R}$ vector spaces, and in SMan the differential objects are also $\mathscr{R}$ vector spaces. We will see that in a coherently closed tangent category the differential objects are always a model of the differential λ-calculus.

Recent work of Leung (Leung, 2017) makes it possible to view tangent categories by *Weil prolongation* – the tangent functor can be formally viewed as infinitesimally extending an object $TM \simeq M \otimes R[x]/(x^2)$. This result makes it possible to connect tangent categories to the work of Nishimura. Nishimura introduced a concept he called axiomatic differential geometry in a series of notes (Nishimura, 2012a; Nishimura, 2012b; Nishimura, 2012c; Nishimura, 2013; Nishimura, 2012d; Nishimura, 2012e; Nishimura, 2017a). In these notes, Nishimura essentially asked for Weil prolongation – the difference between Nishimura’s setting and that of tangent categories is that tangent categories require Weil prolongation to be coherent, whereas Nishimura missed the coherences required of an action. Similarly Nishimura’s axiomatics asked for a (non-canonical) isomorphism

$$T[A,B] \simeq [A,TB]$$

Nishimura needed this isomorphism so that the object of vector fields can be formulated as an object intrinsic to his axiomatics. Also interestingly, while not Cartesian closed, the category of manifolds modelled on convenient vector spaces does have some homs; in particular when M is compact and N has local addition, there is a hom $[M,N]$ and $T[M,N] \simeq [M,TN]$ (Kriegl and Michor, 1997) chapter IX, theorem 42.17. In Kriegl and Michor, the isomorphism is the canonical one described here. Thus, there seems to be a mathematical role that our isomorphism plays.

Nishimura's axiomatic differential geometry was an abstraction of some of his earlier work on Frölicher spaces (Nishimura, 2009; Nishimura, 2010). He introduced an idea of Weil exponentiation. In this part of thesis we will make his idea of Weil exponentation formal using coherently closed tangent categories and prove that a coherently closed tangent category may be extracted from Frölicher spaces as an exponential ideal that contains the convenient manifolds. This gives a categorically presented alternative to Michor's cartesian closed category of smooth manifolds (Michor, 1984a; Michor, 1984b).

Strength and enrichment are intimately related. In (Garner, 2018), Leung's theorem relating tangent categories and Weil prolongation is extended; Garner shows that tangent categories are categories enriched in a monoidally reflective subcategory of Weil spaces. We will extend Garner's result, and prove results relating transverse limits and enriched limits, and coherently closed tangent categories and enriched closed categories. We will then use the enrichment to prove properties about closed tangent categories.

Chapter 5

Coherently Closed Tangent Categories

There have been many attempts to formalize "convenient" or monoidal closed categories that generalize the category of smooth manifolds in some suitable sense. Fréchet manifolds and convenient manifolds have all internal homs of finite dimensional manifolds, but are not themselves Cartesian closed. Frölicher spaces, Sikorski (also known as differential) spaces, diffeological spaces, and other *smootheologies* Stacey, 2011 define smoothness using curves into a space (e.g. functions $\mathscr{R} \longrightarrow X$) or functionals out of a space (e.g. functions $X \longrightarrow \mathscr{R}$) rather than by topological methods. These categories are Cartesian closed, but it is not clear to what extent these settings have tangent structure. Synthetic differential geometry provides differential geometry in a Cartesian closed category, and does have tangent structure.

A tangent category is a category equipped with structure that abstracts essential properties of the tangent bundle on smooth manifolds. We will provide a definition in the sequel. In this chapter, we will address what it means to have a tangent category that is cartesian closed and in which the closed structure is coherent with the tangent bundle.

It turns out that strength, and a related notion we call exponential strength play an important role. In this chapter we will first develop tangent categories as strong, and prove that all the structural transformations are strong. Before doing this, we develop a few ideas about strength that we will need. We then develop coherently closed tangent categories; these are tangent categories that are also Cartesian closed, where a certain exponential strength is an isomorphism. We prove that the differential objects of a coherently closed tangent category are always a differential λ-category, thus simplifiying the coherence condition on differential λ-categories. We then characterize when the coKleisli category of a tangent category is a tangent category; to do so we characterize how tangent structure lifts across a comonadic adjunction. This gives another view on the strength; all the natural transformations lift precisely because they are strong. We then show that for coherently closed tangent categories homs into the total space of a differential bundle are a differential bundle. This property of differential bundles has application to internalizing the object of vector fields. We finally discuss locally closed tangent categories.

5.1 Tangent categories

In this section we introduce tangent categories which serve as our abstract setting for differential geometry.

Let $\mathbb{X}$ be a category with products, and let $E \xrightarrow{q} M$ be any map. q provides an **additive bundle** over M when there are finite pullbacks of q (so that there are finite products of q in $\mathbb{X}/M$) and there are maps $E_2 \xrightarrow{+} E$ and $M \xrightarrow{0} E$ such that

$$\begin{tikzcd}
 & E_2 \arrow[rr, "+"] \arrow[dl] \arrow[dr] & & E \arrow[ddll, bend left, "q"] \\
E \arrow[dr, "q"'] & & E \arrow[dl, "q"] & \\
 & M & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M \arrow[r, "0"] \arrow[dr, equal] & E \arrow[d, "q"] \\
 & M
\end{tikzcd}$$

provide the structure of a commutative monoid in the slice $\mathbb{X}/M$.

Given bundles, $(E \xrightarrow{q} M, +, 0)$ and $(E' \xrightarrow{q'} M', +, 0)$ a pair of maps $(E \xrightarrow{f} E', M \xrightarrow{g} M')$ is an **additive bundle** homomorphism when they commute with the bundles, the addition, and the zero.

Definition 5.1.1. $\mathbb{X}$ *is a* ***tangent category*** *Cockett and Cruttwell, 2014b when there is an endofunctor* $\mathbb{X} \xrightarrow{T} \mathbb{X}$ *together with natural transformations:*

Tangent bundle: $T(M) \xrightarrow{p} M$

Canononical flip: $T^2(M) \xrightarrow{c} T^2(M)$

Vertical lift: $T(M) \xrightarrow{l} T^2(M)$

Tangent addittion: $M \xrightarrow{0} T(M)$ *and* $T_2(M) \xrightarrow{+} T(M)$

such that

1. *All pullbacks of* p *along* p *exist and are preserved by* T. *Call the pullback of a single* p *along* p $T_2(M)$.
2. *For each* M, $T(M) \xrightarrow{p} M$ *is an additive bundle with respect to* 0 *and* $+$.
3. $(l, 0) : (TM \xrightarrow{p} M, +, 0) \longrightarrow (T^2M \xrightarrow{Tp} TM, T(+), T(0))$ *is an additive bundle morphism.*
4. $(c, 1) : (T^2M \xrightarrow{Tp} TM, T(+), T(0)) \longrightarrow (T^2M \xrightarrow{p} TM, +, 0)$ *is an additive bundle morphism.*
5. l *and* c *are coherent:*

$$\begin{tikzcd} T \arrow[r, "l"] \arrow[d, "l"'] & T^2 \arrow[d, "T(l)"] \\ T^2 \arrow[r, "l_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T(c)"] \arrow[d, "c_T"'] & T^3 \arrow[r, "c_T"] & T^3 \arrow[d, "T(c)"] \\ T^3 \arrow[r, "T(c)"'] & T^3 \arrow[r, "c_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "l_T"] \arrow[d, "c"'] & T^3 \arrow[r, "T(c)"] & T^3 \arrow[d, "c_T"] \\ T^2 \arrow[rr, "T(l)"'] & & T^3 \end{tikzcd}$$

6. *The lift is universal: the following is an equalizer diagram, and the equalizer is preserved by* T^k.

$$\begin{tikzcd} T_2M \arrow[r, "v:=\langle \pi_0 l, \pi_1 0\rangle T(+)"] & T^2M \arrow[r, shift left, "T(p)"] \arrow[r, shift right, "pp0"'] & TM \end{tikzcd}$$

Lemma 5.1.2. $(l, 0) : (p, +, 0) \longrightarrow (p_T, +_T, 0_T)$ *is an additive bundle homomorphism.*

Proof. See Cockett and Cruttwell, 2014b. □

We have the following useful lemma that shows that l is itself an equalizer.

Lemma 5.1.3 (Cockett and Cruttwell, 2014b)**.** *In any tangent category, the following is a triple equalizer diagram.*

$$TM \xrightarrow{l} T^2M \underset{pp0}{\overset{T(p)}{\underset{p}{\rightrightarrows}}} TM$$

A category with finite products and a tangent structure $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is a **cartesian tangent category** when T preserves products, and when the isomorphism $T(A) \times T(B) \xrightarrow{m_T} T(A \times B)$ preserves all the tangent structure maps. For the full details, see Cockett and Cruttwell, 2014b.

Definition 5.1.4. *In a tangent category, we call a limit $\lim_i X_i$ **transverse** when $T(\lim_i X_i) \simeq \lim_i T(X_i)$.*

Observation 5.1.5. *In a tangent category, the pullback powers of p are transverse limits as is the equalizer for the universality of the lift. In a Cartesian tangent category, the product is transverse.*

Example 5.1.6. *The category of smooth manifolds is a tangent category with the usual tangent bundle of smooth manifolds.*

Example 5.1.7. *Every Cartesian differential category is a Cartesian tangent category. The tangent functor on objects is $T(A) := A \times A$ and on arrows $A \xrightarrow{f} B$ is*

$$A \times A \xrightarrow{\langle D[f], \pi_1 f \rangle} B \times B$$

and for example the lift is

$$A \times A \xrightarrow{\langle 1,0,0,1 \rangle} A \times A \times A \times A$$

and the canonical flip is

$$A \times A \times A \times A \xrightarrow{\langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle} A \times A \times A \times A$$

A few additional examples are presented in sections 6.2 and 6.7.

5.2 Strength and exponential strength

We express the coherence for closed tangent categories in terms of strength. For endofunctors on a monoidal closed category strength encodes what it means for a functor to be internalizable. However strength can be phrased independently of closed monoidal structure, and on just monoidal structure. In this section, we will introduce monoidal strength and a few related notions we will need.

Monoidal categories and *symmetric monoidal categories* were introduced by **journal:monoidal-original** and their coherence was exposited by **journal:kelly-maclane-coherence** Monoidal structure on a category is meant to generalize structure like the tensor product of vector spaces, which is neither product nor coproduct. In particular there is a bifunctor $\mathbb{X}\times\mathbb{X} \xrightarrow{\otimes} \mathbb{X}$ and an object I with an natural associativity $(A\otimes B)\otimes C \xrightarrow{a_\otimes} A\otimes(B\otimes C)$ and unit $A\otimes I \xrightarrow{u_R} A$, $I\otimes A \xrightarrow{u_L} A$ isomorphisms that satisfy coherences (see **journal:kelly-maclane-coherence**). These maps are often called the associator and unitors respectively. A *monoidal functor* $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ is a functor $\mathbb{X}\longrightarrow\mathbb{Y}$ between monoidal categories together with natural transformations $F(A)\otimes F(B)\longrightarrow F(A\otimes B)$ and $I\longrightarrow F(I)$ that satisfy coherences **journal:kelly-doctrinal-adj** A monoidal functor between symmetric monoidal categories is called *symmetric* when the following commutes:

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r] \arrow[d, "c"'] & F(A\otimes B) \arrow[d, "Fc"] \\
F(B)\otimes F(A) \arrow[r] & F(B\otimes A)
\end{tikzcd}$$

A natural transformation $\alpha : F \longrightarrow G$ is a *monoidal tranformation* when

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "\alpha\otimes\alpha"] \arrow[d] & G(A)\otimes G(B) \arrow[d] \\
F(A\otimes B) \arrow[r, "\alpha"'] & G(A\otimes B)
\end{tikzcd}$$

a monoidal natural transformation between symmetric monoidal functors is sometimes called a *symmetric monoidal transformation* – that is, there is no additional coherence for a monoidal transformation to be symmetric provided that it is between symmetric monoidal functors.

Let $\mathscr{V}$ be a symmetric monoidal category. A **monoidal closed category** is a generalization of cartesian closed categories that were introduced in 3.1: there is an isomorphism $\mathscr{V}(A \otimes B, C) \simeq \mathscr{V}(B, [A, C])$ that is natural in B.

Definition 5.2.1. *A $\mathscr{V}$-enriched category Kelly, 2005 or $\mathscr{V}$-category $\mathbb{X}$ is given by*

- *A collection of objects $\mathbb{X}_0$;*
- *For each pair of $\mathbb{X}$-objects $A, B \in \mathbb{X}_0$, a $\mathscr{V}$-object $\mathbb{X}(A, B)$;*
- *For each $\mathbb{X}$-object A, a $\mathscr{V}$-map $I \xrightarrow{\mathsf{Id}} \mathbb{X}(A, A)$;*
- *For any three $\mathbb{X}$-objects A, B, C, a $\mathscr{V}$-map $\mathbb{X}(A, B) \otimes \mathbb{X}(B, C) \xrightarrow{\mathsf{c}} \mathbb{X}(A, C)$.*

such that appropriate diagrams commute that make composition associative and unital.

Definition 5.2.2. *A $\mathscr{V}$-functor from a $\mathscr{V}$-cat $\mathbb{X}$ to a $\mathscr{V}$-cat $\mathbb{Y}$ is given by:*

- *A function $\mathbb{X}_0 \xrightarrow{F} \mathbb{Y}_0$;*
- *For each pair of $\mathbb{X}$-objects, a $\mathscr{V}$-map, $\mathbb{X}(A, B) \xrightarrow{F_{A,B}} \mathbb{Y}(FA, FB)$;*

such that $\mathsf{c}; F_{A,C} = (F_{A,B} \otimes F_{B,C}); \mathsf{c}$ *and* $\mathsf{Id}; F = \mathsf{Id}$.

Definition 5.2.3. *A $\mathscr{V}$-natural transformation α is a family of $\mathscr{V}$-maps:*

$$\mathbb{X}(A, B) \xrightarrow{\alpha} \mathbb{Y}(FA, GB)$$

such that "$\alpha(f; g) = \alpha(f); G(g) = F(f); \alpha(g)$":

$$\begin{tikzcd}
\mathbb{X}(A, B) \otimes \mathbb{X}(B, C) \arrow[r, "\alpha \otimes G_{B,C}"] \arrow[d, "F_{A,B} \otimes \alpha"'] & \mathbb{Y}(FA, GB) \otimes \mathbb{Y}(GB, GC) \arrow[d, "\mathsf{c}"] \\
\mathbb{Y}(FA, FB) \otimes \mathbb{Y}(FB, GC) \arrow[r, "\mathsf{c}"'] & \mathbb{Y}(FA, GB)
\end{tikzcd}$$

Note that the above diagram implies that $\mathsf{c}; \alpha$ factors the square from the top left to the bottom right.

To get the natural transformation in the form $\alpha_A : FA \longrightarrow GA$ take $\alpha_A := \alpha(1_A)$.

Definition 5.2.4. *Let $\mathscr{V}$ be a symmetric monoidal category, and let $\underline{C}$ be a $\mathscr{V}$-category. A **power** of $X \in \underline{C}$ by $V \in \mathscr{V}$ is an object $V \pitchfork X$ together with a family of $\mathscr{V}$-natural isomorphisms*

$$\underline{C}(Y, V \pitchfork X) \simeq \mathscr{V}(V, \underline{C}(Y, X))$$

*Likewise a **copower** is $V \cdot X$ with $\mathscr{V}$ natural isomorphisms*

$$\underline{C}(V \cdot X, Y) \simeq \mathscr{V}(V, \underline{C}(X, Y))$$

Lemma 5.2.5. *Let $\mathscr{V}$ be a symmetric monoidal closed category. Then*

1. *$\mathscr{V}$ is self-enriched. $\mathscr{V}(A, B) := [A, B]$. The identity is $\lambda x.x$ and composition is given by $\lambda f g x.g(f(x))$.*

2. *$\mathscr{V}$ has powers. The power is $[V, Y]$. The required isomorphism $[Y, [V, X]] \simeq [V, [Y, X]]$ is the swap map: $\lambda f v y.f(y)(v)$.*

3. *$\mathscr{V}$ has copowers. The copower is $V \otimes Y$. The required isomorpism $[A \otimes B, C] \simeq [A, [B, C]]$ is the internal curry, which is the last map in the following sequence of inferences:*

$$\dfrac{\dfrac{B \otimes A \otimes [A \otimes B, C] \xrightarrow{c_\otimes} A \otimes B \otimes [A \otimes B, C] \xrightarrow{\mathsf{ev}} C}{A \otimes [A \otimes B, C] \xrightarrow{\lambda(c_\otimes; \mathsf{ev})} [B, C]}}{[A \otimes B, C] \xrightarrow{\lambda(\lambda(c_\otimes; \mathsf{ev}))} [A, [B, C]]}$$

Proof.

1. That the described gives a self-enrichment, we must show associativity and the identity laws, which we prove using λ-notation. For associativity

   $$(fg)h = \lambda y.h(\lambda x.g(f(x))y) = \lambda y.h(g(f(y))) = \lambda y.(\lambda z.h(g(z)))(f(y)) = f(gh)$$

   The unit laws follow from similar calculuations.

2. As $[Y, [V, X]] \simeq [V, [Y, X]]$ as witnessed by the swap isomorphism, the proof is done.

3. The curry map in the theorem is easily shown to be an isomorphism.

$\square$

In the above proof, points *2,3* depend on the symmetry explicitly.

5.2.1 Monoidal strength

Kock, 1972 demonstrated that an endofunctor $\mathscr{V} \xrightarrow{T} \mathscr{V}$ on a monoidal closed category $\mathscr{V}$ gives rise to a $\mathscr{V}$-enriched functor $\mathscr{V} \longrightarrow \mathscr{V}$ precisely when it is a *monoidally strong* functor.

Let $\mathbb{X}$ be a monoidal category. A **(right) monoidal strength** for an endofunctor $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is a natural transformation:

$$A \otimes TB \xrightarrow{\theta} T(A \otimes B)$$

that is coherent with the associator and unitor:

$$\begin{tikzcd}
(A\otimes B)\otimes TC \arrow[rr, "\theta"] \arrow[d, "a_\otimes"'] & & T((A\otimes B)\otimes C) \arrow[d, "Ta_\otimes"] \\
A\otimes(B\otimes TC) \arrow[r, "1\otimes\theta"'] & A\otimes T(B\otimes C) \arrow[r, "\theta"'] & T(A\otimes(B\otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes TA \arrow[rr, "\theta"] \arrow[dr, "u_L"'] & & T(I\otimes A) \arrow[dl, "Tu_L"] \\
& TA &
\end{tikzcd}$$

There is a dual notion of **left monoidal strength**. This is a map

$$\theta_L : TA \otimes B \longrightarrow T(A \otimes B)$$

that satisfies:

$$\begin{tikzcd}
TA\otimes(B\otimes C) \arrow[rr, "\theta_L"] \arrow[d, "a^{-1}"'] & & T(A\otimes(B\otimes C)) \arrow[d, "Ta^{-1}"] \\
(TA\otimes B)\otimes C \arrow[r, "\theta_L\otimes 1"'] & T(A\otimes B)\otimes C \arrow[r, "\theta_L"'] & T((A\otimes B)\otimes C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
TA\otimes I \arrow[rr, "\theta_L"] \arrow[dr, "u_R"'] & & T(A\otimes I) \arrow[dl, "Tu_R"] \\
& TA &
\end{tikzcd}$$

A well known result is:

Proposition 5.2.6. *In a symmetric monoidal category, to have a monoidal strength is equivalent to having a left strength.*

Lemma 5.2.7. *When $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is a monoidal functor, and θ is a strength, then the following commutes:*

$$\begin{tikzcd}
I \arrow[rr, "m_I"] \arrow[d, "u_R^{-1}"'] & & TI \\
I\otimes I \arrow[r, "1\otimes m_I"'] & I\otimes TI \arrow[r, "\theta"'] & T(I\otimes I) \arrow[u, "Tu_R"']
\end{tikzcd}$$

When T is a symmetric monoidal functor, we may also ask a strength to satisfy the following coherence which we call **symmetric strength**.

$$\begin{tikzcd}
(A\otimes TB)\otimes(C\otimes TD) \arrow[r, "\theta\otimes\theta"] \arrow[d, "\mathsf{ex}"'] & T(A\otimes B)\otimes T(C\otimes D) \arrow[d, "m_T"] \\
(A\otimes C)\otimes(TB\otimes TD) \arrow[d, "1\otimes m_T"'] & T((A\otimes B)\otimes(C\otimes D)) \arrow[d, "T\mathsf{ex}"] \\
(A\otimes C)\otimes T(B\otimes D) \arrow[r, "\theta"'] & T((A\otimes C)\otimes(B\otimes D))
\end{tikzcd}$$

The strength we deal with in this writeup arises canonically from a symmetric monoidal transformation. The following gives an equivalence between symmetric monoidal functors with a unit $1 \xrightarrow{\eta} T$ and symmetrically strong functors.

Proposition 5.2.8. *When $\mathbb{X}$ is a symmetric monoidal category and $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is a monoidal functor (T, m_T, m_I) then*

1. *For any monoidal natural transformation: $1 \xrightarrow{\eta} T$,*

$$\theta := A\otimes TB \xrightarrow{\eta\otimes 1} TA\otimes TB \xrightarrow{m_T} T(A\otimes B)$$

is a strength.

2. *When T is additionally a symmetric monoidal functor, the induced θ from proposition 5.2.8.1 is a symmetric strength.*

3. *Conversely, when θ is a symmetric strength, then*

$$\eta := B \xrightarrow{u_R^{-1}} B \otimes I \xrightarrow{1 \otimes m_I} B \otimes TI \xrightarrow{\theta} T(B \otimes I) \xrightarrow{Tu_R} TB$$

is a monoidal natural transformation.

4. *The constructions of proposition 5.2.8.2,3 are inverse to each other.*

Proof. Suppose $\mathbb{X} \xrightarrow{T} \mathbb{X}$ is a symmetric monoidal functor.

1. Let $1 \xrightarrow{\eta} T$ be a monoidal natural transformation. Then consider the following diagram.

$$\begin{tikzcd}
(X \otimes Y) \otimes TZ \arrow[rr, "\eta\otimes 1"] \arrow[dr, "(\eta\otimes 1)\otimes 1"] \arrow[dd, "a"'] & & T(X \otimes Y) \otimes TZ \arrow[r, "m_T"] & T((X \otimes Y) \otimes Z) \arrow[dddd, "Ta"] \\
& (TX \otimes Y) \otimes TZ \arrow[r, "(1\otimes\eta)\otimes 1"] \arrow[d, "a"] & (TX \otimes TY) \otimes TZ \arrow[u, "m_T\otimes 1"'] \arrow[dd, "a"] & \\
X \otimes (Y \otimes TZ) \arrow[r, "\eta\otimes 1"] \arrow[d, "1\otimes(\eta\otimes 1)"'] & TX \otimes (Y \otimes TZ) \arrow[dr, "1\otimes(\eta\otimes 1)"] & & \\
X \otimes (TY \otimes TZ) \arrow[rr, "\eta\otimes 1"] \arrow[d, "1\otimes m_T"'] & & TX \otimes (TY \otimes TZ) \arrow[d, "1\otimes m_T"] & \\
X \otimes T(Y \otimes Z) \arrow[rr, "\eta\otimes 1"] & & TX \otimes T(Y \otimes Z) \arrow[r, "m_T"] & T(X \otimes (Y \otimes Z))
\end{tikzcd}$$

For the unitor coherence, consider that $I \xrightarrow{\eta} TI = I \xrightarrow{m_I} TI$. Then the required coherence:

$$\begin{tikzcd}
I \otimes TA \arrow[r, "\eta\otimes 1"] \arrow[dr, "u_L"'] & TI \otimes TA \arrow[r, "m_T"] & T(I \otimes A) \arrow[dl, "Tu_K"] \\
& TA &
\end{tikzcd}$$

is the fact that T is a monoidal functor.

2. When η is a symmetric monoidal transformation:

$$\begin{tikzcd}
\begin{array}{c} A\otimes TB \\ \bigotimes \\ C\otimes TD \end{array} \arrow[r, "\begin{array}{c} \eta\otimes 1 \\ \otimes \\ \eta\otimes 1 \end{array}"] \arrow[d, "\mathsf{ex}"'] & \begin{array}{c} TA\otimes TB \\ \bigotimes \\ TC\otimes TD \end{array} \arrow[rr, "m_T\otimes m_T"] \arrow[d, "\mathsf{ex}"] & & \begin{array}{c} T(A\otimes B) \\ \bigotimes \\ T(C\otimes D) \end{array} \arrow[dd, "m_T"] \\
\begin{array}{c} A\otimes C \\ \bigotimes \\ TB\otimes TD \end{array} \arrow[r, "\begin{array}{c} \eta\otimes\eta \\ \otimes \\ 1 \end{array}"] \arrow[d, "1\otimes m_T"'] & \begin{array}{c} TA\otimes TC \\ \bigotimes \\ TB\otimes TD \end{array} \arrow[d, "m_T\otimes m_T"] & & \\
\begin{array}{c} A\otimes C \\ \bigotimes \\ T(B\otimes D) \end{array} \arrow[r, "\eta\otimes 1"'] & \begin{array}{c} T(A\otimes C) \\ \bigotimes \\ T(B\otimes D) \end{array} \arrow[r, "m_T"'] & T\begin{pmatrix} A\otimes C \\ \bigotimes \\ B\otimes D \end{pmatrix} \arrow[r, "T\mathsf{ex}"'] & T\begin{pmatrix} A\otimes B \\ \bigotimes \\ C\otimes D \end{pmatrix}
\end{tikzcd}$$

The top left square is naturality. The bottom left square is that η is monoidal. The right square follows as T is a symmetric monoidal functor.

3. Suppose T is a symmetric monoidal functor. First, consider the following diagram.

$$\begin{tikzcd}
(B\otimes C)\otimes(I\otimes I) \arrow[r, "1\otimes(m_I\otimes m_I)"] \arrow[d, "1\otimes(1\otimes m_I)"'] & (B\otimes C)\otimes(TI\otimes TI) \arrow[r, "1\otimes m_T"] & (B\otimes C)\otimes T(I\otimes I) \arrow[dd, "\theta"] \arrow[dddll, "1\otimes Tu_L"'] \\
(B\otimes C)\otimes(I\otimes TI) \arrow[d, "1\otimes\theta"'] \arrow[dd, bend left, "1\otimes u_L"] & & \\
(B\otimes C)\otimes T(I\otimes I) \arrow[d, "1\otimes Tu_L"'] & & T((B\otimes C)\otimes(I\otimes I)) \arrow[d, "T(1\otimes u_L)"] \\
(B\otimes C)\otimes TI \arrow[rr, "\theta"'] & & T((B\otimes C)\otimes I)
\end{tikzcd}$$

The bottom left corner is an axiom of strength. The right square is naturality of θ. The top left diagram follows as T is a monoidal functor.

Now, consider the following diagram

$$
\begin{tikzcd}
B\otimes C \arrow[r, "u_R^{-1}\otimes u_R^{-1}"] \arrow[d, "u_R^{-1}"'] & (B\otimes I)\otimes(C\otimes I) \arrow[r, "\begin{smallmatrix}1\otimes m_I\\ \otimes\\ 1\otimes m_I\end{smallmatrix}"] \arrow[d, "\mathsf{ex}"] & (B\otimes TI)\otimes(C\otimes TI) \arrow[r, "\theta\otimes\theta"] \arrow[d, "\mathsf{ex}"] & T(B\otimes I)\otimes T(C\otimes I) \arrow[r, "Tu_R\otimes Tu_R"] \arrow[d, "m_T"] & TB\otimes TC \arrow[dddd, "m_T"] \\
(B\otimes C)\otimes I \arrow[r, "1\otimes u_R^{-1}"] \arrow[ddd, "u_R"'] & (B\otimes C)\otimes(I\otimes I) \arrow[r, "1\otimes(m_I\otimes m_I)"] \arrow[d, "1\otimes(1\otimes m_I)"'] & (B\otimes C)\otimes(TI\otimes TI) \arrow[d, "1\otimes m_T"] & T((B\otimes I)\otimes(C\otimes I)) \arrow[dd, "T\mathsf{ex}"] & \\
 & (B\otimes C)\otimes(I\otimes TI) \arrow[d, "1\otimes\theta"'] & (B\otimes C)\otimes T(I\otimes I) \arrow[dr, "\theta"] & & \\
 & (B\otimes C)\otimes T(I\otimes I) \arrow[dr, "1\otimes Tu_R"'] & & T((B\otimes C)\otimes(I\otimes I)) \arrow[d, "T(1\otimes u_L)"] & \\
B\otimes C \arrow[r, "u_R^{-1}"'] & (B\otimes C)\otimes I \arrow[r, "1\otimes m_I"'] & (B\otimes C)\otimes TI \arrow[r, "\theta"] & T((B\otimes C)\otimes I) \arrow[r, "Tu_R"] & T(B\otimes C)
\end{tikzcd}
$$

The top left square is a monoidal coherence. The next square along the top is naturality. The next diagram along the top is the symmetry condition. The right square commutes as T is a symmetric monoidal functor. That the bottom left square commutes is lemma 5.2.7. The middle square is the first diagram in this proof.

$\square$

An **exponential strength** for $\mathbb{X} \xrightarrow{T} \mathbb{X}$, when $\mathbb{X}$ is monoidal closed, is a natural map

$$T[A,B] \xrightarrow{\psi} [A,TB]$$

that satisfies:

$$
\begin{tikzcd}
T[A\otimes B,C] \arrow[rr, "\psi"] \arrow[d, "T\mathsf{cur}"'] & & {[A\otimes B,TC]} \arrow[d, "\mathsf{cur}"] \\
T[B,[A,C]] \arrow[r, "\psi"'] & {[B,T[A,C]]} \arrow[r, "{[B,\psi]}"'] & {[B,[A,TC]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[d, "T\mathsf{cur}_I"'] \arrow[dr, "\mathsf{cur}_I"] & \\
T[I,A] \arrow[r, "\psi"'] & {[I,TA]}
\end{tikzcd}
$$

Kock proved the second point below.

Proposition 5.2.9 (Kock, 1972)**.** *For a monoidal closed category:*

1. *T has a strength precisely when T has an exponential strength;*

2. *Kock, 1972. T has a left strength iff T induces an $\mathbb{X}$-enriched functor $T : \mathbb{X} \longrightarrow \mathbb{X}$;*

3. *When $\mathbb{X}$ is a symmetric monoidal category, then having a functor with a strength, an exponential strength, or an $\mathbb{X}$-enriched functor are all equivalent.*

In a non-symmetric monoidal closed category note that we curry a map as

$$\begin{tikzcd} A \otimes [A, C] \arrow[r, "\mathsf{ev}"] & C \\ A \otimes B \arrow[u, "\lambda(f) \otimes 1"] \arrow[ur, "f"'] & \end{tikzcd}$$

For the first point then, given a right strength, we can form the following by currying:

$$\frac{A \otimes T[A, B] \xrightarrow{\theta} T(A \otimes [A, B]) \xrightarrow{T\mathsf{ev}} TB}{T[A, B] \longrightarrow [A, TB]}$$

Given a left strenght then, we can form the enriched functor by currying:

$$\frac{T(A) \otimes [A, B] \xrightarrow{\theta_L} T(A \otimes [A, B]) \xrightarrow{T\mathsf{ev}} TB}{[A, B] \longrightarrow [TA, TB]}$$

When one has the symmetry map, then a right strength can be expressed as a left strength, and hence an enriched functor, and indeed, all three notions: strength, exponential strength, and a self-enriched functor, become equivalent.

Proof. We give the proof of *1* here. The proof of *2* is similar, and moreover *2* can also be found in Kock, 1972 theorem 1.3. For *3*: in a symmetric monoidal category, strengths and left strengths are equivalent, together with *2*.

From a strength θ, define an exponential strength ψ as follows:

$$\frac{A \otimes T[A, B] \xrightarrow{\theta} T(A \otimes [A, B]) \xrightarrow{T\mathsf{ev}} TB}{T[A, B] \xrightarrow[\psi := \lambda(\theta T(\mathsf{ev}))]{} [A, TB]}$$

To see that this map is natural, use the naturality of λ together with the naturality of θ. To see that it satisfies the requirements of being an exponential strength, first consider the cur_I law.

$$
\begin{aligned}
& T\mathsf{cur}_I \psi \\
&= T\mathsf{cur}_I \lambda(\theta T(\mathsf{ev})) \\
&= \lambda((1 \otimes T\mathsf{cur}_I)\theta T(\mathsf{ev})) \qquad \text{naturality of } \lambda \\
&= \lambda(\theta T(1 \otimes \mathsf{cur}_I) T(\mathsf{ev})) \qquad \text{naturality of } \theta \\
&= \lambda(\theta T u_L) = \lambda(u_L) \qquad \text{coherence of } \theta \\
&= \mathsf{cur}_I
\end{aligned}
$$

Next, consider the curry law: we want to show $\psi\mathsf{cur} = T(\mathsf{cur})\psi[1,\psi]$. We do this by showing the universal property of maps into $[B,[A,TC]]$, so it suffices to show that

$$(1 \otimes (1 \otimes \psi\mathsf{cur})\mathsf{ev})\mathsf{ev} = (1 \otimes (1 \otimes T(\mathsf{cur})\psi[1,\psi])\mathsf{ev})\mathsf{ev}.$$

Then consider:

$$
\begin{aligned}
& (1 \otimes (1 \otimes \psi\mathsf{cur})\mathsf{ev})\mathsf{ev} \\
&= (1 \otimes (1 \otimes \psi))(1 \otimes \lambda(a^{-1}))\mathsf{ev} \\
&= (1 \otimes (1 \otimes \psi))a^{-1}\mathsf{ev} \\
&= a^{-1}(1 \otimes \psi)\mathsf{ev} \\
&= a^{-1}\theta T(\mathsf{ev}) \\
&= (1 \otimes \theta)\theta T(a^{-1})T(\mathsf{ev}) \qquad \theta \text{ is a strength} \\
&= (1 \otimes \theta)\theta T((1 \otimes \lambda(a^{-1}\mathsf{ev}))\mathsf{ev}) \\
&= (1 \otimes \theta)(1 \otimes T(\lambda(a^{-1}\mathsf{ev})))\theta T(\mathsf{ev}) \\
&= (1 \otimes \theta)(1 \otimes T((1 \otimes \mathsf{cur})\mathsf{ev}))\theta T(\mathsf{ev}) \\
&= (1 \otimes (1 \otimes T(\mathsf{cur})))(1 \otimes \theta T(\mathsf{ev}))\theta T(\mathsf{ev}) \\
&= (1 \otimes (1 \otimes T(\mathsf{cur})))(1 \otimes (1 \otimes \psi)\mathsf{ev})(1 \otimes \psi)\mathsf{ev} \\
&= (1 \otimes (1 \otimes T(\mathsf{cur})\psi))(1 \otimes \mathsf{ev}\psi)\mathsf{ev} \\
&= (1 \otimes (1 \otimes T(\mathsf{cur})\psi))(1 \otimes (1 \otimes [1,\psi])\mathsf{ev})\mathsf{ev} \\
&= (1 \otimes (1 \otimes T(\mathsf{cur})\psi[1,\psi])\mathsf{ev})\mathsf{ev}
\end{aligned}
$$

as required. Thus, ψ is an exponential strength.

Conversely, from an exponential strength ψ define a strength θ as follows. First recall the unit of the adjunction of the monoidal closed structure.

$$\frac{A \otimes B \xrightarrow{1} A \otimes B}{B \xrightarrow[\eta := \lambda(1)]{} [A, A \otimes B]}$$

Next define θ:

$$\frac{TB \xrightarrow{T(\eta)} T[A, A \otimes B] \xrightarrow{\psi} [A, T(A \otimes B)]}{A \otimes TB \xrightarrow[\theta := \lambda^{-1}(T(\eta)\psi)]{} T(A \otimes B)}$$

To see that this map is natural, use the naturality of λ^{-1} together with the naturality of ψ. To see that this map is a monoidal strength consider first the unit law which is proved by the following calculation:

$$\begin{aligned}
&\theta T(u_L) \\
&= \lambda^{-1}(T(\eta)\psi)T(u_L) \\
&= \lambda^{-1}(T(\eta)\psi[1, Tu_L]) && \text{naturality of } \lambda^{-1} \\
&= \lambda^{-1}(T(\eta)T[1, u_L]\psi) && \text{naturality of } \psi \\
&= \lambda^{-1}(T(\lambda(1)[1, u_L])\psi) \\
&= \lambda^{-1}(T(\mathsf{cur}_I)\psi) \\
&= \lambda^{-1}(\mathsf{cur}_I) && \text{coherence of } \psi \\
&= u_L
\end{aligned}$$

Next, we show the associativity law: $\theta T(a_\otimes) = a_\times(1 \otimes \theta)\theta$. However, before we will show the following identity: $\lambda(a)\mathsf{cur} = \lambda(\eta)$. This is proved by the universal property for maps into $[X,Y]$.

$$
\begin{aligned}
&(1\otimes\lambda(a)\mathsf{cur})\mathsf{ev}\\
&=(1\otimes\lambda(a))\lambda(a^{-1}\mathsf{ev})\\
&=\lambda(1\otimes(1\otimes\lambda(a))a^{-1}\mathsf{ev})\\
&=\lambda(a^{-1}(1\otimes\lambda(a))\mathsf{ev})\\
&=\lambda(a^{-1}a)=\lambda(1)=\eta\\
&=(1\otimes\lambda(\eta))\mathsf{ev}
\end{aligned}
$$

By uniqueness $\lambda(a)\mathsf{cur}=\lambda(\eta)$.

Next, we expand $a_\otimes(1\otimes\theta)\theta$ to obtain:

$$
\begin{aligned}
&a_\otimes(1\otimes\theta)\theta\\
&=a_\otimes(1\otimes\theta)\lambda^{-1}(T(\eta)\psi)\\
&=a_\otimes\lambda^{-1}(\theta\,T(\eta)\psi)\\
&=a_\otimes\lambda^{-1}(\lambda^{-1}(T(\eta)\psi)T(\eta)\psi)\\
&=a_\otimes\lambda^{-1}(\lambda^{-1}(T(\eta)\psi[1,T(\eta)\psi]))\\
&=a_\otimes\lambda^{-1}(\lambda^{-1}(T(\eta)T[1,\eta]\psi[1,\psi])) \qquad \text{naturality of } \psi\\
&=a_\otimes\lambda^{-1}(\lambda^{-1}(T(\lambda(\eta))\psi[1,\psi]))\\
&=a_\otimes(1\otimes(1\otimes T(\lambda(\eta))))(1\otimes(1\otimes\psi)\mathsf{ev}\psi)\mathsf{ev}
\end{aligned}
$$

On the other hand, we expand $\theta\,T(a_\otimes)$. First note that the cur is an isomorphism, and its inverse is $\mathsf{cur}^{-1}:=\lambda(a_\otimes(1\otimes\mathsf{ev})\mathsf{ev})$. The curry coherence on exponential strengths can be restated as $\psi=T(\mathsf{cur})\psi[1,\psi]\mathsf{cur}^{-1}$. Now consider the calculation:

$$
\begin{aligned}
&\theta T(a_\otimes)\\
&=\lambda^{-1}(T(\eta)\psi)T(a_\otimes)\\
&=\lambda^{-1}(T(\eta)\psi[1,T(a_\otimes)]) \qquad \text{naturality of } \lambda^{-1}\\
&=\lambda^{-1}(T(\eta)T[1,a_\otimes]\psi) \qquad \text{naturality of } \psi\\
&=\lambda^{-1}(T(\lambda(a_\otimes))\psi)\\
&=(1\otimes T(\lambda(a_\otimes)))\lambda^{-1}(\psi) \qquad \text{naturality of } \lambda^{-1}\\
&=(1\otimes T(\lambda(a_\otimes)))\lambda^{-1}(T(\mathsf{cur})\psi[1,\psi]\mathsf{cur}^{-1}) \qquad \text{coherence for } \psi\\
&=(1\otimes T(\lambda(a_\otimes)))(1\otimes T(\mathsf{cur}))\lambda^{-1}(\psi[1,\psi]\mathsf{cur}^{-1})
\end{aligned}
$$

Now, we separately consider, $\lambda^{-1}(\psi[1,\psi]\mathsf{cur}^{-1})$:

$$
\begin{aligned}
&\lambda^{-1}(\psi[1,\psi]\mathsf{cur}^{-1})\\
&=(1\otimes\psi[1,\psi])(1\otimes\mathsf{cur}^{-1})\mathsf{ev} \qquad \text{naturality of } \lambda^{-1}\\
&=(1\otimes\psi[1,\psi])a_\otimes(1\otimes\mathsf{ev})\mathsf{ev}\\
&=(1\otimes\psi)a_\otimes(1\otimes(1\otimes[1,\psi])\mathsf{ev})\mathsf{ev}\\
&=(1\otimes\psi)a_\otimes(1\otimes\mathsf{ev}\psi)\mathsf{ev}\\
&=a_\otimes(1\otimes(1\otimes\psi)\mathsf{ev}\psi)\mathsf{ev}
\end{aligned}
$$

Putting this calculation together with the calculation for $\theta T(a_\otimes)$, followed by the calculation for $\lambda(\eta)=\lambda(a_\otimes)\mathsf{cur}$ shows that, followed by the calculation for $a_\otimes(1\otimes\theta)\theta$ shows that

$$
\begin{aligned}
&\theta T(a_\otimes)\\
&=(1\otimes T(\lambda(a_\otimes)\mathsf{cur}))a_\otimes(1\otimes(1\otimes\psi)\mathsf{ev}\psi)\mathsf{ev}\\
&=(1\otimes T(\lambda(\eta)))a_\otimes(1\otimes(1\otimes\psi)\mathsf{ev}\psi)\mathsf{ev}\\
&=a_\otimes(1\otimes(1\otimes T(\lambda(\eta))))(1\otimes(1\otimes\psi)\mathsf{ev}\psi)\mathsf{ev}\\
&=a_\otimes(1\otimes\theta)\theta
\end{aligned}
$$

as required. Thus, θ is a strength.

To see that these constructions are inverses of each other:

$$
\begin{aligned}
&\lambda^{-1}(T(\eta)\lambda(\theta T(\mathsf{ev})))\\
&= \lambda^{-1}(\lambda((1\otimes T(\eta))\theta T(\mathsf{ev}))) \qquad \text{naturality } \lambda\\
&= (1\otimes T(\eta))\theta T\mathsf{ev}\\
&= \theta T((1\otimes\eta)\mathsf{ev}) \qquad \text{naturality } \phi\\
&= \theta T(1) = \theta
\end{aligned}
$$

For the other way around:

$$
\begin{aligned}
&\lambda(\lambda^{-1}(T(\eta)\psi)T(\mathsf{ev}))\\
&= \lambda(\lambda^{-1}(T(\eta)\psi[1, T\mathsf{ev}])) \qquad \text{naturality}\lambda^{-1}\\
&= T(\eta)\psi[1, T\mathsf{ev}]\\
&= T(\eta[1, \mathsf{ev}])\psi \qquad \text{naturality } \psi\\
&= T(\lambda(\mathsf{ev}))\psi \qquad \text{naturality } \lambda\\
&= T(1)\psi = \psi
\end{aligned}
$$

□

5.2.2 Strong transformations

As strong endofunctors on a monoidally closed category may be internalized, strong natural transformation express what it means for a natural transformations to be internalizable.

Let S have a strength θ_S and let T have a cartesian strength θ_T. Let $S \overset{\alpha}{\Longrightarrow} T$ be natural. α is a **monoidal strong natural transformation** when the following commutes.

$$
\begin{tikzcd}
A\otimes SB \arrow[r, "1\times\alpha"] \arrow[d, "\theta_S"'] & A\otimes TB \arrow[d, "\theta_T"] \\
S(A\otimes B) \arrow[r, "\alpha"'] & T(A\otimes B)
\end{tikzcd}
$$

When $\mathbb{X}$ is monoidal closed, S and T also have exponential strengths, and we say that α is an **exponentially strong natural transformation** when the following commutes

$$\begin{tikzcd}
S([A,B]) \arrow[r, "\alpha"] \arrow[d, "\psi_S"'] & T([A,B]) \arrow[d, "\psi_T"] \\
{[A,S(B)]} \arrow[r, "{[A,\alpha]}"'] & {[A,T(B)]}
\end{tikzcd}$$

In fact,

Proposition 5.2.10. *When $\mathbb{X}$ is monoidal closed, (S,θ_S) and (T,θ_T) are strong functors, and $S \overset{\alpha}{\Longrightarrow} T$ is natural, then α is monoidal strong if and only if it is exponentially strong.*

Proof. Suppose that α is monoidal strong. We show it is exponentially strong by showing the following commutes

$$\begin{tikzcd}
S([A,B]) \arrow[r, "\alpha"] \arrow[d, "\lambda(\theta S(\mathsf{ev}))"'] & T([A,B]) \arrow[d, "\lambda(\theta T(\mathsf{ev}))"] \\
{[A,S(B)]} \arrow[r, "{[1,\alpha]}"'] & {[A,T(B)]}
\end{tikzcd}$$

$$\begin{aligned}
&\alpha\lambda(\theta_T T(\mathsf{ev})) \\
&= \lambda((1\otimes\alpha)\theta_T T(\mathsf{ev})) \\
&= \lambda(\theta_S \alpha T(\mathsf{ev})) && \alpha \text{ is strong} \\
&= \lambda(\theta_S S(\mathsf{ev})\alpha) \\
&= \lambda(\theta_S S(\mathsf{ev}))[1,\alpha] && \text{naturality of } \lambda
\end{aligned}$$

Next, suppose that α is exponentially strong. We show that it is monoidal strong by showing the following diagram commutes.

$$\begin{tikzcd}
A\otimes SB \arrow[r, "1\otimes\alpha"] \arrow[d, "\lambda^{-1}(T(\eta)\psi)"'] & A\otimes TB \arrow[d, "\lambda^{-1}(T(\eta)\psi)"] \\
S(A\otimes B) \arrow[r, "\alpha"'] & T(A\otimes B)
\end{tikzcd}$$

$$
\begin{aligned}
&(1\otimes\alpha)\lambda^{-1}(T(\eta)\psi)\\
&=\lambda^{-1}(\alpha T(\eta)\psi)\\
&=\lambda^{-1}(S(\eta)\alpha\psi)\\
&=\lambda^{-1}(S(\eta)\psi[1,\alpha]) && \alpha \text{ is exponentially strong}\\
&=\lambda^{-1}(S(\eta)\psi)\alpha
\end{aligned}
$$

□

Proposition 5.2.11 (Kock, 1972)**.** *When $\mathbb{X}$ is a closed monoidal category with left strong functors T and S and a strong natural transformation $\alpha : T \longrightarrow S$, then we get an induced $\mathbb{X}$-enriched natural transformation between the $\mathbb{X}$-enriched functors T and S.*

The above then states that $\mathbb{X}$-enriched natural transformations correspond to strong natural transformations of left strong monoidal functors.

5.2.3 Cartesian Strength

One can say a bit more about strength when the monoidal structure is the product. We call strengths with respect to $\times$ *Cartesian strength.*

When the monoidal structure on a category is the product, and a functor T preserves the product, obtaining a strength reduces to obtaining a natural transformation $1 \longrightarrow T$. Moreover the strength is symmetric.

First observe:

Observation 5.2.12. *Suppose T preserves products. Then any natural transformation $\eta : 1 \longrightarrow T$ satisfies $(\eta\times\eta)m_T = \eta$*

Proof. Note that m_T is the inverse of $\langle T\pi_0, T\pi_1\rangle$. Then consider,

$$
\begin{aligned}
&\eta\langle T\pi_0, T\pi_1\rangle\\
&=\left\langle \eta T\pi_0, \eta T\pi_1\right\rangle\\
&=\left\langle \pi_0\eta, \pi_1\eta\right\rangle = (\eta\times\eta) && \text{naturality}
\end{aligned}
$$

Then, multiplying both sides by m_T gives the desired result. □

Also note that $T(1)$ is a terminal object as T preserves products, hence, $\eta : 1 \longrightarrow T(1)$ is the unique map. Thus when T preserves products, any natural transformation is monoidal.

Proposition 5.2.13. *When $\mathbb{X}$ has products, and $\mathbb{X} \xrightarrow{T} \mathbb{X}$ preserves finite products, then for any natural $\eta : 1 \longrightarrow T$,*

$$A \times TB \xrightarrow{\eta \times 1} T(A) \times T(B) \xrightarrow{m_T} T(A \times B)$$

is a symmetric strength.

Proof. This follows immediately from proposition 5.2.8 and the above observation that η is monoidal. □

5.3 Tangent categories are strong

In this section we exhibit a strength for tangent categories and show that all the natural transformations of tangent structure are strong.

We specialize 5.2.13 to cartesian tangent categories.

Proposition 5.3.1. *In any cartesian tangent category, the following:*

$$\theta_T := A \times TB \xrightarrow{0 \times 1} T(A) \times T(B) \xrightarrow{m_T} T(A \times B)$$

is a strength.

Lemma 5.3.2. *In any cartesian tangent category, $\theta p = 1 \times p$, and the map θ_2 defined by the universal property in:*

$$\begin{tikzcd}
& B \times T_2(M) \arrow[rr, dashed, "\theta_2"] \arrow[dr] \arrow[ddl] & & T_2(B \times M) \arrow[dr] \arrow[ddl] & \\
& & B \times T(M) \arrow[rr, "\theta"] \arrow[ddl, "1\times p"] & & T(B \times M) \arrow[ddl, "p"] \\
B \times T(M) \arrow[rr, "\theta"] \arrow[dr, "1\times p"'] & & T(B \times M) \arrow[dr, "p"] & & \\
& B \times M \arrow[rr, equal] & & B \times M &
\end{tikzcd}$$

has $\theta_2 = B \times T_2(M) \xrightarrow{\langle 0,0\rangle \times 1} T_2B \times T_2M \xrightarrow{m_{T_2}} T_2(B \times M)$*, and is a strength.*

That $\theta p = 1 \times p$ uses that m_T is a morphism of tangent structure

$$(0 \times 1)m_T p = (0 \times 1)(p \times p) = 1 \times p$$

Proof. Since $\mathbb{X}$ is a cartesian tangent category, T_2 preserves products. The map $B \xrightarrow{\langle 0,0\rangle} T_2(B)$ is natural. Hence by proposition 5.2.13, the map $(\langle 0,0\rangle \times 1)m_{T_2}$ is a strength.

It remains to show that $\theta_2 = (\langle 0,0\rangle \times 1)m_{T_2}$. Consider the construction of m_{T_2}:

$$\begin{tikzcd}
 & T_2(M)\times T_2(B) \arrow[rr, dashed, "m_{T_2}"] \arrow[dr, "\pi_1\times\pi_1"] \arrow[ddl, "\pi_0\times\pi_0"'] & & T_2(M\times B) \arrow[dr] \arrow[ddl] & \\
 & & T(M)\times T(B) \arrow[rr, "m_T"] \arrow[ddl, "p\times p"] & & T(M\times B) \arrow[ddl, "p"] \\
T(M)\times T(B) \arrow[rr, "m_T"] \arrow[dr, "p\times p"'] & & T(M\times B) \arrow[dr, "p"] & & \\
 & M\times B \arrow[rr, equal] & & M\times B &
\end{tikzcd}$$

Then

$$\begin{aligned}
&(\langle 0,0\rangle \times 1)m_{T_2}\pi_1 \\
&= (\langle 0,0\rangle \times 1)(\pi_1 \times \pi_1)m_T \\
&= (1 \times \pi_1)(0 \times 1)m_T \\
&= (1 \times \pi_1)\theta
\end{aligned}$$

Similarly, $(\langle 0,0\rangle \times 1)m_{T_2}\pi_0 = (1 \times \pi_0)\theta$. Thus, by the universality of θ_2 with respect to this property, $\theta_2 = (\langle 0,0\rangle \times 1)m_{T_2}$. □

Our goal is now to show that every one of the structural transformations for tangent categories is strong.

Theorem 5.3.3. *In any tangent category,* $p, l, c, +, 0$ *are all strong natural transformations.*

Proof. Note that $m_T \langle T(\pi_0), T(\pi_1)\rangle = 1_{T(A)\times T(B)} = \langle \pi_0, \pi_1\rangle$. This implies that:

$$\theta T(\pi_0) = \pi_0 0 \qquad \text{and} \qquad \theta T(\pi_1) = \pi_1$$

The required diagram for p is

$$\begin{tikzcd}
A \times TB \arrow[r, "1\times p"] \arrow[d, "\theta"'] & A \times B \arrow[d, equal] \\
T(A \times B) \arrow[r, "p"'] & A \times B
\end{tikzcd}$$

Which is done in lemma 5.3.2.

The required diagram for c is

$$\begin{tikzcd}
A \times T^2 B \arrow[r, "1\times c"] \arrow[d, "\theta T(\theta)"'] & A \times T^2 B \arrow[d, "\theta T(\theta)"] \\
T^2(A \times B) \arrow[r, "c"'] & T^2(A \times B)
\end{tikzcd}$$

As T preserves products, $T^2(A \times B)$ is a product, and it then suffices to check that the two maps are simultaneously equal when postcomposed with $T^2(\pi_0)$ and $T^2(\pi_1)$.

Then,

$$\begin{aligned}
&(1 \times c)\theta T(\theta) T^2(\pi_0) \\
&= (1 \times c)\theta T(\pi_0) T(0) \\
&= (1 \times c)\pi_0 0 T(0) \\
&= \pi_0 0 T(0) \\
&= \pi_0 00 \qquad \text{naturality } 0 \\
&= \theta T(\pi_0) 0 \\
&= \theta T(\pi_0) T(0) c \qquad c \text{ is a bundle morphism} \\
&= \theta T(\theta) T^2(\pi_0) c \\
&= \theta T(\theta) c T^2(\pi_0)
\end{aligned}$$

and

$$
\begin{aligned}
&(1 \times c)\theta T(\theta)T^2(\pi_1)\\
&= (1 \times c)\theta T(\pi_1)\\
&= (1 \times c)\pi_1\\
&= \pi_1 c\\
&= \theta T(\pi_1)c\\
&= \theta T(\theta)T^2(\pi_1)c\\
&= \theta T(\theta)c T^2(\pi_1)
\end{aligned}
$$

Thus, c, is strong.
The required diagram for 0 is

$$
\begin{tikzcd}
A \times B \arrow[r, "1\times 0"] \arrow[d, equal] & A \times TB \arrow[d, "\theta"] \\
A \times B \arrow[r, "0"'] & T(A \times B)
\end{tikzcd}
$$

This is just observation 5.2.12, $(1 \times 0)\theta = (0 \times 0)m_T = 0$; thus 0 is strong.
The required diagram for $+$ is the outer square in the following.

$$
\begin{tikzcd}
A \times T_2 B \arrow[r, "1\times +"] \arrow[d, "\langle 0,0\rangle \times 1"'] & A \times TB \arrow[d, "0\times 1"] \\
T_2(A) \times T_2(B) \arrow[r, "+\times +"] \arrow[d, "m_{T_2}"'] & T(A) \times T(B) \arrow[d, "m_T"] \\
T_2(A \times B) \arrow[r, "+"'] & T(A \times B)
\end{tikzcd}
$$

The top square commutes as

$$\langle 0,0\rangle + = \left\langle 0p0, 0\right\rangle + = 0\left\langle p0, 1\right\rangle + = 0$$

For the bottom square, first note that:

$$
\begin{aligned}
&+\langle T(\pi_0), T(\pi_1)\rangle \\
&= \langle +T(\pi_0), +T(\pi_1)\rangle \\
&= \langle T_2(\pi_0)+, T_2(\pi_1)+\rangle \\
&= \langle T_2(\pi_0), T_2(\pi_1)\rangle(+\times+)
\end{aligned}
$$

Then precomposing both sides with m_{T_2} and postcomposing both sides with m_T we get,

$$m_{T_2}+ = (+\times+)m_T$$

so that $+$ is a strong transformation. Finally, consider the diagram for l.

$$
\begin{tikzcd}
A \times TB \arrow[r, "1\times l"] \arrow[d, "\theta"'] & A \times T^2B \arrow[d, "\theta T(\theta)"] \\
T(A\times B) \arrow[r, "l"'] & T^2(A\times B)
\end{tikzcd}
$$

We prove the diagram commutes by showing

$$\theta l \left\langle T^2(\pi_0), T^2(\pi_1)\right\rangle = (l\times 1)\theta T(\theta)\left\langle T^2(\pi_0), T^2(\pi_1)\right\rangle$$

so that postcomposing with the inverse, m_{T_2} gives the desired result. Also note that from lemma 5.1.2, we have $0l = 00$. Now,

$$
\begin{aligned}
&(1\times l)\theta T(\theta)\left\langle T^2\pi_0, T^2\pi_1\right\rangle \\
&= \left\langle (1\times l)\theta T(\theta)T^2\pi_0, (1\times l)\theta T(\theta)T^2\pi_1\right\rangle \\
&= \langle (1\times l)\theta T(\pi_0)T0(1\times l)\theta T\pi_1\rangle \\
&= \langle (1\times l)\pi_0 0T0, (1\times l)\pi_1\rangle \\
&= \langle \pi_0 00, \pi_1 l\rangle \\
&= \langle \pi_0 0l, \pi_1 l\rangle \\
&= \langle \theta T(\pi_0)l, \theta T(\pi_1)l\rangle \\
&= \theta l\left\langle T^2\pi_0, T^2\pi_1\right\rangle
\end{aligned}
$$

Thus l is a strong natural transformation. Hence all of the structural transformation of a tangent category are strong. $\square$

5.4 Cartesian closed tangent categories

In this section, we combine the theorems we've proved about strength with the fact that the tangent functor of a Cartesian tangent category is strong and that all the structural natural transformations are strong transformations. This implies that when a Cartesian tangent category is Cartesian closed, the tangent functor is exponentially strong, and all the structural natural transformations are exponentially strong. We will state explicitly what this means.

Corollary 5.4.1. *Let $\mathbb{X}$ be a Cartesian tangent category and also a Cartesian closed category. Then every structural transformation is exponentially strong.*

We elaborate on the above, and write out explicitly what the exponential strength means for each of the transformations.

For p we have

$$\begin{tikzcd}
T([A,B]) \arrow[r, "p"] \arrow[d, "\psi"'] & {[A,B]} \arrow[d, equal] \\
{[A,T(B)]} \arrow[r, "{[A,p]}"'] & {[A,B]}
\end{tikzcd}$$

For l we have

$$\begin{tikzcd}
T([A,B]) \arrow[r, "l"] \arrow[dd, "\psi"'] & T^2([A,B]) \arrow[d, "T(\psi)"] \\
 & T([A,T(B)]) \arrow[d, "\psi"] \\
{[A,T(B)]} \arrow[r, "{[A,l]}"'] & {[A,T^2(B)]}
\end{tikzcd}$$

For c we have

$$\begin{tikzcd} T^2([A,B]) \arrow[r, "c"] \arrow[d, "T(\psi)\psi"'] & T^2([A,B]) \arrow[d, "T(\psi)\psi"] \\ {[A,T^2(B)]} \arrow[r, "{[A,c]}"'] & {[A,T^2(B)]} \end{tikzcd}$$

For $+$ we have

$$\begin{tikzcd} T_2([A,B]) \arrow[r, "+"] \arrow[d, "\psi_2"'] & T([A,B]) \arrow[d, "\psi"] \\ {[A,T_2(B)]} \arrow[r, "{[A,+]}"'] & {[A,T_2(B)]} \end{tikzcd}$$

For 0 we have

$$\begin{tikzcd} {[A,B]} \arrow[r, "0"] \arrow[d, equal] & T([A,B]) \arrow[d, "\psi"] \\ {[A,B]} \arrow[r, "{[A,0]}"'] & {[A,T(B)]} \end{tikzcd}$$

In a cartesian closed tangent category consider the partial tangent:

$$\dfrac{\dfrac{\dfrac{A \times B \xrightarrow{f} C}{B \xrightarrow{\lambda(f)} [A,C]}}{TB \xrightarrow{T(\lambda(f))} T[A,C] \xrightarrow{\psi} [A,TC]}}{A \times TB \xrightarrow[\lambda^{-1}(T(\lambda(f))\psi)]{} TC}$$

Lemma 5.4.2. *In a cartesian closed tangent category,*

$$\lambda^{-1}(T(\lambda(f))\psi) = \theta T(f)$$

Proof. Consider the following calculation:

$$
\begin{aligned}
&\lambda^{-1}(T(\lambda(f))\psi) \\
&= \lambda^{-1}(T(\lambda(f))\lambda(\theta T(\mathsf{ev}))) \\
&= \lambda^{-1}(\lambda((1 \times T(\lambda(f)))\theta T(\mathsf{ev}))) \\
&= (1 \times T(\lambda(f)))\theta T(\mathsf{ev}) \\
&= \theta T((1 \times \lambda(f))\mathsf{ev}) \\
&= \theta T(f)
\end{aligned}
$$

$\square$

In the sequel, we will sometimes refer to $\theta T(f)$ as above by $T_B(f)$.

5.5 Differential objects and coherent closure

The notion of differential object of a tangent category was introduced by Cockett and Cruttwell, 2014b to play the role that $\mathscr{R}$ vector spaces play in SMan. In this section we show that if a cartesian closed tangent category satisfies a coherence, then its differential objects form a differential λ-category.

Definition 5.5.1. *A* ***differential object*** *in a cartesian tangent category is a commutative monoid* (A, σ, ζ) *and a map* $TA \xrightarrow{\hat{p}} A$ *such that* $A \xleftarrow{\hat{p}} TA \xrightarrow{p} A$ *is a product diagram, and where the tangent addition and monoid structure on* A *are compatible in the following sense:*

- *The following diagrams commute:*

$$
\begin{tikzcd}
T1 \arrow[r, "!"] \arrow[d, "T\zeta"'] & 1 \arrow[d, "\zeta"] \\
TA \arrow[r, "\hat{p}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T(A \times A) \arrow[r, "\langle T\pi_0, T\pi_1 \rangle"] \arrow[d, "T\sigma"'] & TA \times TA \arrow[r, "\hat{p} \times \hat{p}"] & A \times A \arrow[d, "\sigma"] \\
TA \arrow[rr, "\hat{p}"'] & & A
\end{tikzcd}
$$

- *The following diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "0"'] & 1 \arrow[d, "\zeta"] \\ TA \arrow[r, "\hat{p}"'] & A \end{tikzcd} \qquad \begin{tikzcd} T_2A \arrow[r, "\langle \pi_0\hat{p}, \pi_1\hat{p}\rangle"] \arrow[d, "+"'] & A \times A \arrow[d, "\sigma"] \\ TA \arrow[r, "\hat{p}"'] & A \end{tikzcd}$$

- *The lift and $\hat{p}$ are coherent:*

$$\begin{tikzcd} TA \arrow[r, "l"] \arrow[d, "\hat{p}"'] & T^2A \arrow[d, "T\hat{p}"'] \\ A & TA \arrow[l, "\hat{p}"] \end{tikzcd}$$

The following theorem summarizes the role that differential objects in a Cartesian tangent category play.

Theorem 5.5.2 (Cockett and Cruttwell, 2014b theorem 4.11)**.** *The differential objects of a tangent category have the structure of a Cartesian differential category whose differential combinator is*

$$\frac{A \xrightarrow{f} B}{A \times A \xrightarrow[\langle \pi_0, \pi_1\rangle_T]{} TA \xrightarrow[Tf]{} TB \xrightarrow[\hat{p}]{} B}$$

Recall, that in any cartesian differential category the partial derivative of $A \times B \xrightarrow{f} C$ with respect to B is defined:

$$D_B[f] := A \times (B \times B) \xrightarrow{\langle 0 \times \pi_0, 1 \times \pi_1\rangle} (A \times B) \times (A \times B) \xrightarrow{D[f]} C$$

Which for the differential objects of a tangent category comes out to

$$A \times (B \times B) \xrightarrow{\langle !_A\zeta_A \times \pi_0, 1 \times \pi_1\rangle} (A \times B)^2 \xrightarrow{\langle \pi_0, \pi_1\rangle_T} T(A \times B) \xrightarrow{T(f)} T(C) \xrightarrow{\hat{p}} C$$

For the above, recall that the zero section 0 of p for differential objects is $!_A\zeta_A$ Cockett and Cruttwell, 2014b.

Lemma 5.5.3. *When A, B, C are differential objects then for each $A \times B \xrightarrow{f} C$ we have*

$$D_B[f] = A \times (B \times B) \xrightarrow{1 \times \langle \pi_0, \pi_1\rangle_T} A \times T(B) \xrightarrow{T_B(f)} T(C) \xrightarrow{\hat{p}} C$$

Proof. Consider:

$$\begin{tikzcd}
A\times(B\times B) \arrow[r, "1\times\langle\pi_0,\pi_1\rangle_T"] \arrow[drr, bend right, "\langle !\zeta,1\rangle\times 1"'] & A\times TB \arrow[r, "0\times 1"] \arrow[dr, "\left\langle 0\hat{p},0p\right\rangle\times\left\langle \hat{p},p\right\rangle"'] & TA\times TB \arrow[r, "m_T"] \arrow[d, "\left\langle \hat{p},p\right\rangle\times\left\langle \hat{p},p\right\rangle"] & T(A\times B) \arrow[r, "T(f)"] & TC \arrow[d, "\hat{p}"] \\
 & & (A\times A)\times(B\times B) \arrow[r, "\mathsf{ex}"'] & (A\times B)\times(A\times B) \arrow[r, "D[f]"'] \arrow[u, "\langle\pi_0,\pi_1\rangle_T"'] & C
\end{tikzcd}$$

The right and middle squares are definition. The middle triangle is obvious. The left triangle follows as $\langle\pi_0,\pi_1\rangle_T\left\langle\hat{p},p\right\rangle = 1_{A\times A}$ together with the fact that $0p = 1$ from tangent structure, and finally $0\hat{p} = !\zeta$ as B is a differential object. Finally, note that

$$(\langle !\zeta,1\rangle\times 1)\mathsf{ex} = \langle !\zeta\times\pi_0, 1\times\pi_1\rangle$$

so that around the bottom is $D_B[f]$, and hence $D_B[f] = D'_B[f]$. $\square$

Lemma 5.5.4. *Let $\mathbb{X}$ be a cartesian closed tangent category, and V a differential object. Then*

$$[M,V] \xleftarrow{[M,\hat{p}]} [M,TV] \xleftarrow{\psi} T[M,V] \xrightarrow{p} [M,V]$$

is a product diagram if and only if ψ is an isomorphism.

Proof. Suppose ψ is an isomorphism. From corollary 5.4.1, $p = \psi[M,p]$. Since $[M,\_]$ preserves products:

$$\begin{tikzcd}
 & T[M,V] \arrow[d, "\psi", "\simeq"'] & \\
{[M,V]} & {[M,TV]} \arrow[l, "{[M,\hat{p}]}"'] \arrow[r, "{[M,p]}"] & {[M,V]}
\end{tikzcd}$$

is a product diagram.

Suppose that

$$[M,V] \xleftarrow{[M,\hat{p}]} [M,TV] \xleftarrow{\psi} T[M,V] \xrightarrow{p} [M,V]$$

is a product diagram. Then there is a unique ψ^{-1} that makes the following diagram commute.

$$\begin{tikzcd}
 & & {[M,TV]} \arrow[dll, "{[M,\hat{p}]}"'] \arrow[d, "\exists!.\psi^{-1}"] \arrow[drr, "{[M,p]}"] & & \\
{[M,V]} & {[M,TV]} \arrow[l, "{[M,\hat{p}]}"] & T[M,V] \arrow[l, "\psi"] \arrow[r, "\psi"'] & {[M,TV]} \arrow[r, "{[M,p]}"'] & {[M,V]}
\end{tikzcd}$$

But as $[M,TV]$ is also a product diagram, ψ^{-1} is the unique map that makes

$$\begin{tikzcd}
 & {[M,TV]} \arrow[dl, "{[M,\hat{p}]}"'] \arrow[d, "\psi^{-1}"] \arrow[dr, "{[M,p]}"] & \\
 & T[M,V] \arrow[d, "\psi"] & \\
{[M,V]} & {[M,TV]} \arrow[l, "{[M,\hat{p}]}"] \arrow[r, "{[M,p]}"'] & {[M,V]}
\end{tikzcd}$$

Thus $\psi^{-1}\psi = 1$. A similar proof shows that $\psi\psi^{-1} = 1$; thus, ψ is an isomorphism. $\square$

Definition 5.5.5. *A tangent category is **coherently closed** when for each N, and any object M, $T[M,N] \xrightarrow{\psi} [M,TN]$ is an isomorphism.*

Proposition 5.5.6. *Representable tangent categories are coherently closed.*

Proof. The canonical isomorphism

$$T([M,V]) = [D,[M,V]] \xrightarrow{\mathsf{sw}} [M,[D,V]] = [M,T(V)]$$

is $\psi = \lambda(\theta T(\mathsf{ev}))$. Consider $\theta T(\mathsf{ev})$ in the internal logic:

$$\begin{array}{ccccccc}
A \times [D,[A,B]] & \xrightarrow{[!,A]\times 1} & [D,A]\times[D,[A,B]] & \xrightarrow{\simeq} & [D,A\times[A,B]] & \xrightarrow{[D,\mathsf{ev}]} & [D,B] \\
(x, \lambda da.t\ d\ a) & \longmapsto & (\lambda d.x, \lambda da.t\ d\ a) & \longmapsto & \lambda d.(x, \lambda a.t\ d\ a)) & \longmapsto & \lambda d.t\ d\ x
\end{array}$$

If we curry the x in the above composite:

$$\lambda da.t\ d\ a \mapsto \lambda ad.t\ d\ a$$

which is precisely $[D,[A,B]] \xrightarrow{\mathsf{sw}} [A,[D,B]]$ as required. $\square$

Thus every model of SDG is a coherently closed tangent category.

Proposition 5.5.7. *A differential λ-category is coherently closed as a tangent category.*

Proof. Consider the map

$$A \times ([A,B] \times [A,B]) \xrightarrow{\langle 0,1\rangle \times 1} A^2 \times [A,B]^2 \xrightarrow{\mathsf{ex}} (A \times [A,B])^2 \xrightarrow{\langle D\mathsf{ev}, \pi_1 \mathsf{ev}\rangle} B \times B$$

We have

$$\begin{aligned}
&(\langle 0,1\rangle \times 1)\mathsf{ex}\,\langle D[\mathsf{ev}], \pi_1\mathsf{ev}\rangle \\
&= \langle 0 \times \pi_0, 1 \times \pi_1\rangle\,\langle D[\mathsf{ev}], \pi_1\mathsf{ev}\rangle \\
&= \langle \langle 0 \times \pi_0, 1 \times \pi_1\rangle\, D[\mathsf{ev}], (1 \times \pi_1)\mathsf{ev}\rangle \\
&= \left\langle \lambda^{-1}(D[\lambda(\mathsf{ev})]), (1 \times \pi_1)\mathsf{ev}\right\rangle \\
&= \left\langle \lambda^{-1}(D[1]), (1 \times \pi_1)\mathsf{ev}\right\rangle \\
&= \left\langle \lambda^{-1}(\pi_0), (1 \times \pi_1)\mathsf{ev}\right\rangle \\
&= \langle (1 \times \pi_0)\mathsf{ev}, (1 \times \pi_1)\mathsf{ev}\rangle
\end{aligned}$$

is the map that sends

$$(x, f, g) \mapsto (f\,x, g\,x)$$

Currying the x

$$(f, g) \mapsto \lambda x.(f\,x, g\,x)$$

This is the canonical isomorphism

$$[M,V] \times [M,V] \xrightarrow[\simeq]{m_\times} [M, V \times V]$$

$\square$

Proposition 5.5.8. *Let $\mathbb{X}$ be a coherently closed tangent category, and V be a differential object. For every M, $[M,V]$ may be given the structure of a differential object. That is, the differential objects are an exponential ideal of $\mathbb{X}$.*

Proof. Equip $[M,V]$ with structure of a monoid by

$$[M,V] \times [M,V] \xrightarrow{m_\times} [M, V \times V] \xrightarrow{[M,\sigma]} [M,V]$$

and zero given by $\lambda x.0$.

From 5.5.4, we have a product diagram:

$$[M,V] \xleftarrow{[M,\hat{p}]} [M,TV] \xleftarrow{\psi} T[M,V] \xrightarrow{p} [M,V]$$

It remains to show the coherences. First, we show that the commutative monoid structure on $[M,V]$ compatible with the bundle 0. Note the following diagram:

$$\begin{tikzcd}
{[M,V]} \arrow[rr, "0"] \arrow[dd, "{[M,!]}"'] \arrow[drr, "{[M,0]}"'] \arrow[ddd, bend right, "!"'] & & T([M,V]) \arrow[d, "\psi"] \arrow[dd, bend left, "\hat{p}"] \\
& & {[M,T(V)]} \arrow[d, "{[M,\hat{p}]}"] \\
{[M,1]} \arrow[rr, "{[M,\zeta]}"'] & & {[M,V]} \\
1 \arrow[u, "m_1"] \arrow[urr, "\zeta"'] & &
\end{tikzcd}$$

The side bits commute by definition. The top triangle commutes from corollary 5.4.1. The main square commutes by the assumption that V is a differential object. The bottom triangle is the definition of ζ.

Next, we show the compatibility of the commutative monoid structure with the bundle addition.

First, note that $[M,T_2(V)]$ is a pullback, and consider the commutativity of the following diagram:

$$\begin{tikzcd}
T_2([M,V]) \arrow[r, "\psi_2"] \arrow[d, "{\langle\pi_0,\pi_1\rangle}"'] & {[M,T_2(V)]} \arrow[ddl, "{\langle\pi_0,\pi_1\rangle}"'] \arrow[dd, "{[M,\langle\pi_0,\pi_1\rangle]}"] \\
T([M,V])\times T([M,V]) \arrow[d, "\psi\times\psi"'] & \\
{[M,T(V)]\times[M,T(V)]} \arrow[r, "m_\times"'] & {[M,T(V)\times T(V)]}
\end{tikzcd}$$

For the top square, ψ_2 is the unique map for which $\psi_2\pi_0 = \pi_0\psi$ and $\psi_2\pi_1 = \pi_1\psi$; the bottom square is definitional. The above square is the upper left corner in the following diagram, which proves the compatibility of the commutative monoid structure with the bundle addition.

$$
\begin{tikzcd}
T_2[M,V] \arrow[rr, "+"] \arrow[d, "\langle\pi_0,\pi_1\rangle"'] \arrow[dr, "\psi_2"] & & T[M,V] \arrow[d, "\psi"] \arrow[ddd, bend left, "\hat{p}"] \\
T[M,V]\times T[M,V] \arrow[d, "\psi\times\psi"'] \arrow[dd, bend right, "\hat{p}\times\hat{p}"'] & {[M,T_2V]} \arrow[r, "{[M,+]}"] \arrow[d, "{[M,\langle\pi_0,\pi_1\rangle]}"] & {[M,TV]} \arrow[dd, "{[M,\hat{p}]}"] \\
{[M,TV]\times[M,TV]} \arrow[r, "m_\times"'] \arrow[d, "{[M,\hat{p}]\times[M,\hat{p}]}"'] & {[M,TV\times TV]} \arrow[d, "{[M,\hat{p}\times\hat{p}]}"] & \\
{[M,V]\times[M,V]} \arrow[r, "m_\times"'] \arrow[rr, bend right, "\sigma"'] & {[M,V\times V]} \arrow[r, "{[M,\sigma]}"'] & {[M,V]}
\end{tikzcd}
$$

For the remaining parts of the diagram: the upper right square follows from corollary 5.4.1; the bottom right square is the assumption; the bottom left square is naturality; the edge pieces are just the definitions of $\hat{p}$ and σ.

Next, we show the linearity of σ. Consider the following diagram:

$$
\begin{tikzcd}
T([M,V]\times[M,V]) \arrow[r, "{\langle T(\pi_0),T(\pi_1)\rangle}"] \arrow[d, "T(m_\times)"'] & T([M,V])\times T([M,V]) \arrow[r, "\psi\times\psi"] & {[M,T(V)]\times[M,T(V)]} \arrow[r, "{[M,\hat{p}]\times[M,\hat{p}]}"] \arrow[d, "m_\times"'] & {[M,V]\times[M,V]} \arrow[d, "m_\times"] \\
T([M,V\times V]) \arrow[r, "\psi"] \arrow[d, "{T([M,\sigma])}"'] & {[M,T(V\times V)]} \arrow[r, "{[M,\langle T(\pi_0),T(\pi_1)\rangle]}"] \arrow[d, "{[M,T(\sigma)]}"] & {[M,T(V)\times T(V)]} \arrow[r, "{[M,\hat{p}\times\hat{p}]}"] & {[M,V\times V]} \arrow[d, "{[M,\sigma]}"] \\
T([M,V]) \arrow[r, "\psi"'] & {[M,T(V)]} \arrow[rr, "{[M,\hat{p}]}"'] & & {[M,V]}
\end{tikzcd}
$$

The top right and bottom left squares commute by naturality. The bottom right square commutes by the assumption that V is a differential object. The top left square also commutes, and we will give the calculation for it below.

Before beginning, note that

$$\theta_\times := A\times(B\times C)\xrightarrow{\Delta\times 1}(A\times A)\times(B\times C)\xrightarrow{\mathsf{ex}}(A\times B)\times(A\times C)$$

is a strength for $A\times\_$. Note the naturality:

$$\begin{array}{ccc} A \times (B \times C) & \xrightarrow{\theta_\times} & (A \times B) \times (A \times C) \\ {\scriptstyle f\times(g\times h)}\downarrow & & \downarrow{\scriptstyle (f\times g)\times(f\times h)} \\ A' \times (B' \times C') & \xrightarrow[\theta_\times]{} & (A' \; x B') \times (A' \times C') \end{array}$$

And finally note that

- $(1_A \times \Delta_B)\theta_\times = \Delta_{A\times B}$;
- $\theta_\times \pi_0 = 1 \times \pi_0$;
- $\theta_\times \pi_1 = 1 \times \pi_1$.

Also note that this strength allows us to re-express the map $m_\times : [M, V] \times [M, V] \longrightarrow [M, V \times V]$.

$$\frac{M \times ([M, v] \times [M, V]) \xrightarrow{\theta_\times} (M \times [M, V]) \times (M \times [M, V]) \xrightarrow{\mathsf{ev} \times \mathsf{ev}} V \times V}{[M, V] \times [M, V] \xrightarrow[m_\times := \lambda(\theta_\times(\mathsf{ev} \times \mathsf{ev}))]{} [M, V \times V]}$$

$T(m_\times^{-1})\psi[M, \langle T(\pi_0), T(\pi_1)\rangle] = \langle T(\pi_0), T(\pi_1)\rangle(\psi \times \psi)m_\times^{-1}$ using the universal property of maps into $[M, T(V) \times T(V)]$.

$$
\begin{aligned}
&(1\times T(m_\times)\psi[M,\langle T\pi_0,T\pi_1\rangle])\mathsf{ev}\\
&=(1\times T(m_\times)\psi)\mathsf{ev}\,\langle T\pi_0,T\pi_1\rangle\\
&=(1\times T(m_\times))\theta_T\,T(\mathsf{ev})\,\langle T\pi_0,T\pi_1\rangle\\
&=\theta_T\,T(\theta_\times)T(\mathsf{ev}\times\mathsf{ev})\,\langle T\pi_0,T\pi_1\rangle\\
&=\theta_T\,T(\theta_\times)T(\mathsf{ev}\times\mathsf{ev})\,\langle T\pi_0,T\pi_1\rangle\\
&=\theta_T\,T(\theta_\times)\,\langle T(\pi_0\mathsf{ev}),T(\pi_1\mathsf{ev})\rangle\\
&=\theta_T\,\langle T(\theta_\times\pi_0\mathsf{ev}),T(\theta_\times\pi_1\mathsf{ev})\rangle\\
&=\theta_T\,\langle T((1\times\pi_0)\mathsf{ev}),T((1\times\pi_1)\mathsf{ev})\rangle\\
&=\Delta(\theta_T\,T((1\times\pi_0)\mathsf{ev})\times\theta_T\,T((1\times\pi_1)\mathsf{ev}))\\
&=(1\times\Delta)\theta_\times((1\times T\pi_0)\times(1\times T\pi_1))(\theta_T\,T(\mathsf{ev})\times\theta_T\,T(\mathsf{ev}))\\
&=(1\times\Delta)(1\times(T\pi_0\times T\pi_1))\theta_\times(\theta_T\,T(\mathsf{ev})\times\theta_T\,T(\mathsf{ev}))\\
&=(1\times\langle T\pi_0,T\pi_1\rangle)\theta_\times(\theta_T\,T(\mathsf{ev})\theta_T\,T(\mathsf{ev}))\\
&=(1\times\langle T\pi_0,T\pi_1\rangle)\theta_\times((1\times\psi)\mathsf{ev}\times(1\times\psi)\mathsf{ev})\\
&=(1\times\langle T\pi_0,T\pi_1\rangle)(1\times(\psi\times\psi))\theta_\times(\mathsf{ev}\times\mathsf{ev})\\
&=(1\times\langle T\pi_0,T\pi_1\rangle(\psi\times\psi)m_x)\mathsf{ev}
\end{aligned}
$$

as required. Finally, for the coherence with l we have

$$
\begin{tikzcd}
T([M,V]) \arrow[rr, "l"] \arrow[d, "\psi"'] & & T^2([M,V]) \arrow[d, "T(\psi)"] \arrow[dd, bend left, "T(\hat{p})"] \\
{[M,T(V)]} \arrow[r, "{[M,l]}"] \arrow[d, "{[M,\hat{p}]}"'] & {[M,T^2(V)]} \arrow[d, "{[M,T(\hat{p})]}"] & T([M,T(V)]) \arrow[l, "\psi"'] \arrow[d, "{T([M,\hat{p}])}"] \\
{[M,V]} & {[M,T(V)]} \arrow[l, "{[M,\hat{p}]}"] & T([M,V]) \arrow[l, "\psi"] \arrow[ll, bend left, "\hat{p}"]
\end{tikzcd}
$$

The top rectangle follows from corollary 5.4.1. The bottom right square is naturality. The bottom left square is assumption that V is a differential object.

This completes the proof that $[M,V]$ is a differential object. □

The above holds for coherently closed tangent categories.

Corollary 5.5.9. *Let $\mathbb{X}$ be a coherently closed tangent category. Then*

$$[\_,\_] : \mathbb{X}^{op} \times \mathsf{Diff}(\mathbb{X}) \longrightarrow \mathsf{Diff}(\mathbb{X})$$

The above corollary says that differential objects are closed to taking powers by arbitrary objects in $\mathbb{X}$.

Theorem 5.5.10. *For a coherently closed tangent category, the differential objects are a differential λ-category.*

Proof. We need to show that the differential structure put on differential objects always satisfies the coherence required for differential λ-categories; namely, for $A \times B \xrightarrow{f} C$

$$D_B[f] = \lambda^{-1} D[\lambda(f)].$$

Consider,

$$\begin{aligned}
&\lambda^{-1}(D[\lambda(f)]) \\
&= \lambda^{-1}(\langle \pi_0, \pi_1 \rangle_T \, T(\lambda(f))\psi[M, \hat{p}]) \\
&= \lambda^{-1}(\langle \pi_0, \pi_1 \rangle_T \, \lambda(T_B(f))[M, \hat{p}]) \qquad \text{lemma 5.4.2} \\
&= \lambda^{-1}(\langle \pi_0, \pi_1 \rangle_T \, \lambda(T_B(f)\hat{p})) \\
&= \lambda^{-1}(\lambda((1 \times \langle \pi_0, \pi_1 \rangle_T) T_B(f)\hat{p})) \\
&= (1 \times \langle \pi_0, \pi_1 \rangle) T_B(f)\hat{p} \\
&= D_B[f] \qquad \text{lemma 5.5.3}
\end{aligned}$$

□

To tie into chapter 3 we have the following corollary.

Corollary 5.5.11.

1. *The differential λ-calculus admits a sound interpretation into the differential objects of a coherently closed tangent category.*

2. *The classifying category of a differential λ-theory is a coherently closed tangent category.*

Proof.

1. From 5.5.10, the differential objects are a differential λ-category. From 3.2.8, any differential λ-category is a model of the differential λ-calculus.

2. The classifying category of the differential λ-calculus is a differential λ-category 3.2.10, and proposition 5.5.7 says that this is a coherently closed tangent category.

$\square$

5.6 CoKleisli categories and the simple slice

When $\mathbb{X}$ is a category with products, the **simple slice** of $\mathbb{X}$ by an object $A \in \mathbb{X}$, denoted $\mathbb{X}[A]$, is the category with the same objects as $\mathbb{X}$ and for which arrows $B \longrightarrow C$ in $\mathbb{X}[A]$ correspond to $A \times B \longrightarrow C$ in $\mathbb{X}$. It is also the coKleisli category of the comonad $A \times \_ : \mathbb{X} \longrightarrow \mathbb{X}$. $\mathbb{X}[A]$ is thought of as having A in context. In a cartesian closed category, the A can be curried, by the monad $[A, \_]$. The simple slice category is equivalent to the Kleisli category of this monad.

In this section, we will prove that the simple slice of a Cartesian tangent category is always a Cartesian tangent category. In fact, we characterize precisely when the coKleisli category of a comonad on a tangent category is again a tangent category. This can be potentially useful for obtaining new tangent categories.

Let $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ be a functor, let $\mathbb{X}$ have a monad T, and $\mathbb{Y}$ have a monad T'. F lifts to the Kleisli category when there is a F_T such that:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "I"] \arrow[d, "F"'] & \mathbb{X}_T \arrow[d, "F_T"] \\ \mathbb{Y} \arrow[r, "I"'] & \mathbb{Y}_{T'} \end{tikzcd}$$

Functor liftings were investigated by Applegate, 1965, Johnstone, 1975, and Mulry, 1993. We recall some of the basic ideas for an endofunctor $\mathbb{X} \xrightarrow{K} \mathbb{X}$ and for coKleisli categories of comonads.

For a comonad (S,δ,ϵ), a functor $K : \mathbb{X} \longrightarrow \mathbb{X}$ lifts to the coKleisli category $\mathbb{X}_S$ when the following square commutes:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "J"] \arrow[d, "K"'] & \mathbb{X}_S \arrow[d, "K_S"] \\ \mathbb{X} \arrow[r, "J"'] & \mathbb{X}_S \end{tikzcd}$$

where J is the identity on objects, and sends f to $S(f)\epsilon$. There is a map

$$\frac{SA \xrightarrow{1_{SA}} SA \qquad \mathbb{X}}{A \xrightarrow[\eta]{} SA \qquad \mathbb{X}_S}$$

that is natural in $\mathbb{X}_S$. Then by whiskering, we get another natural map in $\mathbb{X}_S$

$$\frac{K(A) \xrightarrow{\underline{\lambda}} K(SA) \qquad \mathbb{X}_S}{S(KA) \xrightarrow[K_S(\eta)]{} K(SA) \qquad \mathbb{X}}$$

As J is a right adjoint, it has a left adjoint, L that sends A to $S(A)$ and sends $A \xrightarrow{f} B$ in $\mathbb{X}_S$ to $SA \xrightarrow{\delta} S^2A \xrightarrow{Sf} SB$ in $\mathbb{X}$. We then use the left adjoint to lower $\underline{\lambda}$ to a natural map in $\mathbb{X}$:

$$\frac{L(KA \xrightarrow{\underline{\lambda}} KSA) = S(KA) \xrightarrow{\delta} S^2(KA) \xrightarrow{S(\underline{\lambda})} S(K(SA))}{\lambda := S(KA) \xrightarrow[L(\underline{\lambda})]{} S(K(SA)) \xrightarrow[\epsilon]{} K(SA)}$$

It is relatively straightforward to show that for the map λ, the following two diagrams commute:

$$\begin{tikzcd} S(KA) \arrow[r, "\lambda"] \arrow[dr, "\epsilon"'] & K(SA) \arrow[d, "K(\epsilon)"] \\ & KA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S(KA) \arrow[rr, "\lambda"] \arrow[d, "\delta"'] & & K(SA) \arrow[d, "K\delta"] \\ S^2(KA) \arrow[r, "S(\lambda)"'] & S(K(SA)) \arrow[r, "\lambda"'] & K(S^2A) \end{tikzcd}$$

The commutativity of the diagram for lifting gives on objects $K_S(A) = K(A)$. In fact, any natural $\lambda : KS \longrightarrow SK$ that satisfies the two diagrams above, determines a functor $K_S(f) := S(KA) \xrightarrow{\lambda} K(SA) \xrightarrow{K(f)} KB$, that gives $K_SA \xrightarrow{K_S(f)} K_SB$ in $\mathbb{X}_S$. Such a λ is called a **lifting law**.

Proposition 5.6.1. *Let $\mathbb{X}$ be a category with a comonad (S,δ,ϵ) and an endofunctor K. K lifts to the coKleisli category if and only if there is a lifting law $\lambda : KS \longrightarrow SK$.*

For a full proof of proposition 5.6.1, see Borceaux, 2008 lemma 4.5.1.

We also have:

Proposition 5.6.2. *Suppose that $\mathbb{X}$ has a comonad (S,δ,ϵ), and K, K' are endofunctors that have lifting laws $\lambda^K, \lambda^{K'}$ repsectively. A natural transformation $\alpha : K \longrightarrow K'$ lifts to the coKleisli category if and only if $\alpha;S;\lambda^{K'} = \lambda^K;S;\alpha$[1] or in components:*

$$\begin{array}{ccc} S(KB) & \xrightarrow{S(\alpha_B)} & S(K'B) \\ {\scriptstyle \lambda^K}\downarrow & & \downarrow{\scriptstyle \lambda^{K'}} \\ K(SB) & \xrightarrow[\alpha_{SB}]{} & K'(SB) \end{array}$$

Proof. If a natural transformation lifts, then the diagram commutes.

Suppose the diagram commutes, we show that $\alpha^S := J(\alpha)$, which on components is $S(KB) \xrightarrow{S(\alpha_B)} S(K'B) \xrightarrow{\epsilon} K'B$, is natural $K_S B \longrightarrow K'_S B$ in $\mathbb{X}_S$. Recall that we must show composites in a coKleisli category are equal; the desired equality is:

$$S(\alpha_B);\epsilon_{K'B};\lambda^{K'}_{B'};K'(f) = \lambda^K_B;K(f);S(\alpha_{B'});\epsilon_{K'B'}$$

The proof is

[1] We use semicolons to separate long, hard-to-parse strings of symbols.

$$
\begin{tikzcd}
SKB \arrow[rr, "\delta"] \arrow[drr, equal] \arrow[ddr, "\lambda^K_B"] \arrow[dd, "\delta"'] & & S^2KB \arrow[r, "S^2\alpha_B"] \arrow[d, "S\epsilon"] & S^2K'B \arrow[d, "S\epsilon_{K'B}"] \\
& & SKB \arrow[r, "S\alpha_B"] \arrow[d, "\lambda^K_B"] & SK'B \arrow[d, "\lambda^{K'}_B"] \\
S^2KB \arrow[d, "S\lambda^K_B"'] & KSB \arrow[r, equal] \arrow[d, "K\delta"] \arrow[dd, equal, bend left] & KSB \arrow[r, "\alpha_{SB}"'] \arrow[ddd, "Kf"] & K'SB \arrow[dddd, "K'f"] \\
SKSB \arrow[r, "\lambda^K_{SB}"] \arrow[dr, "\epsilon"'] \arrow[ddd, "SKf"'] & KS^2B \arrow[d, "K\epsilon"] & & \\
& KSB \arrow[d, "Kf"] & & \\
& KB' \arrow[r, equal] & KB' \arrow[dr, "\alpha_{B'}"] & \\
SKB \arrow[ur, "\epsilon"] \arrow[rr, "S\alpha_B"'] & & SK'B' \arrow[r, "\epsilon"'] & K'B'
\end{tikzcd}
$$

The shapes inside the diagram consist of naturality, defining properties of λ, the commuting diagram from the assumption, and comonad laws. □

Of interest to us is when $\mathbb{X}$ is a tangent category so that the endofunctor is T.

Lemma 5.6.3. *Suppose $\mathbb{X}$ is a tangent category, and (S,δ,ϵ) is a comonad where T and p lift. Then T_n lifts, and more generally, all the T_n^m lift.*

Proof. We have

$$
\begin{tikzcd}
& ST_nA \arrow[rrr, dotted, "\exists!.\lambda"] \arrow[ddl] \arrow[dr] & & & T_nSA \arrow[ddl] \arrow[dr] & \\
& & STA \arrow[rrr, "\lambda"] \arrow[ddl, "Sp"] & & & TSA \arrow[ddl, "p"] \\
STA \arrow[rrr, "\lambda"] \arrow[dr, "Sp"'] & \cdots & & TSA \arrow[dr, "p"] & \cdots & \\
& SA \arrow[rrr, equal] & & & SA &
\end{tikzcd}
$$

The bottom two faces commute because p lifts. The coherences follow from the universal property.

A similar argument shows that T_n^k lifts to $\mathbb{X}_S$. $\square$

Thus when p lifts, it always makes sense to ask $+$ to lift, because T_2 lifts.

The functor $\mathbb{X} \xrightarrow{J} \mathbb{X}_S$ defined at the beginning of the section is a right adjoint, hence preserve limits. We can use the continuity [2] of J to show that diagrams in $\mathbb{X}_S$ are indeed limits.

Proposition 5.6.4. *Suppose $\mathbb{X}$ is a tangent category, and (S, δ, ϵ) is a comonad where T lifts, and all the structural transformations lift. Then $\mathbb{X}_S$ is a tangent category.*

Proof. The right adjoint J preserves products strictly. The projections in $\mathbb{X}_S$ are $J(\pi_i)$, and the pairing of f, g is $J(\langle f, g\rangle)$. All the equations between the natural transformations required for tangent structure in $\mathbb{X}_S$ may be put in the form $J(e_1) = J(e_2)$ which holds because $e_1 = e_2$ in $\mathbb{X}$ and functors preserve equations.

It remains to show that the required limits exist. As J is continuous the pullback power of $p^S = J(p)$ exists and is $J(T_n(M))$. The square for the universality of the lift may be put into the form $J(\langle \pi_0 l, \pi_0\rangle T(+) T p) = J(\pi_0 p 0)$; which commutes as J is a functor, but as J is continuous, the square is a pullback in $\mathbb{X}_S$. $\square$

The simple slice category $\mathbb{X}[A]$ is the coKleisli category of the comonad $A \times \_$. It has the same objects as $\mathbb{X}$ and maps $B \longrightarrow C$ in $\mathbb{X}[A]$ are maps $A \times B \longrightarrow C$ in $\mathbb{X}$. The identity is $A \times B \xrightarrow{\pi_1} B$ and the composite of $B \xrightarrow{f} C$ and $C \xrightarrow{g} D$ is

$$A \times B \xrightarrow{\Delta \times 1} A \times A \times B \xrightarrow{A \times f} A \times C \xrightarrow{g} D$$

Corollary 5.6.5. *Let $\mathbb{X}$ be a cartesian tangent category, and A be an object of $\mathbb{X}$. The simple slice category $\mathbb{X}[A]$ is a tangent category.*

Proof. The lifting law $A \times T(B) \xrightarrow{\lambda} T(A \times B)$ is the strength $A \times TB \xrightarrow{\theta} T(A \times B)$. Thus the tangent functor lifts: on arrows: $T_{A\times\_}(f) := A \times TB \xrightarrow{\theta} T(A \times B) \xrightarrow{Tf} TC$.

That all the structural transformations lift comes from theorem 5.3.3: all the structural transformations are strong, hence they all lift. Then proposition 5.6.4 shows that $\mathbb{X}[A]$ has tangent structure. $\square$

[2] We often call a limit preserving functor continuous; similarly colimit preserving functors are called cocontinuous.

Observation 5.6.6. *Let $\mathbb{X}$ be a cartesian tangent category and A any object. Then the differential objects of $\mathbb{X}$ are the same as the differential objects of $\mathbb{X}[A]$. Moreover* $\mathsf{Diff}(\mathbb{X}[A])$ *is a cartesian differential category.*

Corollary 5.6.7. *Let $\mathbb{X}$ be a cartesian category and A any object in $\mathbb{X}$. The category* $\mathsf{Diff}[A](\mathbb{X})$ *whose objects are differential objects of $\mathbb{X}$ and whose maps $V \longrightarrow W$ correspond to maps $A \times V \longrightarrow W$ in $\mathbb{X}$ is a cartesian differential category with*

$$\frac{A \times V \xrightarrow{f} W}{D[f] := A \times V \times V \longrightarrow A \times TV \xrightarrow[\theta]{} T(A \times V) \xrightarrow[Tf]{} TW \xrightarrow[\hat{p}]{} W}$$

This is the coKleisli category of the comonad $A \times \_$ on the category $\mathsf{Diff}(\mathbb{X})$.

The simple slice category has a counter part. In a cartesian closed category $\mathbb{X}$, for any object A, the functor $[A, \_] : \mathbb{X} \longrightarrow \mathbb{X}$ is a monad. The unit and multiplication for the monad, are presented by

$$m \mapsto \lambda x.m \qquad f \mapsto \lambda x.f\,x\,x$$

respectively (these are k and w from combinatory logic **book:lambda-calc-combinators**). That the unit monad laws hold is the η-rule, or the universal property of the internal hom: $\lambda y.m\,y = m$ and the associativity is trivial. The Kleisli category of this monad is denoted $\mathbb{X}^A$.

Observation 5.6.8. *The categories $\mathbb{X}[A]$ and $\mathbb{X}^A$ are equivalent.*

Proof. The functor $\mathbb{X}[A] \xrightarrow{\lambda} \mathbb{X}^A$ sends B to B, and given a map $B \xrightarrow{f} C$, which in $\mathbb{X}$ is a map $A \times B \longrightarrow C$, $\lambda(f) : B \longrightarrow [A, C]$. The functor going in the other direction is given by uncurrying. □

If $\mathbb{X}$ is a tangent category, then $\mathbb{X}^A$ is too. Let $f : B \longrightarrow [A, C]$; $T(f) := TB \xrightarrow{Tf} T[A, C] \xrightarrow{\psi} [A, TC]$. Provided how coherence works for Cartesian differential categories, the following may seem surprising as we do not require ψ to be an isomorphism.

Observation 5.6.9. *Let $\mathbb{X}$ be a cartesian closed tangent category. The functor λ : $\mathbb{X}[A] \longrightarrow \mathbb{X}^A$ sends tangent structure to tangent structure. Similarly for λ^{-1}. That is λ is an equivalence of tangent categories.*

Proof. Suppose $A \times B \xrightarrow{f} C$ is a map in $\mathbb{X}[A]$. Then

$$\begin{aligned}
&\lambda(T_{A\times_}(f)) \\
&= \lambda(\theta T(f)) = \lambda(\theta T(1 \times \lambda(f)) T\mathsf{ev}) \\
&= \lambda((1 \times T\lambda(f))\theta T\mathsf{ev}) = T\lambda(f)\lambda(\theta T\mathsf{ev}) \\
&= T\lambda(f)\psi = T_{\Rightarrow}(\lambda(f))
\end{aligned}$$

$$\begin{tikzcd}
\mathbb{X}[A] \arrow[r, "\lambda"] \arrow[d, "T_{A\times_}"'] & \mathbb{X}^A \arrow[d, "T_{\Rightarrow}"] \\
\mathbb{X}[A] \arrow[r, "\lambda"'] & \mathbb{X}^A
\end{tikzcd}$$

Similarly, λ sends the structural transformations to the structural transformations. See 5.4.1 for the proof. That λ^{-1} is a morphism of tangent structure is similar.

$$\begin{aligned}
\lambda^{-1}(T_{\Rightarrow}(f)) &= (1 \times T(f)\psi)\mathsf{ev} = (1 \times Tf)\theta T\mathsf{ev} \\
&= \theta T(1 \times f)T\mathsf{ev} = \theta T\lambda^{-1}(f) = T_{A\times_}(\lambda^{-1}(f))
\end{aligned}$$

□

We should point out that generally Kleisli categories of monads on tangent categories will not have tangent structure that lifts: the left adjoint $\mathbb{X} \xrightarrow{F} \mathbb{X}_T$ does not preserve limits.

A sense in which ψ being an isomorphism is a coherence is:

Corollary 5.6.10. *When $\mathbb{X}$ is a coherently closed tangent category, the map*

$$\lambda : \mathsf{Diff}[A](\mathbb{X}) \longrightarrow \mathsf{Diff}(\mathbb{X})^A$$

is an equivalence of cartesian differential categories.

In the next chapter, we will see that requiring the exponential strength of a tangent category, $\psi : T[A,B] \longrightarrow [A, TB]$, to be an isomorphism allows one to extract cartesian closed tangent categories from cartesian tangent categories.

5.7 Homs into differential bundles

Differential objects are generalized by differential bundles. In this section, we show that differential bundles are also closed to powers in a coherently closed tangent category.

Definition 5.7.1. *A* ***differential bundle*** $(E,q,B,\lambda,\sigma,\zeta)$ *is a commutative monoid* (q,σ,ζ) *in* $\mathbb{X}/B$*; together with a lift* $E \xrightarrow{\lambda} TE$ *such that*

1. *All pullback powers of q are preserved by T_n and T^m;*
2. $(\lambda,0):(E,q) \longrightarrow (TE,Tq)$ *is an additive map of bundles;*
3. $(\lambda,\zeta):(E,q) \longrightarrow (TE,p)$ *is an additive map of bundles;*
4. *The following diagram is a pullback and is preserved by T^n for all n:*

$$\begin{tikzcd} E_2 \arrow[r, "\lambda\times 0"] \arrow[d, "\pi_0 q=\pi_1 q"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(E_2) \arrow[r, "T\sigma"] & TE \arrow[d, "Tq"] \\ B \arrow[rr, "0"'] & & TB \end{tikzcd}$$

A generalization of 5.5.8 is the following:

Proposition 5.7.2. *Let $\mathbb{X}$ be a coherently closed tangent category, (E,q,λ,B) be a differential bundle, and A be any object. Then $[A,E]$ may be given the structure of a differential bundle over $[A,B]$. That is differential bundles are closed to powers.*

Proof. The projection is

$$\begin{tikzcd} {[A,E]} \arrow[d, "{[A,q]}"] \\ {[A,B]} \end{tikzcd}$$

And the lift is

$$[A,E] \xrightarrow{[A,\lambda]} [A,TE] \xrightarrow{\psi^{-1}} T[A,E]$$

Note the use of ψ^{-1}.

As $[A,\_]$ is continuous, $[A,E_2]$ is the pullback of $[A,q]$ with itself.

The sum and zero are

$$[A,E_2] \xrightarrow{[A,\sigma]} [A,E] \qquad [A,B] \xrightarrow{[A,\zeta]} [A,E]$$

That this is a commutative monoid over $[A,B]$ is immediate as functors preserve equations.

That all pullback powers of $[A,q]$ are preserved by T_n and T^m comes from coherence:

$$T_n^m[A,q] \simeq [A, T_n^m q]$$

and then use that T_n^m preserves pullbacks of q, and that $[A,\_]$ is continuous.

That $(\lambda,0): q \longrightarrow Tq$:

$$\begin{tikzcd}
{[A,E]} \arrow[r, "{[A,\lambda]}"] \arrow[d, "{[A,q]}"'] & {[A,TE]} \arrow[r, "\psi^{-1}"] \arrow[d, "{[A,Tq]}"] & T[A,E] \arrow[d, "{T[A,q]}"] \\
{[A,B]} \arrow[r, "{[A,0]}"'] & {[A,TB]} \arrow[r, "\psi^{-1}"'] & T[A,B]
\end{tikzcd}$$

The left square commutes because q is a differential bundle and the right square is naturality. Note that by 5.4.1, $[A,0]\psi^{-1} = 0$, and definitionally, $[A,\lambda]\psi^{-1}$ giving the coherence as required.

That this is an additive map:

$$\begin{tikzcd}
{[A,E_2]} \arrow[r, "{[A,\sigma]}"] \arrow[d, "{[A\lambda\times\lambda]}"'] & {[A,E]} \arrow[d, "{[A,\lambda]}"] \\
{[A,TE_2]} \arrow[r, "{[A,T\sigma]}"] \arrow[d, "\psi^{-1}"'] & {[A,TE]} \arrow[d, "\psi^{-1}"] \\
T[A,E_2] \arrow[r, "{T[A,\sigma]}"'] & T[A,E]
\end{tikzcd}$$

The top square commutes because λ is a differential bundle and the bottom is naturality. Thus, $(\lambda,0): q \longrightarrow Tq$ is an additive map.

That $(\lambda,\zeta): q \longrightarrow p$:

$$\begin{tikzcd}
{[A,E]} \arrow[r, "{[A,\lambda]}"] \arrow[d, "{[A,q]}"'] & {[A,TE]} \arrow[r, "\psi^{-1}"] \arrow[dr, "{[A,p]}"'] & T[A,E] \arrow[d, "p"] \\
{[A,B]} \arrow[rr, "{[A,\zeta]}"'] & & {[A,E]}
\end{tikzcd}$$

The left shape is that q is a differential bundle. The right triangle is 5.4.1.

Finally we show the universality of the lift. First we exhibit the commutativity of the square.

$$
\begin{tikzcd}
{[A,E_2]} \arrow[r, "{[A,\lambda\times 0]}"] \arrow[dd, "{[A,\pi_0 q]}"'] & {[A,TE_2]} \arrow[r, "\psi^{-1}"] \arrow[d, "{[A,T\sigma]}"] & T[A,E_2] \arrow[r, "{T[A,\sigma]}"] & T[A,E] \arrow[dd, "{T[A,q]}"] \\
& {[A,TE]} \arrow[urr, "\psi^{-1}"'] \arrow[d, "{[A,Tq]}"] & & \\
{[A,B]} \arrow[r, "{[A,0]}"'] & {[A,TB]} \arrow[rr, "\psi^{-1}"'] & & T[A,B]
\end{tikzcd}
$$

The left square is that q is a differential bundle, and functoriality of $[A,\_]$. The squares on the right are both naturality; thus, the outer square commutes.

Note that the left square is a pullback as $[A,\_]$ is continuous. The right square can be put into the form

$$(\psi^{-1},\psi^{-1}):[A,T(\sigma q)]\longrightarrow T[A,\sigma q]$$

as an isomorphism in the arrow category, and hence we extend a pullback square by an isomorphism square, and the outer square is a pullback.

Thus $[A,E]$ is a differential bundle over $[B,E]$.

□

5.8 Vector Fields

The section on differential bundles in a coherently closed tangent category allows formulating an object of vector fields of a differential bundle as an object in the category. In this section, we show how to construct the vector fields of a bundle.

First, we make use of the following lemma:

Lemma 5.8.1 (Cockett and Cruttwell, 2016 Corollary 3.5)**.** *Let $\mathbb{X}$ be a tangent category, and $E \xrightarrow{q} B$ a differential bundle. For any point $1 \xrightarrow{b} B$, if the following* *pullback is transverse*

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

then E_b is a differential object.

We have a corollary of proposition 5.7.2 and lemma 5.8.1.

Corollary 5.8.2. *Let $\mathbb{X}$ be a coherently closed tangent category. Let $E \xrightarrow{q} B$ be a differential bundle. If the following pullback is transverse,*

$$\begin{tikzcd} \Gamma(q) \arrow[r] \arrow[d] & {[B,E]} \arrow[d, "{[B,q]}"] \\ 1 \arrow[r, "\mathsf{Id}_B"'] & {[B,B]} \end{tikzcd}$$

then $\Gamma(q)$ is a differential object.

Proof. From proposition 5.7.2, $[B,q]$ is a differential bundle. The identity Id_B is always an element $1 \longrightarrow [B,B]$ which is given by currying $B \xrightarrow{1_B} B$, and under the assumption that the limit exists and is transverse, then this follows immediately from 5.8.1. □

Observation 5.8.3. *The object $\Gamma(q)$ of 5.8.2 can be formulated as the equalizer*

$$\Gamma(q) \rightarrowtail [B,E] \underset{\lambda s.\mathsf{Id}_B \equiv \lambda(\pi_0)}{\overset{[B,q]}{\rightrightarrows}} [B,B]$$

A **section of a differential bundle** (E,q,B) is a map $B \xrightarrow{s} E$ such that

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[dr, equal] & E \arrow[d, "q"] \\ & B \end{tikzcd}$$

Proposition 5.8.4. *Let $\mathbb{X}$ be coherently closed and $E \xrightarrow{q} B$ be a differential bundle. Then*

$$\mathbb{X}(1,\Gamma(q)) \simeq \{s : B \longrightarrow E \mid sq = 1_B\}$$

That is, elements of $\Gamma(q)$ are sections of q.

Proof. Suppose $\widetilde{s} : 1 \longrightarrow \Gamma(q)$. Then we may uncurry $\widetilde{s}$ to obtain

$$s := B \xrightarrow{\langle B, \widetilde{s} \rangle} B \times \Gamma(q) \rightarrowtail B \times [B, E] \xrightarrow{\mathsf{ev}} E$$

Note that $\widetilde{s}[B,q] = \widetilde{s}\lambda(\pi_0)$, and this means that $\lambda((1 \times \widetilde{s})\mathsf{ev}q) = \lambda(1 \times \widetilde{s}\pi_0)$, and as λ is an isomorphism we have $(1 \times \widetilde{s})\mathsf{ev}q = (1 \times \widetilde{s})\pi_0$ which implies that

$$\begin{aligned} sq &= \langle 1_B, \widetilde{s} \rangle \, \mathsf{ev}q \\ &= \langle 1_B, 1_B \rangle (1 \times \widetilde{s})\mathsf{ev}q \\ &= \langle 1_B, 1_B \rangle (1 \times \widetilde{s})\pi_0 \\ &= 1_B \end{aligned}$$

as required.

Conversely, if $sq = 1_B$, then curry s to obtain a map $1 \xrightarrow{\lambda(s)} [B, E]$. By assumption $\lambda(s)[B,q] = \lambda(s)\lambda(\pi_0)$ and by the univerality of the equalizer, we induce a map

$$1 \xrightarrow{\widetilde{s}} \Gamma(q)$$

as required. □

The sections of the tangent bundle are of particular interest. Sections of $TM \xrightarrow{p} M$ are called **vector fields**, and $\Gamma(p)$ is denoted $\mathscr{X}(M)$.

In the construction of $\mathscr{X}(M)$ we take the pullback of $[M, p]$ and Id_M. However, for the tangent bundle $[M, p]$ is related to p by a natural isomorphism. In other words, $\mathscr{X}(M)$ is the pullback:

$$\begin{tikzcd} \mathscr{X}(M) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & T[M,M] \arrow[d, "\psi^{-1}"] \arrow[dd, bend left, "p"] \\ & {[M,TM]} \arrow[d, "{[M,p]}"] \\ 1 \arrow[r, "\mathsf{Id}_M"'] & {[M,M]} \end{tikzcd}$$

The pullback of the tangent bundle along a point of the base is called the tangent space. Generally this is denoted $T_x M$.

Observation 5.8.5.

$$\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}[M,M]$$

A **differential bundle with negatives** is a differential bundle, that is also a group in the slice category; in particular there is a negation map $-(\_) : q \longrightarrow q$ in the slice.

Observation 5.8.6. *When $E \xrightarrow{q} B$ is a differential bundle where addition has inverses, then $\Gamma(q)$ is a group (under addition). The inverse of a section is given by*

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[dr, equal] & E \arrow[r, "-"] \arrow[d, "q"] & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Hence when $TM \xrightarrow{p} M$ is a group bundle, the object $\mathscr{X}(M)$ is a group, so that $T_{\mathsf{Id}_M}[M,M]$ is a group.

5.9 Locally coherently closed tangent categories

Often, one wants to describe spaces of functions between vector bundles; for example, we have already seen that $\Gamma(p)$, the space of sections of the tangent bundle $TM \xrightarrow{p} M$, can be viewed as an equalizer of the internal hom $[M, TM]$. However, it can also be viewed as $[1_M, p]$ in the slice category $\mathbb{X}/M$. To make use of this structure, we need the slice category of a tangent category to be Cartesian closed: that is, we require a locally cartesian closed tangent category.

We also require the differential bundles to form a sub-closed tangent category. For this, we need to formulate how to have a locally Cartesian closed subcategory, and to do this we introduce the notion of a display system.

A **display system** $\mathscr{D}$ for a category $\mathbb{X}$ is a class of maps such that for any $d \in \mathscr{D}$ and any map f in $\mathbb{X}$ whose codomain is the same as d, the pullback of f along d exists and is in $\mathscr{D}$ (see **book:taylor-practical-foundations** chapter 8).

Given a display system, there is an associated fibration, called the display fibration. The total category of this fibration, is denoted $\mathbb{X}^{\mathscr{D}}$, and is described:

Obj: Maps $A \xrightarrow{d} B \in \mathscr{D}$;

Arr: Commuting squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"'] & A' \arrow[d, "d'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

The projection of the fibration $\mathbb{X}^{\mathscr{D}} \xrightarrow{\mathsf{cod}} \mathbb{X}$ sends a square to its bottom edge:

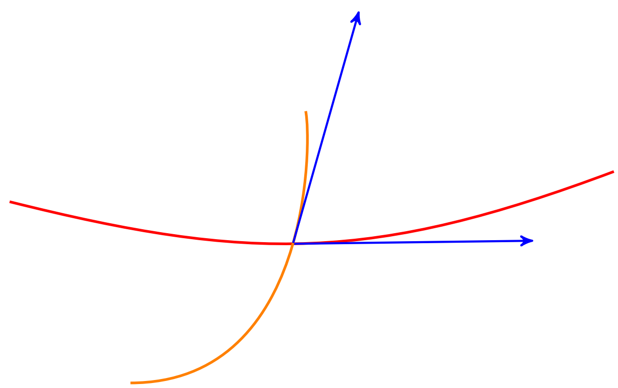
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"'] & B \arrow[d, "d'"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$
$$\mapsto$$
$$\begin{tikzcd} C \arrow[r, "g"'] & D \end{tikzcd}$$

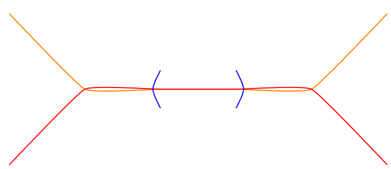
In this section, we will describe conditions on when the fibers of this fibration are tangent categories. We will see that displayed *differential bundles* (vector bundles) in $\mathbb{X}$ become differential objects in $\mathbb{X}/M$. Thus if $\mathbb{X}/M$ is a coherently closed tangent category, then the differential λ-calculus becomes the internal logic, not only of the differential objects, but also of the differential bundles of a tangent category.

5.9.1 Transversally displayed tangent categories

If $\mathbb{X}$ is an arbitrary tangent category, it is not the case that $\mathbb{X}/M$ is a tangent category in a canonical way. The problem is more generally with display systems in a tangent category: when we form the pullback of a map it may not have good geometric properties – in particular the pullback might not be preserved by the tangent functors T^n, T_n.

In the category of smooth manifolds and smooth maps, a pullback that is transverse is preserved by all the tangent functors. We recall the classical definition of transverse limit. Suppose we have two curves, regarded as manifolds, embedded in $\mathscr{R}^2$. For example,





In the first picture, the intersections of the manifolds are *transverse*, the tangent vectors at the intersection span $\mathscr{R}^2$. In the second picture, the manifolds intersect along a line, and their intersection spans only $\mathscr{R}$. This means that their intersection will be degenerate. In physical systems, this kind of degeneracy can lead to catastrophe **journal:thom-original** Zeeman showed, that even for some simple physical systems there are configuration spaces with catastrophic behaviour; a small movement of one aspect of the system will cause erratic motion in other parts of the system that have for example become restricted in dimension, and can lead to a chaotic phase space **journal:zeeman-early**

However, when intersections are of 'full rank', the geometric behaviour is good, and indeed the tangent bundle preserves these pullbacks. Indeed the *Thom transversality theorem* implies the following theorem.

For $\mathscr{R}$-vector spaces, if V, W embed into a vector space E, they are called transverse when V and W span E. For smooth manifolds, submanifolds M, N of L are transverse when for every $x : M \cap N$, $T_x M$, $T_x N$ span $T_x L$.

Theorem 5.9.1 (**journal:thom-trans**)**.** *If M, N are transverse submanifolds of E, then $M \cap N$ is isomorphic to the pullback of the embeddings into E, and the pullback is a smooth manifold. Moreover, this pullback is preserved by the tangent functors T^k.*

Thus, for our display system, we should ask that the pullback along a display map, not just exist, but we should ask that it is a transverse pullback, and transverse pullbacks should have the property that they are preserved by tangent functors.

Cockett and Cruttwell, 2016 considered the notion of a transverse display system for a tangent category. In the paper, the authors setup a *transverse system* as a collection of pullbacks together with closure properties; this was necessary as they were working not just with a display system, but a more general fibration.

We have already introduced the notion of a transverse limit as one that is preserved by all the T^k 5.1.4. A **transverse displayed tangent category** is a tangent category $\mathbb{X}$ together with a display system $\mathscr{D}$ such that pullbacks of maps along $\mathscr{D}$ are transverse.

Example 5.9.2. *In SMan submersions, that is smooth functions $M \xrightarrow{f} N$ for which $T_x M \xrightarrow{T_x f} T_{f\,x} N$ is surjective, form a transverse display system.*

Observation 5.9.3. *The tangent bundle $TM \xrightarrow{p} M$ need not be in $\mathscr{D}$. In smooth manifolds it is, but it need not be in SMan/M.*

Example 5.9.4. *In synthetic differential geometry, as every pullback is transverse, and all limits exists, $\mathscr{D}$ may be taken to be all maps.*

When $\mathbb{X}$ has a tangent transverse display system, then define $\mathsf{bun}_{\mathscr{D}}(\mathbb{X}) := \mathbb{X}^{\mathscr{D}}$.

Proposition 5.9.5 (Cockett and Cruttwell, 2016)**.** *When $\mathbb{X}$ is a transverse displayed tangent category, and each tangent bundle $TM \xrightarrow{p} M$ is displayed, then $\mathsf{bun}_{\mathscr{D}}(\mathbb{X})$ is always a tangent category.*

We will sometimes refer to the fiber of the display fibration as a relative slice, or sometimes just slice, and we abuse notation in referring to it as $\mathbb{X}/M$. Of particular interest is the following:

Proposition 5.9.6 (Cockett-Cruttwell)**.** *When* $\mathbb{X}$ *is a transverse displayed tangent category, then each fiber* $\mathbb{X}/M$*, with respect to* $\mathsf{bun}_{\mathscr{D}}(\mathbb{X})$*, is a (cartesian) tangent category.*

For a full proof see Cockett and Cruttwell, 2016.

sketch. The tangent structure is given by pullback. Let M be an object of $\mathbb{X}$, and let $q : E \longrightarrow M \in \mathscr{D}$. As $Tq \in \mathscr{D}$ we can pullback along it; this is $T_M(q) : T_M(E) \longrightarrow M$:

$$\begin{tikzcd}
T_M(E) \arrow[r] \arrow[d, "T_M(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tq"] \\
M \arrow[r, "0"'] & TM
\end{tikzcd}$$

Suppose $h : q_1 \longrightarrow q_2$ is a map in the slice:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "q_1"'] & & Y \arrow[dl, "q_2"] \\
& M &
\end{tikzcd}$$

Then we define $T_M(h)$ by universal property:

$$\begin{tikzcd}
T_M(X) \arrow[rrr] \arrow[dr] \arrow[dd, dotted, "\exists!.T_M h"'] & & & TM \arrow[dl, "Tq_1"'] \arrow[dd, "Th"] \\
& M \arrow[r, "0"] & TM & \\
T_M(Y) \arrow[rrr] \arrow[ur] & & & TY \arrow[ul, "Tq_2"]
\end{tikzcd}$$

Define $p_M : T_M(q) \longrightarrow q$ as the following map in $\mathbb{X}/M$:

$$
\begin{tikzcd}
T_M E \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[dr, "p"] \arrow[dd, "Tq"] & \\
& & E \arrow[dd, "q"] \\
M \arrow[r, "0"] \arrow[drr, equal] & TM \arrow[dr, "p"] & \\
& & M
\end{tikzcd}
$$

The lift is uniquely determined by the following diagram

$$
\begin{tikzcd}
& & TE \arrow[d, "Tq"] \arrow[drr, "l"] & & \\
T_M E \arrow[urr] \arrow[d, "T_M q"'] \arrow[dr, dotted, "\exists!.l_M"] \arrow[r, phantom, "\lrcorner", very near start] & & TM \arrow[drr, "l"] & & T^2 E \arrow[d, "T^2 q"] \\
M \arrow[urr, "0"] \arrow[dr, equal] & T_M T_M E \arrow[rr] \arrow[d, "T_M^2 q"] \arrow[dr, phantom, "\lrcorner", very near start] & & T T_M E \arrow[ur] \arrow[d, "T T_M q"] \arrow[r, phantom, "\lrcorner", very near start] & T^2 M \\
& M \arrow[rr, "0"'] & & TM \arrow[ur, "T0"'] &
\end{tikzcd}
$$

The other transformations are defined similarly. Various coherences, like $0_M l_m = 0_M T(0_M)$ are hold by universality of pullback. $\square$

5.9.2 Transversally displayed differential bundles

This subsection establishes that differential bundles are differential objects in the slice tangent category of a Cartesian tangent category.

Differential bundles are objects in a tangent category that behave like vector bundles in SMan.

A **transversally displayed differential bundle** in a transversally displayed tangent category $\mathbb{X}$, is an additive bundle (E, B, q, σ, ζ) together with a lift $E \xrightarrow{\lambda} TE$ where

1. $E \xrightarrow{q} B$ is a display map, and also an additive bundle;

2. The pullback powers of q are transverse;

3. $(\lambda, 0) : (E, q) \longrightarrow (TE, Tq)$ is an additive map;

4. $(\lambda, \zeta) : (E, q) \longrightarrow (TE, p)$ is an additive map;

5. The following diagram is a transverse pullback

$$\begin{tikzcd} E_2 \arrow[r, "\lambda\times 0"] \arrow[d, "\pi_0 q = \pi_1 q"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(E_2) \arrow[r, "T\sigma"] & TE \arrow[d, "Tq"] \\ B \arrow[rr, "0"'] & & TB \end{tikzcd}$$

Lemma 5.9.7 (Cockett and Cruttwell, 2016)**.** *Suppose (E, q, λ, B) is a differential bundle, and $C \xrightarrow{f} B$ is a map. If the pullback of q along f exists, then it is a differential bundle.*

Lemma 5.9.8 (Cockett and Cruttwell, 2016)**.** *Suppose $(E, q_1, \lambda_1, B), (E_2, q_2, \lambda_2, B)$ are differential bundles over the same base. Their pullback is a differential bundle, and is the product in the category of differential bundles over B.*

The lift is induced by the universal property of the pullback given by applying T to the pullback of q_1 and q_2. This bundle is called the Whitney sum, and is the chosen product functor on the category of differential bundles over B.

The pullback of a differential bundle is indeed a differential bundle, but it need not be displayed. We say the differential bundles are displayed when this is the case. As a corollary of proposition 5.9.6, we then get:

Proposition 5.9.9. *If $\mathbb{X}$ is a transversally displayed tangent category in which the differential bundles are displayed, then the class of differential bundles forms a display system, and hence the fibers of the display fibration are tangent categories.*

One should point out that we need the differential bundles to be a transverse display system, but this is guaranteed as each differential bundle is displayed in $\mathscr{D}$. Thus, the pullbacks are in $\mathscr{Q}$, and hence $\mathscr{Q}$ is a transverse system for the differential bundles. The relative slice of differential bundles over B is denoted $\mathsf{DBun}(\mathbb{X})[B]$.

This brings us to the main theorem of differential bundles:

Theorem 5.9.10 (Cockett and Cruttwell, 2016 theorems 5.12,14)**.** *Let $\mathbb{X}$ be a transversally displayed tangent category. Then the following are equivalent:*

1. *A differential bundle in $\mathbb{X}$ over B;*
2. *A differential bundle over 1_B in the relative slice $\mathbb{X}/B$ with respect to the display fibration;*
3. *A differential object in the relative slice $\mathbb{X}/B$ with respect to the display fibration.*

Moreover, $\mathsf{DBun}(\mathbb{X})[B]$ *is always a cartesian differential category.*

5.9.3 Locally cartesian closed categories

We introduce locally coherently closed categories to allow constructions in a tangent category beyond spaces of sections of differential bundles. We describe a bit of machinery for working with locally cartesian closed categories that allows expressing the coherence for locally closed tangent categories.

One way to describe locally cartesian closed categories is by *pullback extensions.* These were investigated in Weber, 2015 under the name *pullback around*, and the right adjoint to pullback was called *distributivity pullback.*

Let f, g be composable arrows in a category $\mathbb{X}$.

$$\begin{tikzcd} C \arrow[d, "g"'] & \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

A **pullback extension** is a pullback of the form:

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "h"] \\
A \arrow[d, "f"'] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

A **morphism of pullback extensions** consists of two maps α, β as in the following diagram:

$$\begin{tikzcd}
& \cdot \arrow[rr, "k'"] \arrow[ddl, bend left, "e'"] & & \cdot \arrow[dddl, bend left, "h'"] \\
\cdot \arrow[ur, bend left, "\alpha"] \arrow[rr, "k"] \arrow[d, "e"'] & & \cdot \arrow[ur, "\beta"] \arrow[dd, "h"] & \\
\cdot \arrow[d, "f"'] & & & \\
\cdot \arrow[rr, "g"'] & & \cdot &
\end{tikzcd}$$

$e = \alpha e'$, $h = \beta h'$ and $\alpha k' = k\beta$. Note that as the square $\alpha k' = k\beta$ commutes, by the pullback pasting lemma, the square is in fact a pullback.

Pullback extensions of f along g form a category $\mathsf{Pull}(f, g)$.

When $\mathsf{Pull}(f, g)$ has a terminal object, we say that f **has an exponential along** g, and we denote the object,

$$\begin{tikzcd}
g^*(\Pi_g(f)) \arrow[r] \arrow[d, "\epsilon"'] & \Pi_g(f) \arrow[dd, "\Pi_g(f)"] \\
A \arrow[d, "f"'] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

We say that a map g is **exponentiable** when for each f, $\mathsf{Pull}(f,g)$ has a terminal object.

In a category with pullbacks, one typically thinks of a map $B \xrightarrow{g} C$ as a dependent type. Given $x : C$, we have $B(x)$ which one thinks of as $B(x) := \{y \in B \,|\, g(y) = x\}$. Given a pair of composable arrows $A \xrightarrow{f} B \xrightarrow{g} C$ then one views f as a dependent type, that depends on g.

$$\cfrac{\cfrac{x : C \vdash x : C}{x : C \vdash B(x)\ \mathsf{ty}}}{x : C, a : B(x) \vdash A(a)\ \mathsf{ty}}$$

We then think of $\Pi_g(f)$ as the implication for dependent types – this is called the dependent product:

$$\left[\!\!\left[x : C \vdash \prod_{a:B(x)} A(a) \right]\!\!\right] := \Pi_g(f) : \Pi_g(f) \longrightarrow C$$

We make this a bit more precise in what follows, in that we recall how locally cartesian closed categories are characterized by pullback extensions. The application to type theory may be found in Seely, 1984.

Lemma 5.9.11 (Weber, 2015)**.** *Let $\mathbb{X}$ be a category. Let $\mathscr{E}$ be the class of exponentiable maps. Then $\mathscr{E}$ is closed to isomorphism, compositition, and pullback.*

Thus, the exponentiable maps form a display system. In Seely, 1984, Seely defined a category to be locally cartesian closed when it has a terminal object and every map is exponentiable. More generally, we call a category **locally cartesian closed** when it has a display system $\mathscr{D}$ for which every map is exponentiable, and all the $\Pi_g(f) \in \mathscr{D}$.

When $\mathbb{X}$ has a terminal object 1, the product $A \times B$ is the pullback of the projections of A, B into 1. The following lemma says that an internal hom can be characterized, in the presence of a terminal object, by pullback extensions.

Lemma 5.9.12. *When $\mathbb{X}$ has a final object, $[A,B]$ exists iff $A \times B \xrightarrow{\pi_0} A$ has an exponential along $A \xrightarrow{!} 1$.*

Proof. The pullback over the terminal object is always a product, thus the pullback extension of the above, named suggestively, is

$$
\begin{tikzcd}
A \times [A,B] \arrow[r, "\pi_1"] \arrow[d, "\langle \pi_0, \mathsf{ev} \rangle"] \arrow[dd, bend right, "\pi_0"'] & {[A,B]} \arrow[dd, "!"] \\
A \times B \arrow[d, "\pi_0"] & \\
A \arrow[r, "!"'] & 1
\end{tikzcd}
$$

Let $A \times C \xrightarrow{f} B$. Note, we can package this up as a pullback extension:

$$
\begin{tikzcd}
A \times C \arrow[r, "\pi_1"] \arrow[d, "\langle \pi_0, f \rangle"'] & C \arrow[dd, "!"] \\
A \times B \arrow[d, "\pi_0"'] & \\
A \arrow[r, "!"'] & 1
\end{tikzcd}
$$

Hence there is a unique morphism such that the following diagram commutes.

$$
\begin{tikzcd}
& A \times [A,B] \arrow[rr, "\pi_1"] \arrow[ddl, bend left, "\langle \pi_0, \mathsf{ev} \rangle"] & & {[A,B]} \arrow[dddl, bend left, "!"] \\
A \times C \arrow[ur, bend left, "1 \times \lambda(f)"] \arrow[rr, "\pi_1"] \arrow[d, "\langle \pi_0, f \rangle"'] & & C \arrow[ur, "\lambda(f)"] \arrow[dd, "!"] & \\
A \times B \arrow[d, "\pi_0"'] & & & \\
A \arrow[rr, "!"'] & & 1 &
\end{tikzcd}
$$

The right triangle does not say much but the left says $(1 \times \lambda(f))\mathsf{ev} = f$.

Conversely, if $[A,B]$ exists, then the first diagram is a pullback extension. That it is terminal follows from the universal property of $[A,B]$. $\square$

Corollary 5.9.13. *When $\mathbb{X}$ has a final object, each $!_A$ is exponentiable iff $\mathbb{X}$ is Cartesian closed.*

Lemma 5.9.14. *When g sits over A, $\mathbb{X}/A$ has g exponentiable iff $\mathbb{X}$ has g exponentiable.*

Proof. Note that once g sits over A, the pullback

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[d] & & \cdot \arrow[dd] \\
\cdot \arrow[d, "f"'] & & \\
\cdot \arrow[rr, "g"'] \arrow[dr, "p_0"'] & & \cdot \arrow[dl, "p_1"] \\
& \cdot &
\end{tikzcd}$$

is both a pullback in $\mathbb{X}$ and $\mathbb{X}/A$. It is the terminal extension in $\mathbb{X}$ iff it is terminal in $\mathbb{X}/A$. $\square$

Combining lemma 5.9.14 with corollary 5.9.13, we obtain:

Corollary 5.9.15. *If $\mathbb{X}$ has a final object and each g is exponentiable then each $\mathbb{X}/A$ is cartesian closed.*

Proof. In $\mathbb{X}/A$ the terminal object is $A \xrightarrow{1_A} A$; the map from $X \xrightarrow{g} A$ to 1_A is g. By lemma 5.9.14, it suffices then to show that $X \xrightarrow{g} A$ is exponentiable in $\mathbb{X}/A$, and by corollary 5.9.13, it suffices to show that g is exponentiable in $\mathbb{X}$; however, this is our assumption. Thus, we have shown that $\mathbb{X}/A$ is cartesian closed. $\square$

We unpack this a bit. Let f, g be objects in $\mathbb{X}/A$. The internal hom $[f, g]$ can be constructed by first considering the pullback extension:

$$\begin{tikzcd}
f \times g \arrow[r, "\pi_1"'] \arrow[d, equal] & Z_1 \arrow[dd, "g"] \\
f \times g \arrow[d, "\pi_0 = f^*(g)"'] & \\
Z_2 \arrow[r, "f"'] & X
\end{tikzcd}$$

And taking the terminal object in $\mathsf{Pull}(f^*(g), f)$. That is, $[f, g] := \Pi_f(f^*(g))$

We can also show the converse:

Lemma 5.9.16. *A map $A \xrightarrow{g} B$ is exponentiable iff $\mathbb{X}/B \xrightarrow{g^*} \mathbb{X}/A$ has a right adjoint.*

Proof. Consider the pullback extension diagram:

$$\begin{tikzcd}
g^*(\Pi_g(f)) \arrow[r] \arrow[d, "\epsilon"'] & \Pi_g(f) \arrow[dd, "\Pi_g(f)"] \\
A \arrow[d, "f"'] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

Then since the square is a pullback, we have that $\epsilon f = g^*(\pi_g(f))$. But consider this in the slice $\mathbb{X}/B$:

$$\begin{tikzcd}
g^*(\Pi_g(f)) \arrow[rr, "\epsilon"] \arrow[dr, "\epsilon"'] \arrow[ddrr, bend right, "g^*(\Pi_g(f))"'] & & A \arrow[dd, "f"] \\
& A \arrow[dr, "f"'] & \\
& & B
\end{tikzcd}$$

Which is precisely the counit of the adjunction $g^*(\Pi_g(f)) \xrightarrow{\epsilon} f$. The universal property required for this to be an adjunction is precisely the terminality of the pullback extension. $\square$

Corollary 5.9.17. *If $\mathbb{X}$ has a final object and each $\mathbb{X}/B$ is cartesian closed, then every $g : A \longrightarrow B$ is exponentiable.*

Corollary 5.9.18. *The Beck-Chevalley condition holds. Given a commuting square (up to iso) of pullback functors*

$$\begin{tikzcd}
\cdot \arrow[r, "f^*"] \arrow[d, "g^*"'] & \cdot \arrow[d, "k^*"] \\
\cdot \arrow[r, "h^*"'] & \cdot
\end{tikzcd}$$

then the map:

$$\Pi_f; g^* \xrightarrow{1;\eta_h} \Pi_f; g^*; h^*; \Pi_h \simeq \Pi_f; f^*; k^*; \Pi_h \xrightarrow{\epsilon_f;1} k^*; \Pi_h$$

is an isomorphism

Putting everything together,

Proposition 5.9.19. *Let $\mathbb{X}$ be a category, and $\mathscr{E}$ be the class of exponentiable maps in $\mathbb{X}$. Then $\mathbb{X}^{\mathscr{E}} \xrightarrow{\mathsf{cod}} \mathbb{X}$ is a fibered Cartesian closed category. In particular, each* $\mathsf{cod}^{-1}(A)$ *is Cartesian closed.*

5.9.4 Locally coherently closed tangent categories

Let $\mathbb{X}$ be a tangent category. Let $\mathscr{E}$ be a transverse display system where every map is exponentiable and the exponents are displayed. Denote the fiber $\mathsf{cod}^{-1}(M)$ by $\mathbb{X}/M$. Denote the internal hom in the slice $\mathbb{X}/M$ by $[\_,\_]_M$, and the tangent functor by T_M.

Definition 5.9.20. $\mathbb{X}$ *is* ***locally coherently closed*** *when $T_M[f,g]_M \xrightarrow{\psi_M} [f, T_M g]_M$ is an isomorphism.*

Proposition 5.9.21. *When $\mathbb{X}$ is a locally coherently closed tangent category then the differential objects of $\mathbb{X}/M$ are a differential λ-category.*

Corollary 5.9.22. *If the map into the terminal object $M \xrightarrow{!} 1$ is in $\mathscr{E}$ for every M, then $\mathbb{X}/1 \simeq \mathbb{X}$, and $\mathbb{X}$ is coherently closed.*

The differential λ-calculus admits an interpretation into the differential objects of each slice of a relatively locally coherently closed tangent category. But, as the differential objects are precisely differential bundles by 5.5.10 we have:

Corollary 5.9.23. *In a locally coherently closed tangent category, the differential λ-calculus admits an interpretation into the smooth maps between differential bundles over a fixed base.*

What is also interesting is that immediately from proposition 5.5.8

Proposition 5.9.24. *Let f be a differential bundle, and g any map. Then in the terminal pullback extension:*

$$\begin{tikzcd}
g^*(\Pi_g(f)) \arrow[r] \arrow[d, "\epsilon"'] \arrow[ddr, phantom, "\lrcorner", very near start] & \Pi_g(f) \arrow[dd, "\Pi_g(f)"] \\
A \arrow[d, "f"'] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

$\Pi_g(f)$ *is a differential bundle.*

Nishimura introduced a setting, called *axiomatic differential geometry* Nishimura, 2012a, containing axioms on a category for doing differential geometry. In Nishimura, 2012c, he "refurbishes" his axioms, and the end result is a locally cartesian closed category, where each slice admits an action by Weil algebras. If we strengthen his notion of action of Weil algebras, and drop the exactness criterion, we obtain exactly a locally coherently closed tangent category. It is interesting that Nishimura obtained the coherence from a completely different point of view: our perspective was to provide models for the differential λ-calculus, and his was for extracting good subcategories of Frölicher spaces. Thus, the settings investigated in this thesis, using the internal logic of differential bundles, could potentially make a reworking and formalization of Nishimura's ideas, using tangent categories, straightforward.

Chapter 6

Weil algebras, closed tangent categories, and constructing models

In this section, we make use of a technique employed first in synthetic differential geometry of extracting microlinear spaces (Reyes and Wraith, 1978a). In a smooth topos, or model of SDG, not all of the objects have the property that TM (which is $[D,M]$) behave locally like an $\mathscr{R}$-vector space. The microlinear spaces are essentially the ones that do behave locally like an $\mathscr{R}$-vector space.

(Kock, 1986) introduced the Weil prolongation of a convenient vector space, to show that they embed into the $\mathscr{R}$-vector spaces of the Cahiers topos. The Weil prolongation introduced by Kock is an instance of a structure known as an actegory [1], as introduced in (Janelidze and Kelly, 2001). An *actegory* is a monoidal category acting coherently on another category. (Kolar, Michor, and J., 1993) and (Kriegl and Michor, 1997) use Weil prolongations on arbitrary dimension smooth manifolds to characterize product preserving functors.

In this chapter, we will establish a connection between Weil prolongations and tangent categories by using the characterization of tangent categories in (Leung, 2017). We will then revisit the notion of coherently closed tangent category

[1] Kock's paper predates the Janelidze-Kelly paper on actegories. In (Kock, 1986), he calls actegories semidirect product of categories, and his concept of Weil prolongation is an actegory.

from an enriched viewpoint on tangent categories due to (Garner, 2018). Garner proved that a tangent category is precisely a category enriched in what we will call microlinear Weil spaces. The category of microlinear Weil spaces is precisely the category of models of the finite limit theory of Weil algebras in Set. We will then characterize transverse limits and coherent closure in terms of enrichment. We will then show how to extract microlinear spaces from an action by Weil algebras to form a tangent category, and then we will show how to extract a coherently closed tangent category from a Cartesian closed category. We end with a few examples of coherently closed tangent categories.

6.1 Tangent structure from Weil actions

Tangent structure allows forming objects by pullback of the bundle projection, forming e.g. $T_n(M)$ which is the bundle of n tangent vectors at the same point in M, and then iterating this construction forming objects like $T_{n_k}(\cdots T_{n_1}(M)\cdots)$. Leung argued that in a presentation of tangent categories, one may view the iterated tangent constructions using graphs. T_n is viewed as a clique with n nodes, and the iterated tangent functors should be viewed as disjoint union of graphs. To make this formal, these graphs express relations between generators in a Weil algebra. The idea, explored in (Leung, 2017), is that each node corresponds to a generator of a Weil algebra and edges correspond to a relation given by multiplying these generators together; the graphs are all reflexive. The single clique with 3 nodes for example corresponds to $\mathscr{R}[x,y,z]/(x^2,xy,y^2,xz,yz,z^2)$. This Weil algebra then correponds to T_3. In fact, these kinds of relations express a subcategory of Weil algebras, and all the natural maps that arise from tangent structure arise as Weil algebra homomorphisms (Leung, 2017).

6.1.1 Weil Algebras

Intuitively, a Weil algebra is a finite dimensional algebra that captures an infinitesimal extension of R. In other words, Weil algebras have a presentation as $a+d$ where d is so small that $d^k = 0$. We work over a commutative *rig*, instead of a commutative *ring* – these are like rings, but without subtraction.

Definition 6.1.1. *Let R be a commutative rig. A* ***Weil algebra*** *(Dubuc and Kock, 1984; Lavendhomme, 1996) W over R is a commutative, unital R algebra such that*

- *The underlying R-module is R^{n+1};*
- *The unit u_W is $(1,0,\cdots,0)$;*
- *The image of the embedding $R^n \rightarrowtail R^{n+1}$ given by $(a_1,\ldots,a_n) \mapsto (0,a_1,\ldots,a_n)$ is a nilpotent ideal I in R^{n+1}.*

A Weil algebra is canonically augmented: this means there is a unital R-algebra homomorphism $W \xrightarrow{\epsilon} R$. The augmentation on the underlying R-modules is defined $\epsilon(a_1,\ldots,a_{n+1}) := a_1$. The augmentation is a retract of the unit map $\eta : R \longrightarrow W$ where $\eta(a) = (a,0,\ldots,0)$. Also note that the nilpotent ideal I which is the image of the embedding $R^n \longrightarrow R^{n+1}$ is the kernel of ϵ.

A **morphism of Weil algebras** is a morphism of augmented algebras.

Now, let W be any augmented algebra. As $\epsilon : W \longrightarrow R$ is a unital algebra map, $\epsilon(u_W) = 1 \in R$. Then the map $\eta : R \longrightarrow W$ defined by $r \mapsto r\,u_W$ has $\epsilon(\eta(r)) = \epsilon(r\,u_W) = r\epsilon(u_W) = r$, witnessing η as a section of ϵ; in particular η is injective. Thus, for any augmented algebra W, $R \cdot u_W := \{r\,u_W \mid r \in R\}$ is isomorphic to R as an R algebra.

When R is a ring, an augmented algebra A has an important property: elements of A really can be thought of as elements of R extended in some way. Given an element $a \in A$, we may decompose it into $\epsilon(a)u_R + a - \epsilon(a)u_R$ because $a - \epsilon(a)u_R \in \ker\epsilon$. The map $a \mapsto (\epsilon(a)u_A, a - \epsilon(a)u_A)$ defines an isomorphism $A \simeq R \cdot u_A \oplus \ker\epsilon$.

When R is just a commutative rig, we have a weaker result:

Observation 6.1.2. *If R is a commutative rig, and A is an augmented algebra over R with cancellative addition, then $R \cdot u_A \oplus \ker\epsilon \rightarrowtail A$ is an inclusion of the underlying R modules.*

Proof. Define $R \cdot u_A \oplus \ker\epsilon \xrightarrow{f} A$ by $f(r\,u_A, v) \mapsto r\,u_A + v$. This is easily seen to be injective: suppose $r\,u_A + v = s\,u_A + w$.

$$r = \epsilon(r\,u_A + v) = \epsilon(s\,u_A + w) = s$$

And hence $r = s$, and $r u_A = s u_A$. Then $r u_A + v = r u_A + w$ implies $v = w$ as addition is cancellative. Thus the map is injective. $\square$

Observation 6.1.3. *When R is a commutative ring, and A is an augmented algebra, then the map of 6.1.2 is an isomorphism of R-modules.*

Proof. Any element $a \in A$ can be separated as $\epsilon(a)u_A + a - \epsilon(a)u_A$, and note that $a - \epsilon(a)u_A \in \ker\epsilon$. Then the map of 6.1.2 is inverted by $a \mapsto (\epsilon(a)u_A, a - \epsilon(a)u_A)$. $\square$

Thus for augmented algebras A over a commutative ring R, the underlying R-module of A is isomorphic to $R \cdot u_A \oplus \ker\epsilon$. For augmented algebras over commutative rigs, we prefer to restrict to augmented algebras that *split* that is augmented algebras whose underlying R-modules are of the form:

$$\begin{tikzcd} A \simeq R \cdot u_A \oplus \ker\epsilon \arrow[d, "\pi_0"] \arrow[d, "\epsilon"'] \\ R \end{tikzcd}$$

Observation 6.1.4. *For a split augmented algebra over a commutative rig, the multiplicative unit of A is $(u_R, 0)$.*

Proof. The inclusion $R \rightarrowtail R \oplus \ker\epsilon$ given by $r \mapsto (r, 0)$ is an algebra homomorphism; in particular $u_R \mapsto u_A$. $\square$

Observation 6.1.5. *If W is a commutative, unital, split augmented algebra whose augmentation ideal is nilpotent, and whose underlying R-module is finitely generated and free, then under the identification $W \simeq R \cdot u_W \oplus \ker\epsilon$, any element of the form $(0, v)$ is nilpotent. Thus W is precisely a Weil algebra as defined at the beginning of the section.*

For the next result, we recall that commutative algebras over commutative rigs have quotients by kernels. Any ideal I of a commutative rig A gives rise to the *Bourne identification* (Golan, 1999) $a \sim_I b$ if there exist $x, y \in I$ such that $a + x = b + y$. It is routine to show that this is a rig congruence. Further if the rig structure on A arises as an R-algebra structure, then the congruence $\sim_I$ is additionally an R-algebra congruence. Thus one can form $A/I := \{[a] | a \in A\}$

where $[a]$ denotes the equivalence class generated by $\sim_I$ containing a. When $A \xrightarrow{f} B$ is a rig (or algebra) homomorphism then $\ker f$ is an ideal of A, and $a \sim_{\ker f} b$ if and only $f(a) = f(b)$. It is not hard to show that the map $\pi : x \mapsto [x]$ is the coequalizer of the kernel pair.

Thus

$$\begin{tikzcd} A_f \times_f A \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & A \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & A/\ker f \arrow[d, dotted] \\ & & B \end{tikzcd}$$

The map π is guaranteed to be epic; one can also show that the map $A/\ker f \longrightarrow B$ is monic.

Lemma 6.1.6. *A (split) Weil algebra over a commutative rig R has a presentation as $R[x_1,\dots,x_n]/I$ where I is a nilpotent ideal, and the augmentation sends $a + p(x_1,\dots,x_n) \mapsto a$.*

Proof. Let W be a Weil algebra whose underlying R module is R^{n+1}, and whose basis is the standard basis $e_0,\dots,e_n$. By 6.1.4, we choose e_0 to be the multiplicative unit of W. Define a map from the free commutative algebra on n generators to W

$$R[x_1,\dots,x_n] \xrightarrow{\tau} W$$

by $\tau(x_i) := e_i$. τ is surjective and hence $R[x_1,\dots,x_n]/\ker\tau \simeq W$. □

Observation 6.1.7. *Given Weil algebras U, V presented by $R[x_1,\dots,x_n]/I_U$ and $R[y_1,\dots,y_m]/I_V$ the product is:*

$$U \times V := R[x_1,\dots,x_n,y_1,\dots,y_m]/(I_U \cup I_V \cup \{x_i y_j\})$$

and the coproduct is:

$$U \otimes V := R[x_1,\dots,x_n,y_1,\dots,y_m]/(I_U \cup I_V)$$

Note that the coproduct has projections given by the augmentations $U \otimes V \xrightarrow{U \otimes \epsilon_V} U \otimes R \simeq U$. Also R is a zero object.

6.1.2 Real Weil algebras

For certain applications, it is useful to have a different view on Weil algebras over $\mathscr{R}$. These are used in (Kolar, Michor, and J., 1993) and (Kriegl and Michor, 1997) to study product preserving functors and their natural transformations, on the category of manifolds.

Define the Lawvere theory T_∞ whose objects are powers of $\mathscr{R}$, and whose generating maps $\mathscr{R}^n \longrightarrow \mathscr{R}$ are smooth maps in the sense that f is smooth if it is continuously differentiable and its derivative is smooth.

Observation 6.1.8. *For each n, $T_\infty(n,1)$ is an $\mathscr{R}$ algebra by pointwise multiplication.*

Two smooth functions $f,g \in T_\infty(n,1)$ **have the same germ at** x when there is a neighborhood $U \subset \mathscr{R}^n$ containing x and such that for all $v \in U$ we have $f(v) = g(v)$. We write $f \sim_x g$ to denote that f and g have the same germ at x.

We write $\mathscr{E}_n$ or sometimes $C_0^\infty(n,1)$ to denote $T_\infty(n,1)/_{\sim_0}$.

Observation 6.1.9. *$\mathscr{E}_n$ is an $\mathscr{R}$-algebra by $[f][g] = [fg]$*

The following proposition is from (Kolar, Michor, and J., 1993), theorem 35.5. We illustrate the idea with an example. Let $\mathscr{R} \xrightarrow{f} \mathscr{R}$ be smooth, and apply Taylor's theorem around 0 to obtain the k^{th} degree Taylor polynomial and remainder:

$$f(x) = \sum_{m=0}^{k} \frac{1}{m!} \frac{\partial^m f(t)}{\partial t^m}(0) \cdot x^m + R(f,0,k) \cdot x^{k+1}$$

If we work with germs of functions around 0, then for every element $g \in [f]$, applying Taylors theorem results in the same Taylor polynomial of degree k. More generally, one can define the Taylor expansion of a germ at 0: as the Taylor expansion at 0 for any g,f with $g \sim_0 f$ is the same, we can apply the Taylor expansion to any representative of a germ, and this is a well defined construction of a formal power series.

It then makes sense to talk about the quotient of the space of germs by a (set of) polynomials. Take for example x^2 and define an equivalence relation $[f] \sim_{x^2} [g]$ if $f - g \in \langle x^2 \rangle$. Note that $h \in \langle x^2 \rangle$ if and only if its Taylor polynomial of any degree

k has no constant part and no degree 1 part. Then $f - g \in \langle x^2 \rangle$ iff $f - g$ has no constant or degree 1 part. This is true iff the Taylor expansion of f and g agree at the first two terms, meaning that $f(0) = g(0)$ and that they have the same derivative at 0. Since the relation is determined by Taylor expansion $\sim_{x^2}$ is well defined in the sense that it is independent of representative elements of germs. Then if we take $\mathscr{E}_1/\langle x^2 \rangle$ to mean $\mathscr{E}_1/\sim_{x^2}$ it is clear that the following (2 dimensional) algebras are the same: $\mathscr{E}_1/\langle x^2 \rangle$, $R[x_1,\ldots,x_n]/\langle x^2 \rangle$, and $R[[x_1,\ldots,x_n]]/\langle x^2 \rangle$ where the last is the quotient of the algebra of formal power series. More generally,

Proposition 6.1.10 (Kolar, Michor, and J., 1993)**.** *Let W be an $\mathscr{R}$ algebra. The following are equivalent:*

1. *W is a Weil algebra;*
2. *$W \simeq \mathscr{R}[[x_1,\ldots,x_n]]/I$ for some n and I and has a finite dimensional underlying vector space;*
3. *$W \simeq \mathscr{E}_n/I$ for some n and has a finite dimensional underlying vector space.*

The hard part is showing that if $\mathscr{E}_n/I$ if finite dimensional, then I can be generated by a finite set of polynomials containing $x_i^{k_i}$ for each i and some k_i.

6.1.3 Tangent structure on Weil algebras

Leung shows that Weil algebras are a tangent category (Leung, 2017). The idea is that $TA := A \otimes R[x]/(x^2)$. Moreover, Leung shows that Weil algebras over $\mathbb{N}$ are primordial: every tangent category arises from this tangent structure. Actually, the whole category of Weil algebras is not needed, and Leung uses a subcategory of Weil algebras called $\mathscr{W}_1$.

Consider the Weil algebras of the form

$$W^n := R[x_1,\ldots,x_n]/(x_i x_j)_{1\le i\le j\le n}$$

For example $W^1 = R[x]/(x^2)$, $W^1 \otimes W^1 = R[x,y]/(x^2,y^2)$, and $W^2 = R[x,y]/(x^2,xy,y^2)$. Note that $W^2 = W^1 \times W^1$. Not all Weil algebras are of this form; missing for example are objects of the form $W_n := R[x]/(x^n)$.

Take $\mathscr{W}_1$ to be the category whose objects are of the form $W^{n_1} \otimes \cdots \otimes W^{n_k}$, and whose maps are augmented algebra homomorphisms.

Proposition 6.1.11 (Leung)**.** *If R is a ring, then for any Weil algebra W, the functor $\_ \otimes W : \mathsf{Weil} \longrightarrow \mathsf{Weil}$ preserves finite limits. Moreover, if R is a ring or the rig $\mathbb{N}$, and W is in $\mathscr{W}_1$, the functor $\_ \otimes W : \mathscr{W}_1 \longrightarrow \mathscr{W}_1$ preserves finite limits.*

sketch. This holds more generally for finitely presented R-algebras. □

This fact is used for two reasons: the first is to show the existence of certain limits, and the second is that it proves that tangent functors preserve these limits.

Corollary 6.1.12. *In Weil algebras over a ring R, and $\mathscr{W}_1$ over a ring or $\mathbb{N}$, for any product $B \times C$, and any A the diagram below is a pullback*

$$\begin{tikzcd}
A \otimes (B \times C) \arrow[r, "A\otimes\pi_1"] \arrow[d, "A\otimes\pi_0"'] & A \otimes C \arrow[d, "1\otimes\epsilon_C"] \\
A \otimes B \arrow[r, "1\otimes\epsilon_B"'] & A
\end{tikzcd}$$

Weil has all products, but products in $\mathscr{W}_1$ are only of the form W^n. Leung calls pullbacks that arise from 6.1.12 *foundational pullbacks.*

We have already seen the map $W^1 \xrightarrow{\epsilon} R$; explicitly it sends $a + bx \mapsto a$. We also have $R \xrightarrow{\eta} W^1$ which sends $a \mapsto a+0$. There is the exchange map $W^1 \otimes W^1 \xrightarrow{c_\otimes} W^1 \otimes W^1$ which sends $a + bx + cy + dxy \mapsto a + cx + by + dxy$. There is the map $W^1 \times W^1 \xrightarrow{\sigma} W^1$ which sends $a + bx + cy \mapsto a + (b+c)x$. There is the map $W^1 \xrightarrow{\lambda} W^1 \otimes W^1$ which sends the generator x to xy and so sends $a + bx \mapsto a + 0 + 0 + bxy$.

Proposition 6.1.13. *[Leung] When R is a ring or $\mathbb{N}$, the map*

$$W^2 \xrightarrow{\langle \pi_0\lambda, \pi_1\eta\rangle} (W^1 \times W^1) \otimes W^1 \xrightarrow{\sigma \otimes W^1} W^1 \otimes W^1$$

is the equalizer of

$$\begin{tikzcd}
W^1 \otimes W^1 \arrow[rr, bend left, "\epsilon\otimes W^1"] \arrow[r, "\epsilon\otimes\epsilon"'] & R \arrow[r, "\eta"'] & W^1
\end{tikzcd}$$

in $\mathscr{W}_1$.

Proof. Considering a presentation of W^2 as $R[x,y]/(x^2,xy,y^2)$ and $(W^1 \times W^1) \otimes W^1$ by $R[x,y,z]/(x^2,y^2,z^2,xy)$ then $\pi_0\lambda$ sends $a+bx+cy \mapsto a+bxz$ and $\pi_1\eta$ sends $a+bx+cy \mapsto a+cy$. Thus $\langle \pi_0\lambda, \pi_1\eta \rangle$ sends $a+bx+cy \mapsto a+cy+bxz$. Then considering the last $W^1 \otimes W^1$ as $R[x,z]/(x^2,z^2)$ the map $\sigma \otimes W^1$ sends $a+cy+bxz \mapsto a+cx+0y+bxz$.

The map $\epsilon \otimes 1 : R[x,z]/(x^2,z^2) \longrightarrow R[z]/z^2$ sends the generator x to 0, and hence sends $a+bx+cz+dxz \mapsto a+cz$. Similarly, $\epsilon \otimes \epsilon$ sends $x,z \mapsto 0$, and sends $a+bx+cy+dxz \mapsto a$, and hence $(\epsilon \otimes \epsilon)\eta$ sends $a+bx+cz+dxz \mapsto a+0$. Thus both $(\epsilon \otimes \epsilon)\eta$ and $\epsilon \otimes 1$ sends $a+cx+bxz \mapsto a+0$. Therefore $\langle \pi_0\lambda, \pi_1\eta \rangle(\sigma \otimes 1)$ weakly equalizes the map.

When addition is cancellative: $a+b=a+c$ implies $b=c$ then, we may show this universal. Let U be any other object and $h : U \longrightarrow W^1 \otimes W^1 = R[x,z]/(x^2,z^2)$ be another weak equalizer. Note h may be written in the form

$$h(u) = u_0 + u_1 x + u_2 z + u_3 xz$$

Note that

$$(\epsilon \otimes 1)(h(u)) = (\epsilon \otimes 1)(u_0 + u_1 x + u_2 z + u_3 xz) = u_0 + u_2 z$$

and

$$\eta((\epsilon \otimes \epsilon)(u_0 + u_1 x + u_2 z + u_3 xz)) = u_0$$

and by cancellativity of addition together with the assumption that h equalizes $\epsilon \otimes 1$ and $(\epsilon \otimes \epsilon)\eta$ implies that $u_2 = 0$. But this means that

$$h(u) = u_0 + u_1 x + 0z + u_3 xz$$

hence $\hat{h}(u) := u_0 + u_1 x + u_3 y$ is a well defined map $\hat{h} : U \longrightarrow W^1 \times W^1 = R[x,y]/(x^2,xy,y^2)$. It is immediate that $\hat{h}\langle \pi_0\lambda, \pi_1\eta \rangle(\sigma \otimes 1) = h$ as the second two maps in the composite simply put the $0z$ back in. Using cancellativity of addition again, if $v\langle \pi_0\lambda, \pi_1\eta \rangle(\sigma \otimes 1) = h$, it is immediate that $v = \hat{h}$ □

Definition 6.1.14. *In* Weil *and* $\mathscr{W}_1$*, a* ***transverse limit*** *is defined inductively by*

- *A product $A \times B$ is a transverse limit;*

- *The equalizer of 6.1.13 is a transverse limit;*
- *If L is a transverse limit, and A is an object, then $A \otimes L$ is a transverse limit.*

We now describe the tangent structure on Weil and $\mathscr{W}_1$.

$$TA := A \otimes W^1 \qquad TA \xrightarrow{p} A := A \otimes W^1 \xrightarrow{A \otimes \epsilon_{W^1}} A$$
$$A \xrightarrow{0} TA := A \xrightarrow{A \otimes \eta} A \otimes W^1$$

Note that T_2A should rise as the pullback of p by p: this is a foundational pullback. $T_2A := A \otimes (W^1 \times W^1) = A \otimes R[x,y]/(x^2, xy, y^2) = A \otimes W^2$ and hence is in $\mathscr{W}_1$ if A is.

$$T_2A \xrightarrow{+} TA := A \otimes W^2 \xrightarrow{A \otimes \sigma} A \otimes W^1$$

It is not hard to see that $+, 0$ form a commutative monoid in the slice.

$T^2A := (A \otimes W^1) \otimes W^1 \simeq A \otimes (W^1 \otimes W^1) \simeq A \otimes R[x,y]/(x^2, y^2)$. The lift is

$$TA \xrightarrow{l} T^2A := A \otimes W^1 \xrightarrow{A \otimes \lambda} A \otimes (W^1 \otimes W^1)$$

Finally

$$T^2A \xrightarrow{c} T^2A := A \otimes (W^1 \otimes W^1) \xrightarrow{1 \otimes c_{\otimes}} A \otimes (W^1 \otimes W^1)$$

Proposition 6.1.15 (Leung, 2017)**.** *The category R-*Weil *is a tangent category when R is a ring, and R-$\mathscr{W}_1$ is a tangent category when R is a ring or $\mathbb{N}$.*

The required data has been defined above. The proof that pullback powers exist is 6.1.12, and the proof that the universality of the vertical lift holds comes from 6.1.13 combined with 6.1.11. The rest of the equality may be checked easily.

And Leung's main result is

Theorem 6.1.16. *[Leung, 2017] To have a tangent structure on a category $\mathbb{X}$ is precisely to have a monoidal functor*

$$(\mathbb{N}\text{-}\mathscr{W}_1, \otimes, \mathbb{N}) \longrightarrow (\mathsf{Fun}(\mathbb{X}, \mathbb{X}), \circ, \mathsf{id})$$

that sends transverse limits to pointwise limits.

One direction of the proof is easy: if one has such a monoidal functor that sends transverse limits to pointwise limits, then it is immediate to see that this creates tangent structure on $\mathbb{X}$. Going the other direction Leung shows that every map in $\mathbb{N}\text{-}\mathscr{W}_1$ can be constructed from tangent category maps; hence proving initiality.

6.2 Weil prolongation: tangent categories as actegories

In this section we will make use of theorem 6.1.16 to generate new tangent categories from old, and to generate tangent categories from things that should be, but aren't quite. The idea comes from formalizing the notion of microlinearity (Kock, 1978; Reyes and Wraith, 1978a) as a general phenomenon for categories that admit an action by $\mathscr{W}_1$.

We first weaken theorem 6.1.16, and "uncurry" it. Throughout this section, we will use R to denote a ring, or the rig $\mathbb{N}$.

Corollary 6.2.1. *If there is a bifunctor (action)*

$$\mathbb{X} \times R\text{-}\mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$$

such that

- *The action is isomonoidal; that is* $A \otimes_\infty R \simeq A$ *and* $(A \otimes_\infty U) \otimes_\infty V \simeq A \otimes_\infty (U \otimes V)$*;*
- *The action satisfies the actegory pentagon and triangle identities (Janelidze and Kelly, 2001):*

$$\begin{tikzcd}
A \otimes_\infty ((U \otimes V) \otimes W) \arrow[rr] \arrow[d] & & (A \otimes_\infty (U \otimes V)) \otimes_\infty W \arrow[d] \\
A \otimes_\infty (U \otimes (V \otimes W)) \arrow[r] & (A \otimes_\infty U) \otimes_\infty (V \otimes W) \arrow[r] & ((A \otimes_\infty U) \otimes_\infty V) \otimes_\infty W
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes_\infty (R \otimes U) \arrow[r] \arrow[dr] & (A \otimes_\infty R) \otimes_\infty U \arrow[d] \\
& A \otimes U
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_\infty (U \otimes R) \arrow[r] \arrow[d] & (A \otimes_\infty U) \otimes_\infty R \arrow[dl] \\
A \otimes_\infty U &
\end{tikzcd}$$

- *For each transverse limit L in $R\text{-}\mathscr{W}_1$, and each object $A \in \mathbb{X}$, $A \otimes_\infty L$ is a limit in $\mathbb{X}$;*

Then $\mathbb{X}$ is a tangent category[2].

We call the structure giving rise to a tangent category expressed by 6.2.1 **transverse Weil prolongation**. Thanks to Leung's theorem, we know that every tangent structure arises from a transverse Weil prolongation. Of course for any ring R, we need not have restricted to $R\text{-}\mathscr{W}_1$: if there is a transverse prolongation by the whole of Weil, then one gets a tangent category (and with some additional features as well).

The moral here is that every tangent functor is of the form $TM \simeq M \otimes_\infty R[x]/(x^2)$.

We recast a few examples of tangent structure

- Every category is a tangent category where $\_ \otimes_\infty U = \mathsf{id}$. A square of identity arrows is a pullback.
- If $\mathbb{X}$ is a cartesian differential category then $M \otimes_\infty W^1 := M \times M$ and

$$M \otimes_\infty W^1 \xrightarrow{f \otimes_\infty W^1 := \langle D[f], \pi_1 f \rangle} N \otimes_\infty W^1$$

- The transverse Weil prolongation of a smooth manifold M by W^1 is defined to be $C^\infty(\mathscr{R}, M)/_\sim$ where $c_1 \sim c_2$ if $c_1(0) = c_2(0)$ and for every smooth function $M \xrightarrow{f} \mathscr{R}$, $(c_1 f)'(0) = (c_2 f)'(0)$. Given a smooth $M \xrightarrow{g} N$

$$M \otimes_\infty W^1 \xrightarrow{g \otimes_\infty W^1} N \otimes_\infty W^1$$

  is the smooth function which sends an equivalence class $[c]$ to $[cg]$.

- Functors in:

  Suppose $\mathbb{Y}$ is a tangent category, and obtain its transverse Weil prolongation, $\mathbb{Y} \times R\text{-}\mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{Y}$. Let $\mathbb{X}$ be any category. We get a bifunctor

$$[\mathbb{X}, \mathbb{Y}] \times R\text{-}\mathscr{W}_1 \xrightarrow{\otimes_\infty} [\mathbb{X}, \mathbb{Y}]$$

[2] The first two points say that the bifunctor $\otimes_\infty$ forms an instance of an **actegory**.

For any U in $R\text{-}\mathscr{W}_1$, and any functor $\mathbb{X} \xrightarrow{F} \mathbb{Y}$, then $F \otimes_\infty U$ is the functor which sends $X \in \mathbb{X}$ to $F(X) \otimes_\infty U$ in $\mathbb{Y}$. This is clearly a monoidal action, as the action in $\mathbb{Y}$ is. Also, if L is a limit in $R\text{-}\mathscr{W}_1$, then for each $X \in \mathbb{X}$, $(F \otimes_\infty L)(X) = F(X) \otimes_\infty L$ is a limit in $\mathbb{Y}$, and hence a $F \otimes_\infty L$ is a limit in $[\mathbb{X}, \mathbb{Y}]$. Thus, we have a transverse Weil prolongation on $[\mathbb{X}, \mathbb{Y}]$.

Next, we consider two important examples that should be tangent categories but are not quite.

- In SDG, the tangent functor is $TM := [D, M]$. In general, for each Weil algebra
$$V \equiv W_1^{n_1} \otimes \cdots W_1^{n_k} \in \mathscr{W}_1$$
The object $D(V) := D(n_1) \times \cdots D(n_k)$ represents it in the sense that the action is $M \otimes_\infty V := [D(V), M]$ where $D(n) := \{(d_1, \ldots, d_n) \in D^n \mid d_i d_j = 0\}$ We give more details in the sequel; for now, the spaces that perceive certain diagrams as colimits are called microlinear, and the microlinear spaces of SDG always form a tangent category (Rosický, 1984; Cockett and Cruttwell, 2014b).

- Functors out:
Let $\mathbb{X}$ be a tangent category, and obtain its transverse Weil prolongation $\mathbb{X} \times R\text{-}\mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$. Let $\mathbb{Y}$ be any category. We obtain an action
$$[\mathbb{X}, \mathbb{Y}] \times R\text{-}\mathscr{W}_1 \xrightarrow{\otimes_\infty} [\mathbb{X}, \mathbb{Y}]$$
for any $U \in R\text{-}\mathscr{W}_1$ and any functor $\mathbb{X} \xrightarrow{F} \mathbb{Y}$, defined as
$$(F \otimes_\infty U)(X) := F(X \otimes_\infty U)$$
This is clearly bifunctorial, and is a monoidal action; however, as F need not preserve limits, $F(X \otimes_\infty L)$ need not be a limit for every limit L in $R\text{-}\mathscr{W}_1$.

Note also that if $\mathbb{X}$ has products, the requirement to be a cartesian tangent category is:

Observation 6.2.2. *A tangent category with products is a cartesian tangent category iff for each $\mathscr{W}_1$ algebra V, the functor $\_ \otimes_\infty V : \mathbb{X} \longrightarrow \mathbb{X}$ preserves products.*

However, we should spell out what it means for an action to preserve products. First, this means that for each $V \in \mathscr{W}_1$ there are isomorphisms

$$(A \otimes_\infty V) \times (B \otimes_\infty V) \xrightarrow{m_\times} (A \times B) \otimes_\infty V \qquad 1 \xrightarrow{m_\top} 1 \otimes_\infty V$$

Moreover, this must be coherent:

$$\begin{tikzcd}
(A \otimes_\infty (U \otimes V)) \times (B \otimes_\infty (U \otimes V)) \arrow[r, "m_\times"] \arrow[d, "\alpha\times\alpha"'] & (A \times B) \otimes (U \otimes V) \arrow[dd, "\alpha"] \\
((A \otimes_\infty U) \otimes_\infty V) \times ((B \otimes_\infty U) \otimes_\infty V) \arrow[d, "m_\times"'] & \\
((A \otimes_\infty U) \times (B \otimes_\infty U)) \otimes_\infty V \arrow[r, "m_\times \otimes_\infty V"'] & ((A \times B) \otimes_\infty U) \otimes_\infty V
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, "m_\top"] \arrow[d, "m_\top"'] & 1 \otimes_\infty (U \otimes V) \arrow[d, "\alpha"] \\
1 \otimes_\infty V \arrow[r, "m_\top \otimes_\infty V"'] & (1 \otimes_\infty U) \otimes_\infty V
\end{tikzcd}$$

6.3 An enriched view on coherent closure

In this section we indicate the meaning of coherently closed from an enriched point of view: the coherence gives enriched cartesian closure.

Let $\mathscr{V}$ be a (small) symmetric monoidal category. There is a monoidal-closed structure on $[\mathscr{V}, \mathsf{Set}]$, which is called the convolution monoidal structure and is defined:

Unit: $\mathscr{Y}(I) = \mathscr{V}(I, \_)$;

Tensor: $(F \otimes G)(X) := \int^{M,N \in \mathscr{V}} \mathscr{V}(M \otimes N, X) \times FM \times GN$;

Hom: $[F,G](X) := \int_{M \in \mathscr{V}} [FM, G(X \otimes M)]$.

The category $[\mathscr{V}, \mathsf{Set}]$ equipped with the above monoidal structure is denoted $\mathscr{P}(\mathscr{V})$.

Theorem 6.3.1 ((Wood, 1978))**.**

1. *Let $\mathscr{V}$ be a (small) symmetric monoidal category. To have a $\mathscr{P}(\mathscr{V})$ enriched category $\underline{C}$ is to have*

   - *A set of objects $\underline{C}_0$;*
   - *For each $x, y \in \underline{C}_0$, a presheaf $\underline{C}(x, y) : \mathscr{V} \longrightarrow \mathsf{Set}$;*
   - *For each $x \in \underline{C}_0$, an identity $\mathsf{Id}_x \in \underline{C}(x, x)(I)$;*
   - *For each $x, y, z \in \underline{C}_0$, a family of compositions*

     $$\underline{C}(x, y)(M) \times \underline{C}(y, z)(N) \longrightarrow \underline{C}(x, z)(M \otimes N)$$

     that is natural in M, N.

2. *There is a correspondence, up to isomorphism, of $\mathscr{V}$-actegories and $\mathscr{P}(\mathscr{V})$-enriched categories that admit powers by representable functors. This correspondence extends to a 2-equivalence.*

Proof. See (Wood, 1978). □

Given a $\mathscr{P}(\mathscr{V})$ category, the action of the corresponding actegory is the power by a representable:

$$M \otimes_\infty V \equiv \mathscr{Y}(V) \pitchfork M$$

Then combining this with the enriched Yoneda lemma followed by the fact that a $\mathscr{P}(\mathscr{V})$ category admits powers by representables we have that the hom in $\mathscr{P}(\mathscr{V})$ enriched category as a functor $\underline{C}(A, B) : M \longrightarrow \mathsf{Set}$ yields

$$\underline{C}(A, B)(U) \simeq [\mathscr{V}, \mathsf{Set}](\mathscr{Y}(U), \underline{C}(A, B)) \simeq \underline{C}_0(A, \mathscr{Y}(U) \pitchfork B) \simeq \underline{C}_0(A, B \otimes_\infty U)$$

where $\underline{C}_0(A, B) := \underline{C}(A, B)(I)$. Thus:

$$\underline{C}(A, B) \simeq \lambda U . \underline{C}_0(A, B \otimes_\infty U) : \mathscr{W}_1 \longrightarrow \mathsf{Set}$$

(Garner, 2018) shows that as the monoidal structure on $\mathscr{W}_1$ is coproduct, then for the convolution monoidal structure on functors $\mathscr{W}_1 \longrightarrow \mathsf{Set}$, every functor is a cocommutative comonoid, and hence the tensor is the product. It follows immediately that the hom is the internal hom for the cartesian closed structure on $[\mathscr{W}_1, \mathsf{Set}]$.

To characterize tangent categories as enriched categories it is not quite sufficient to regard them as $\mathscr{P}(\mathscr{W}_1)$-enriched categories. Doing so will get the $\mathscr{W}_1$-actegory aspect, but will miss out the preservation of transverse limits from $\mathscr{W}_1$.

Garner notes that to characterize tangent categories, one must take a reflective subcategory of $[\mathscr{W}_1, \mathsf{Set}]$. Denote by $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ the category of functors of $\mathscr{W}_1 \longrightarrow \mathsf{Set}$ that preserve all transverse limits. In particular, for a tangent category $\underline{C}$, the hom is continuous in the second variable, so for $V = \mathsf{lim}_i V_i$:

$$\underline{C}_0(A, B \otimes_\infty \mathsf{lim}_i V_i) \simeq \mathsf{lim}_i\ \underline{C}_0(A, B \otimes_\infty V_i)$$

hence the enrichment is indeed into $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$.

Theorem 6.3.2 ((Garner, 2018))**.** *The category of tangent categories is* 2*-equivalent to the category of* $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ *enriched categories that admit powers by representables.*

Garner proves that moreover, $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$-presheaves on a tangent category are always a representable tangent category.

Proposition 6.3.3 ((Garner, 2018) proposition 24)**.** *Let* $\mathbb{X}$ *be any tangent category. The tangent category* $\mathsf{Tan}(\mathbb{X}^{op}, \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$ *is a coherently closed tangent category where the tangent functor is representable; i.e.* $TM = [D, M]$ *for some object* D.

We may now use this theorem to characterize cartesian tangent categories and coherently closed tangent categories.

A limit $\mathsf{lim}_i M_i$ in a tangent category $\mathbb{X}$ is called **transverse** when $(\mathsf{lim}_i M_i) \otimes_\infty U \simeq \mathsf{lim}_i (M_i \otimes_\infty U)$ for every $U \in \mathscr{W}_1$. This is the same definition of transverse limit given earlier.

Proposition 6.3.4. *When* $\mathbb{X}$ *is a tangent category, then the enriched Yoneda embedding* $\mathbb{X} \longrightarrow \mathsf{Tan}(\mathbb{X}^{op}, \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$ *preserves every transverse limit.*

Proof. Let $\mathsf{lim}_i M_i$ be a transverse limit in $\mathbb{X}$. The enriched Yoneda embedding sends

$$\mathsf{lim}_i M_i \mapsto \lambda AU.\mathbb{X}(A, \mathsf{lim}_i M_i)(U)$$

But

$$\begin{aligned}
&\mathbb{X}(A, \mathsf{lim}_i M_i)(U) = \mathbb{X}(A, \mathsf{lim}_i M_i \otimes_\infty U) \\
&= \mathsf{lim}_i \mathbb{X}(A, M_i \otimes_\infty U)
\end{aligned}$$

which completes the proof as limits are computed pointwise. □

As $\mathscr{W}_1$ is a finite limit theory the category $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ is equivalent to the category of models of the theory $\mathscr{W}_1$ in Set. This means that $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ is a locally finitely presented category (**adamek-rosicky**). Thus $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ is complete and cocomplete. This also means that $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ has copowers with set given by coproduct. This proposition allows forming enriched diagram categories from ordinary diagrams.

Limits for enriched categories are treated in depth in (Kelly, 2005). We only need the notion of conical limit. Transverse limits in a tangent category are an instance of the special case of enriched (weighted) limits called conical limits, as we now show.

Proposition 6.3.5. *In a tangent category a transverse limit is a* $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ *enriched (conical) limit.*

Proof. Let $\mathsf{lim}_i A_i$ be a transverse limit, and let X be any object. Then for every $U \in \mathscr{W}_1$ we have

$$\begin{aligned}
&\mathscr{W}_1(X, \mathsf{lim}_i A_i)(U) \simeq \mathscr{W}_1(X, \mathsf{lim}_i A_i \otimes_\infty U) \\
&\simeq \mathscr{W}_1(X, \mathsf{lim}_i (A_i \otimes_\infty U)) \simeq \mathsf{lim}_i \mathscr{W}_1(X, (A_i \otimes_\infty U)) \\
&\simeq \mathsf{lim}_i \mathscr{W}_1(X, A_i)(U)
\end{aligned}$$

Thus $\mathscr{W}_1(X, \mathsf{lim}_i A_i)(\_)$ is a (pointwise) limit in $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$. This is then an instance of a conical limit in the sense of [(Kelly, 2005) 3.8]. □

The preservation of products gives rise to enriched products.

Proposition 6.3.6. *Let* $\mathbb{X}$ *be a cartesian tangent category.*

1. $\mathbb{X}$ *has* $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ *enriched products. Conversely, enriched products make* $\mathbb{X}$ *a cartesian tangent category.*

2. *The Yoneda embedding* $\mathbb{X} \longrightarrow \mathsf{Tan}(\mathbb{X}^{op}, \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$ *preserves products.*

Proof. Let $\mathbb{X}$ be a cartesian tangent category.

1. We compute

$$\mathbb{X}(A, B \times C)(U) \simeq \mathbb{X}(A, (B \times C) \otimes_\infty U) \simeq \mathbb{X}(A, B)(U) \times \mathbb{X}(A, C)(U)$$

   and hence we have enriched products. Likewise if we have enriched products, then

$$\mathbb{X}(A, (B \times C) \otimes_\infty U) \simeq \mathbb{X}(A, B \times C)(U) \simeq \mathbb{X}(A, B)(U) \times \mathbb{X}(A, C)(U)$$
$$\simeq \mathbb{X}(A, B \otimes_\infty U) \times \mathbb{X}(A, C \otimes_\infty U) \simeq \mathbb{X}(A, B \otimes_\infty U \times C \otimes_\infty U)$$

   Hence by faithfulness of the Yoneda embedding, $(B \times C) \otimes_\infty U \simeq (B \otimes_\infty U) \times (C \otimes_\infty U)$.

2. This is immediate.

$\square$

Coherent closure is then enriched closure:

Proposition 6.3.7. *Let* $\mathbb{X}$ *be a coherently closed tangent category.*

1. $\mathbb{X}$ *is a cartesian closed* $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ *enriched category. homs.*

Proof. Let $\mathbb{X}$ be a coherently closed tangent category.

1. For every $U \in \mathscr{W}_1$, we have

$$\mathbb{X}(A, [B, C])(U) \simeq \mathbb{X}(A, [B, C] \otimes_\infty U)$$
$$\simeq \mathbb{X}(A, [B, C \otimes_\infty U]) \simeq \mathbb{X}(B \times A, C \otimes_\infty U)$$
$$\simeq \mathbb{X}(B \times A, C)(U)$$

   Thus as functors

$$\mathbb{X}(B \times A, C)(\_) \simeq \mathbb{X}(A, [B, C])(\_) : \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$$

2. We recall a technique used by Scott in (Scott, 1980b), although the technique was known before this.

   Recall the Yoneda lemma: $F(A) \simeq \mathsf{Nat}(\mathbb{X}(\_, A), F)$ tells us what the internal hom of presheaves must be:

   $$[F, G](X) \simeq \mathsf{Nat}(\mathbb{X}(\_, X), [F, G]) \simeq \mathsf{Nat}(\mathbb{X}(\_, X) \times F, G)$$

   Also, applying the Yoneda lemma to a representable functors gives:

   $$\mathsf{Nat}(\mathbb{X}(\_, A), \mathbb{X}(\_, B)) \simeq \mathbb{X}(A, B)$$

   Finally compute the following sequence of isomorphisms that are natural in X.

   $$\begin{aligned} &[\mathbb{X}(\_, A), \mathbb{X}(\_, B)](X) \\ &\simeq \mathsf{Nat}(\mathbb{X}(\_, X) \times \mathbb{X}(\_, A), \mathbb{X}(\_, B)) \\ &\simeq \mathsf{Nat}(\mathbb{X}(\_, X \times A), \mathbb{X}(\_, B)) \\ &\simeq \mathbb{X}(X \times A, B) \\ &\simeq \mathbb{X}(X, [A, B]) \end{aligned}$$

   Thus

   $$\mathbb{X}(\_, [A, B]) \simeq [\mathbb{X}(\_, A), \mathbb{X}(\_, B)]$$

   so the Yoneda embedding preserves the internal hom of a cartesian closed category. For a $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$ enriched category, replacing Nat in the above calculation, with $\mathsf{Tan}(\mathbb{X}^{op}, \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$,we can make the same moves until $\mathbb{X}(X \times A, B)$ as they are all enriched moves. That is

   $$[\mathbb{X}(\_, A), \mathbb{X}(\_, B)](X) \simeq \mathbb{X}(X \times A, B)$$

   by the enriched Yoneda lemma. For the last move, we use the enrichment of the homs

   $$\mathbb{X}(X \times A, B)(\_) \simeq \mathbb{X}(X, [A, B])(\_) : \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$$

   proved in the first part of this lemma.

$\square$

Thus for coherently closed tangent categories, the Yoneda embedding is a Cartesian closed functor.

6.4 Vector Fields Revisited

Previously we introduced vector fields as objects internal to a coherently closed tangent category. When our tangent category has negatives we showed that $\mathscr{X}(M)$ is a group under addition. In this section we extend this observation and show that $\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$, and thus that the object of vector fields is the canonical Lie algebra associated to the Lie group $\mathsf{Aut}(M)$. This fact is important in studying infinite dimensional Lie groups. For convenient manifolds this is exactly theorem 43.1 of (Kriegl and Michor, 1997).

We will say a tangent category has negatives when all tangent bundles $TM \xrightarrow{p} M$ form not just monoids, but rather groups in the slice category. This means also that differential objects are groups.

In a tangent category, let G be a differential object that is a group under the addition. G is a **Lie algebra** whenever there is a multiplication $[\_,\_] : G \times G \longrightarrow G$ such that

- The multiplication is bilinear:

$$\begin{tikzcd} T(G\times G) \arrow[r, "T[\_,\_]"] & TG \\ G\times G \arrow[u, "\lambda\times 0"] \arrow[r, "{[\_,\_]}"'] & G \arrow[u, "\lambda"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(G\times G) \arrow[r, "T[\_,\_]"] & TG \\ G\times G \arrow[u, "0\times\lambda"] \arrow[r, "{[\_,\_]}"'] & G \arrow[u, "\lambda"'] \end{tikzcd}$$

- The multiplication is antisymmetric:

$$G\times G \xrightarrow{\langle \pi_1,\pi_0\rangle} G\times G \xrightarrow{[\_,\_]} G = G\times G \xrightarrow{[\_,\_]} G \xrightarrow{-} G$$

- The multiplication satisfies the Jacobi identity: on generalized elements this is expressed as:

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$$

In a coherently closed tangent category, we can formulate the space of sections internally, and can use the embedding into a representable tangent category to produce a much shorter proof of the Jacobi identity. It turns out that along the way, we can characterize $\mathscr{X}(M)$ in any coherently closed tangent category mirroring (Kriegl and Michor, 1997) theorem 43.1: $\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$

In this section we sketch how a proof of how the following proposition can be obtained using the embedding theorem.

Proposition 6.4.1. *Let $\mathbb{X}$ be a coherently closed tangent category, let $TM \xrightarrow{p} M$ be a group bundle, and suppose the pullback of $p : T[M,M] \longrightarrow [M,M]$ along $\mathsf{Id} : 1 \longrightarrow [M,M]$ exists. Then $\mathscr{X}(M)$ is a Lie algebra in $\mathbb{X}$.*

We have already seen that when $\mathbb{X}$ is a coherently closed tangent category, $TM \xrightarrow{p} M$ is a group bundle, and the pullback of $T[M,M] \xrightarrow{p} M$ along $1 \xrightarrow{\mathsf{Id}_M} [M,M]$ exists, that the differential object $\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}([M,M])$.

So far, we have been working mostly with monoid-bundle tangent categories. Group-bundle tangent categories are $\mathbb{Z}$-Weil algebra actegories (Leung, 2017). The above section's embedding and enrichment theorems work exactly the same.

Cockett-Cruttwell give the definition for representable monoid-bundle tangent categories (Cockett and Cruttwell, 2014b) definition 5.6. Rosický gives the definition of representable group-bundle categories, and for the full definition see (Rosický, 1984) section 4. In particular, we have an object D with maps

$$1 \xrightarrow{0} D \qquad D \xrightarrow{\delta} D \star D \qquad D \xrightarrow{-} D$$

along with the unique map $D \xrightarrow{!} 1$. Additionally there are maps

$$D^2 \xrightarrow{c} D^2 \qquad D^2 \xrightarrow{\cdot} D$$

that witness D as a commutative semigroup. $D \star D$ is defined to be the object in the following pushout

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "0"'] & D \arrow[d, "i_1"] \\ D \arrow[r, "i_2"'] & D \star D \end{tikzcd}$$

Note that as the above is a pushout, the identity of D on the right side and bottom also make the square commute which forces a unique map

$$\begin{tikzcd}
 & D \arrow[d, "i_1"] \arrow[dr, "1"] & \\
1 \arrow[ur, "0"] \arrow[dr, "0"'] & D \star D \arrow[r, "+"] & D \\
 & D \arrow[u, "i_2"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

such that $i_1;+ = 1_D$ and $i_2;+ = 1_D$. The defining property of $D \xrightarrow{-} D$ is $d + -(d) = 0$.

As the internal hom sends pushouts in the first argument to pullbacks, $D \star D$ represenents T_2M

Otherwise the maps correspond as

Representable	Represents
$1 \xrightarrow{0} D$	$TM \xrightarrow{p} M$
$D \xrightarrow{\sigma} D \star D$	$T_2M \xrightarrow{+} TM$
$D \xrightarrow{!} 1$	$M \xrightarrow{0} TM$
$D \xrightarrow{-} D$	$TM \xrightarrow{-} TM$
$D^2 \xrightarrow{c} D^2$	$T^2M \xrightarrow{c} T^2M$
$D^2 \xrightarrow{\cdot} D$	$TM \xrightarrow{l} T^2M$

For example that 0 is a section of p:

$$M \xrightarrow{[!,M]} [D,M] \xrightarrow{[0,M]} M = M \xrightarrow{[0!,M]} M = M \xrightarrow{1} M$$

because $1 \xrightarrow{0} M \xrightarrow{!} 1$ is $1 \xrightarrow{1_1} 1$.

Lemma 6.4.2. *In a representable tangent category with negatives, a section of the tangent bundle is precisely a map $X : M \longrightarrow [D,M]$ such that $X(m)(0) = m$ It is equivalently, a map $\hat{X} : D \times M \longrightarrow M$ such that $\hat{X}(0,m) = m$. It is equivalently, a map $\tilde{X} : D \longrightarrow [M,M]$ such that $\tilde{X}(0) = 1_M$.*

Proof. Begin with a $\hat{X} : D \times M \longrightarrow M$ such that $\hat{X}(0,m) = m$; currying the D we have $X : M \longrightarrow [D,M]$. Suppressing the isomorphisms $\lambda(1_A) : A \longrightarrow [1,A]$ and

$[1,A] \simeq 1 \times [1,A] \xrightarrow{\mathsf{ev}} A$, we have $X(m)(0) = m$. Alternatively, currying the other direction, we have $\tilde{X}(0) = \lambda m.m = 1_M$. □

Lemma 6.4.3. *In a representable tangent category with negatives, a vector field, regarded as a map $X : D \times M \longrightarrow M$ such that $X(0,m) = m$, has*

$$D{\star}D{\times}M \xrightarrow{+\times M} D{\times}M \xrightarrow{X} M = D{\star}D{\times}M \xrightarrow{i_\star \times 1} D{\times}D{\times}M \xrightarrow{1\times X} D{\times}M \xrightarrow{X} M$$

where $D \star D \longrightarrow D \times D$ is given the universal property of the pushout: the two injections of $D \xrightarrow{D\times D}$ by $\langle 0,1\rangle$ and $\langle 1,0\rangle$ both make the defining square for $D \star D$ commute.

The equality of lemma 6.4.3 in the internal logic of a cartesian closed category is

$$X(d_1 + d_2, m) = X(d_1, X(d_2, m))$$

Proof. In the internal logic we have

$$X(d_1 + 0, m) = X(d_1, m) = X(d_1, X(0, m))$$

and

$$X(0 + d_2, m) = X(d_2, m) = X(0, X(d_2, m))$$

Which by the universal property of the pushout implies $X(d_1{+}d_2, m) = X(d_1, X(d_2, m))$.

To see the pushout property being used more directly, first consider the curry of both sides:

$$D \star D \xrightarrow{\lambda((+\times M);X)} [M,M] \quad \text{and} \quad D \star D \xrightarrow{\lambda((i_\star \times M);(1\times X)X)} [M,M]$$

As they are both maps out of a pushout, then it suffices to show that they are equal on precomposition with both injections. For the first injection $D \xrightarrow{i_1} D \star D$:

$$\begin{aligned}
&i_1\lambda((+\times M);X) = \lambda((i_1 \times M);(+\times M);X)\\
&= \lambda(X) \qquad \text{as } i_1;+ = 1\\
&= \lambda(\langle 1,1\rangle \pi_1 X)\\
&= \lambda((\langle 1,0\rangle \times 1);(1_{D\times D} \times X);X) \quad \text{as } (0\times 1)X = \pi_1\\
&= \lambda((i_1 \times M);(i_\star \times M);(1_{D\times D} \times X);X) \qquad \text{as } i_1 i_\star = \langle 1,0\rangle\\
&= i_1\lambda((i_\star \times M);(1\times X);X)
\end{aligned}$$

As required. Similarly for i_2. Thus,

$$\lambda((+\times M);X) = \lambda((i_\star \times M);(1\times X);X)$$

But, λ is an isomorphism; hence

$$(+\times M);X = (i_\star \times M);(1\times X);X$$

As required. $\square$

Here the proof makes heavy use of the higher order structure.

Lemma 6.4.4. *In a complete, representable tangent category with negatives, let X be a vector field, regarded as a map $D \times M \longrightarrow M$ with $X(0,m) = m$. Then*

$$X(d, X(-d, m)) = m$$

Proof. This is immediate from the above as $X(d, X(-d,m)) = X(d-d, m) = X(0,m) = m$. $\square$

We can now revisit a proposition of Lavendhomme.

Proposition 6.4.5 ((Lavendhomme, 1996) section 3.2.1, proposition 4)**.** *Let $\mathbb{X}$ be a complete, representable tangent category with negatives. Then*

$$\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$$

Proof. This follows from 6.4.4 as $X(d, \_)$ is an automorphism whose inverse is $X(-d, \_)$. $\square$

Provided that the equalizer providing the automorphisms from the endomorphisms is transverse, then a tangent vector to the identity in $[M,M]$ must be an automorphism.

Proposition 6.4.6. *Let $\mathbb{X}$ be a coherently closed tangent category, suppose that $TM \xrightarrow{p} M$ is a group bundle, suppose the pullback of $T[M,M] \xrightarrow{p} M$ along $\mathsf{Id} : 1 \longrightarrow [M,M]$ exists, and the equalizer involved in defining the group of automorphisms:*

$$\begin{tikzcd} \mathsf{Aut}(M) \arrow[r, tail] & {[M,M]\times[M,M]} \arrow[r, shift left=2, "\cdot"] \arrow[r, "\mathsf{sw};\cdot" description] \arrow[r, shift right=2, "!;\mathsf{Id}_M"'] & {[M,M]} \end{tikzcd}$$

is transverse. Then

$$\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$$

Proof. Since the equalizer involved in the definition $\mathsf{Aut}(M)$ is transverse, it is preserved by the Yoneda embedding into $[\mathbb{X}^{\mathrm{op}}, \mathsf{Microl}([\mathscr{W}_1, \mathsf{Set}])]$. As tangent spaces are differential objects, the action with any $\mathscr{W}_1$ algebra is a product, hence a limit, and thus tangent spaces are always transverse limits.

As homs are also preserved by the Yoneda embedding, and tangent functors are sent to tangent functors, both the constructions of $\mathscr{X}(M)$ and $T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$ are preserved by the Yoneda embedding. Thus

$$\mathscr{Y}(T_{\mathsf{Id}_M}(\mathsf{Aut}(M))) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(\mathscr{Y}(M))) \simeq \mathscr{X}(\mathscr{Y}(M)) \simeq \mathscr{Y}(\mathscr{X}(M))$$

using 6.4.5. As the Yoneda embedding is full and faithful:

$$T_{\mathsf{Id}_M}(\mathsf{Aut}(M)) \simeq \mathscr{X}(M)$$

□

That $\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}(\mathsf{Aut}(M))$ holds not just in SDG, but also can be found for convenient manifolds, provided that $\mathsf{Aut}(M)$ exists (Kriegl and Michor, 1997) theorem 43.1.

In a similar spirit, one can obtain the Jacobi identity for $\mathscr{X}(M)$. The idea behind a vector field is that it indicates a direction one can travel in TM. Given two vector fields X, Y on a curved surface and two infinitesimal distances d_1, d_2, it is clear that travelling along X for d_1 followed by Y for d_2, followed by X for $-d_1$ and finally Y for $-d_2$, that there is no reason to land where one started. The Lie bracket of X, Y is supposed to represent the distance we are off.

Before proceeding, we make the following observation: in a representable group tangent category, the multiplication $D \times D \overset{\cdot}{\longrightarrow} D$ is a regular epic. To see this, first note, that in a group tangent category, the universality of the lift can be simplified:

Lemma 6.4.7 ((Cockett and Cruttwell, 2014b) lemma 3.10)**.** *In a tangent category with negatives, the universality of the vertical lift is equivalent to the following* *diagram:*

$$TM \xrightarrow{\quad l \quad} T^2M \;\overset{p}{\underset{pp0}{\substack{\longrightarrow \\ -Tp\rightarrow \\ \longrightarrow}}}\; TM$$

For a representable tangent category with invertible addition this is

$$TM \xrightarrow{\quad [\cdot, M] \quad} T^2M \;\overset{[\langle 0,D\rangle, M]}{\underset{[!\langle 0,0\rangle, M]}{\substack{\longrightarrow \\ -[\langle D,0\rangle, M]\rightarrow \\ \longrightarrow}}}\; TM$$

Thus to represent this equalizer we have

Lemma 6.4.8. *In a representable tangent category with negatives we have that the following is a coequalizer:*

$$D \;\overset{\langle 0,D\rangle}{\underset{!\langle 0,0\rangle}{\substack{\longrightarrow \\ -\langle D,0\rangle\rightarrow \\ \longrightarrow}}}\; D \times D \xrightarrow{\quad \cdot \quad} D$$

sketch. Writing out the above coequalizer property on generalized elements, we have that if $D \times D \xrightarrow{\tau} M$ has $\tau(0,d) = \tau(d,0) = \tau(0,0)$ then there is a unique $D \xrightarrow{t} M$ such that $t(d_1 \cdot d_2) = \tau(d_1, d_2)$. □

Lemma 6.4.9. *Let X, Y be vector fields, regarded as maps $X, Y : D \longrightarrow [M, M]$. There exists one and only one vector field $[X, Y] : D \longrightarrow [M, M]$ such that*

$$[X,Y](d_1 \cdot d_2) = X(d_1); Y(d_2); X(-d_1); Y(-d_2)$$

Proof. Consider the map τ defined as:

$$D^2 \xrightarrow{\Delta\times\Delta} D^4 \xrightarrow{D\times - \times D \times -} D^4 \xrightarrow{\mathsf{ex}} D^4 \xrightarrow{X\times Y\times X\times Y} [M,M]^4 \xrightarrow{c_4} [M,M]$$

where the $-$ denotes the map giving the inverse to addition, ex denotes the middle exchange, and where c_4 is the left to right composition map

$$c_4 := [M,M]\times[M,M]\times[M,M]\times[M,M] \xrightarrow{m\times 1\times 1} [M,M]\times[M,M]\times[M,M] \xrightarrow{m\times 1} [M,M]\times[M,M] \xrightarrow{m} [M,M]$$

and where $[M,M]\times[M,M] \xrightarrow{m} [M,M]$ is the representative of $\lambda f\, g\, x.g(f\,x)$.

This will send (d_1, d_2) first to (d_1, d_1, d_2, d_2), and then to $(d_1, -d_1, d_2, -d_2)$, and then to $(d_1, d_2, -d_1, -d_2)$, and then to $(X(d_1), Y(d_2), X(-d_1), Y(-d_2))$, and then finally applying c_4 will give $\lambda m : M.Y(-d_2)(X(-d_1)(Y(d_2)(X(d_1)(m))))$. Thus we have described essentially the commutator.

Thus, in the internal type theory we have formed the term

$$d_1 : D, d_2 : D \vdash \lambda m : M.Y(-d_2)(X(-d_1)(Y(d_2)(X(d_1)(m))))$$

We claim that the above map coequalizes $\langle 0, D\rangle$, $\langle D, 0\rangle$, and $\langle 0, 0\rangle$.

If we cut 0 in for d_1 we get

$$\begin{aligned} &\lambda m.Y(-d_2)(X(0)(Y(d_2)(X(0)(m)))) \\ &= \lambda m.Y(-d_2)(Y(d_2)(m)) \\ &= \lambda m.m \end{aligned}$$

using $X(0) = 1_M$, and that $Y(-d_2) = Y(d_2)^{-1}$ from the fact that $T_{\mathsf{Id}_M}(\mathsf{Aut}(M)) \simeq \mathscr{X}(M)$.

Likewise if we cut 0 in for d_2 we get $\lambda m.m$, and if we cut both d_1, d_2 with 0, we get $\lambda m.m$. Thus the map defined coequalizes the three maps. Then there is a unique $[X, Y] : D \longrightarrow [M, M]$ such that

$$D \times D \xrightarrow{\cdot} D \xrightarrow{[X,Y]} [M, M] = D \times D \xrightarrow{\tau} [M, M]$$

as required. □

Having gotten the Lie bracket into the same form as used in synthetic differential geometry, we could use their equational proof that it gives a Lie algebra. We state the theorem without proof as we would just be copying the proof here verbatim. The proof of the following is due to Reyes and Wraith (Reyes and Wraith, 1978b) and can be found in Lavendhomme (Lavendhomme, 1996), 3.2.2 proposition 7. Note that the proof is just over one page in length, and this can be contrasted with Rosickýs general proof of the Jacobi identity for a tangent category that is over 40 pages long.

Lemma 6.4.10. *In a complete, representable tangent category with negatives, $\mathscr{X}(M)$ equipped with the Lie bracket of 6.4.9, is an internal Lie algebra.*

Proposition 6.4.11. *Let $\mathbb{X}$ be a coherently closed group tangent category for which the tangent space $\mathscr{X}(M) \simeq T_{\mathsf{Id}_M}([M,M])$ exists. Then $\mathscr{X}(M)$ is an internal Lie algebra.*

Proof. As tangent spaces are transverse, and the Yoneda embedding preserves internal homs, we have

$$\mathscr{Y}(T_{\mathsf{Id}_M}[M,M]) \simeq T_{\mathsf{Id}_{\mathscr{Y}(M)}}[\mathscr{Y}(M), \mathscr{Y}(M)]$$

which is a Lie algebra by 6.4.10. By full and faithfulness of the Yoneda embedding, then so is $T_{\mathsf{Id}_M}([M,M])$. □

6.5 Tangent subcategories of Weil actions

We have established a connection between Weil algebras and tangent categories, and used this to express coherent closure from an enriched point of view. We have seen how to express tangent structure from Weil algebras that preserve limits. In this section we develop a bit of general machinery for extracting a coherently closed tangent category from a Cartesian closed category with just a monoidal action by $\mathscr{W}_1$.

Suppose $\mathbb{X}$ is a category with (just) a monoidal action $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$.

Definition 6.5.1. *An object A is called **microlinear** if for every transverse limit $L = \lim_j V_j$ in $\mathscr{W}_1$, $A \otimes_\infty L$ is a limit in $\mathbb{X}$ over the diagram components $A \otimes_\infty V_j$. In other words, $A \otimes_\infty \lim_j V_j \simeq \lim_j (A \otimes_\infty V_j)$.*

In a tangent category, every object is microlinear, but also:

Proposition 6.5.2. *The microlinear objects with respect to a monoidal action $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$ form a tangent category,* $\mathsf{Microl}(\mathbb{X})$.

Proof. The proof is nearly immediate. One must show the action restricts to $\mathsf{Microl}(\mathbb{X})$; that is, for each $A \in \mathsf{Microl}(\mathbb{X})$ and $U \in \mathscr{W}_1$, we must show that $A \otimes_\infty U$ is in $\mathsf{Microl}(\mathbb{X})$.

Then let L be any transverse limit in $\mathscr{W}_1$, and consider

$$(A \otimes_\infty U) \otimes_\infty L \simeq A \otimes_\infty (U \otimes L)$$

due to the monoidal action. Next, $U \otimes L$ is a transverse limit in $\mathscr{W}_1$ by the inductive definition of transverse limits in $\mathscr{W}_1$. But A is microlinear, hence $A \otimes_\infty (U \otimes L)$ is a limit in $\mathbb{X}$, and thus $A \otimes_\infty U$ is microlinear.

Thus the action restricts to $\mathsf{Microl}(\mathbb{X})$. By definition, the action of a transverse limit on a microlinear object is a limit. Thus $\mathsf{Microl}(\mathbb{X})$ is a tangent category. $\square$

Proposition 6.5.3. *Suppose $\mathbb{X}$ has products, and let $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$ be a monoidal action for which each $V \in \mathscr{W}_1$ preserves products; that is, $(M \times N) \otimes_\infty V \simeq (M \otimes_\infty V) \times (N \otimes_\infty V)$. Then $\mathsf{Microl}(\mathbb{X})$ is a cartesian tangent category.*

Proof. By observation 6.2.2 it suffices to show that if M, N are microlinear, the $M \times N$ is microlinear. Let L be a limit in $\mathscr{W}_1$. Then $M \otimes_\infty L$ and $N \otimes_\infty L$ are limits in $\mathbb{X}$ by microlinearity. The product functor is a right adjont and hence preserves limits; thus, $(M \otimes_\infty L) \times (N \otimes_\infty L)$ is a limit. $\square$

When $\mathbb{X}$ has all limits, one might expect, following SDG, that the microlinear objects have all limits. Indeed, this is the case for many situations; however, in general, for a limit $X = \mathsf{lim} F$ in $\mathbb{X}$, where $F(d)$ is microlinear, the result need not be. The issue is that $(\_ \otimes_\infty W)$ need not preseve limits. We restate the notion of transverse limit even when $\mathbb{X}$ is not a tangent category, but just admits a Weil algebra action.

Definition 6.5.4. *A limit $X = \mathsf{lim}_i F_i$ is called* ***transverse*** *(Cockett and Cruttwell, 2016) if it is preserved by all tangent bundles[3]: for every $W \in \mathscr{W}_1$, we have $(\mathsf{lim}_i F_i \otimes_\infty W) \simeq \mathsf{lim}_i (F_i \otimes_\infty W)$.*

Proposition 6.5.5. *For any $\mathbb{X}$ with a monoidal action $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$, suppose that A is microlinear. Then for any transverse limit $L \in \mathscr{W}_1$, $A \otimes_\infty L$ is a transverse limit in $\mathbb{X}$.*

Moreover, if $\_ \otimes_\infty V$ preserves products for every $V \in \mathscr{W}_1$ then products are transverse limits in $\mathbb{X}$.

[3] This is the same notion of transverse introduced earlier in the thesis, but expressed in terms of Weil actions.

This might seem like a restatement of the definition of microlinear. It is not quite. This says that the limits in $\mathsf{Microl}(\mathbb{X})$ created by $\_ \otimes_\infty (\mathsf{lim}_i V_i)$ where $\mathsf{lim}_i V_i$ is transverse in $\mathscr{W}_1$, are transverse in $\mathbb{X}$.

Proof. Let $L = \mathsf{lim}_j V_j$ be a transverse limit in $\mathscr{W}_1$. Then by the inductive construction of transverse limits in $\mathscr{W}_1$, for any W, $L \otimes W$ is a transverse limit in $\mathscr{W}_1$. Then for any $A \in \mathbb{X}$, $A \otimes_\infty L$ is transverse:

$$(\mathsf{lim}_j A \otimes_\infty V_j) \simeq \otimes_\infty W = (A \otimes_\infty L) \otimes_\infty W \simeq A \otimes_\infty (L \otimes W) \simeq A \otimes_\infty (\mathsf{lim}_j V_j \otimes W)$$

As A is microlinear, the above is $\mathsf{lim}_j A \otimes_\infty (V_j \otimes W)$, as required.

The rest of the proposition is immediate. □

In particular, in a tangent category $\mathbb{X}$, all the limits involved in the tangent structure (pullback powers of p, the equalizer diagram for the universality of the lift) are transverse.

Proposition 6.5.6. *When $\mathbb{X}$ has all limits, then any limit of microlinear objects that is transverse is microlinear.*

Proof. This follows from the fact that in a complete category, limits commute. Suppose that $A = \mathsf{lim}_i F_i$ is a transverse limit of microlinear objects and that $L = \mathsf{lim}_j V_j$ is a limit in $\mathscr{W}_1$.

$$\begin{aligned}
A \otimes_\infty L &\simeq (\mathsf{lim}_i F_i) \otimes_\infty L \\
&\simeq \mathsf{lim}_i (F_i \otimes_\infty L) && \text{transversality} \\
&\simeq \mathsf{lim}_i (F_i \otimes_\infty \mathsf{lim}_j V_j) \\
&=\simeq \mathsf{lim}_i \mathsf{lim}_j (F_i \otimes_\infty V_j) && \text{microlinearity} \\
&=\simeq \mathsf{lim}_j \mathsf{lim}_i (F_i \otimes_\infty V_j) && \text{commutativity of limits} \\
&= \mathsf{lim}_j A \otimes V_j
\end{aligned}$$

as required. □

An immediate consequence of 6.5.6 is:

Corollary 6.5.7. *Suppose $\mathbb{X}$ has all limits. Then if for every $V \in \mathscr{W}_1$, the functor $\_ \otimes_\infty V$ is continuous; that is, every limit is transverse, then* $\mathsf{Microl}(\mathbb{X})$ *has all limits. Also, in this case*

$$\mathsf{Microl}(\mathbb{X}) \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathsf{Microl}(\mathbb{X})$$

is continuous in its first argument, and preserves transverse limits in its second.

6.6 Exponentiability

In this section, we revisit Nishimura's idea of exponentiability from (Nishimura, 2010; Nishimura, 2009). Nishimura's idea is that one should be able to extract a well-behaved cartesian closed subcategory from the microlinear spaces. Making use of the notion of a coherently closed tangent category, we can realize his goal in a precise manner; we will show that if one starts with a cartesian closed actegory, that one may extract a coherently closed tangent category.

Let $\mathbb{X}$ be a cartesian closed category, and let $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$ be an actegory where for each $V \in \mathscr{W}_1$, $\_ \otimes_\infty V$ preserves products. For each $U \in \mathscr{W}_1$, there is a strength:

$$\theta_U := A \times (B \otimes_\infty U) \xrightarrow{(A \otimes_\infty \eta_U) \times 1} (A \otimes_\infty U) \times (B \otimes_\infty U) \xrightarrow{m_\times} (A \times B) \otimes_\infty U$$

An object $B \in \mathbb{X}$ is $\mathscr{W}_1$ **exponentiable** when for every $A \in \mathbb{X}$ and every $U \in \mathscr{W}_1$, the inferred exponential strength $[A, B] \otimes_\infty U \longrightarrow [A, B \otimes_\infty U]$ is an isomorphism:

$$\frac{A \times ([A, B] \otimes_\infty U) \xrightarrow{\theta} (A \times [A, B]) \otimes_\infty U \xrightarrow{\mathsf{ev} \otimes_\infty 1} B \otimes_\infty U}{[A, B] \otimes_\infty U \xrightarrow[\lambda(\theta; (\mathsf{ev} \otimes_\infty 1))]{} [A, B \otimes_\infty U]}$$

Lemma 6.6.1. *Let $\mathbb{X}$ be a cartesian closed category with an actegory $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$. Then*

1. *The strength $\theta_U : A \times (B \otimes_\infty U) \longrightarrow (A \times B) \otimes_\infty U$ satisfies*

$$\begin{tikzcd}
A \times (B \otimes_\infty (U \otimes V)) \arrow[rr, "\theta_{U \otimes V}"] \arrow[d, "1 \times \alpha"] & & (A \times B) \otimes_\infty (U \otimes V) \arrow[d, "\alpha"] \\
(A \times (B \otimes_\infty U)) \otimes_\infty V \arrow[r, "\theta_V"'] & ((A \otimes_\infty U) \times (B \otimes_\infty U)) \otimes_\infty V \arrow[r, "\theta_U \otimes_\infty V"'] & ((A \times B) \otimes_\infty U) \otimes_\infty V
\end{tikzcd}$$

2. *Suppose the exponential strength* $\psi_U := \lambda(\theta_U; (\mathsf{ev} \otimes_\infty U)) : [X, Y \otimes_\infty U] \longrightarrow [X, Y] \otimes_\infty U$ *has an inverse. Then*

$$
\begin{tikzcd}
X \times [X, Y \otimes_\infty U] \arrow[r, "1 \times \psi_U^{-1}"] \arrow[drr, "\mathsf{ev}"'] & X \times ([X, Y] \otimes_\infty U) \arrow[r, "\theta_U"] & (X \times [X, Y]) \otimes_\infty U \arrow[d, "\mathsf{ev} \otimes_\infty U"] \\
 & & Y \otimes_\infty U
\end{tikzcd}
$$

Proof.

1. Consider the following diagram:

$$
\begin{tikzcd}
A \times (B \otimes_\infty (U \otimes V)) \arrow[rr, "(A \otimes_\infty \eta_{U \otimes V}) \times 1"] \arrow[dr, "(A \otimes_\infty \eta_V) \times 1"] \arrow[d, "1 \times \alpha"] & & (A \otimes_\infty (U \otimes V)) \times (B \otimes_\infty (U \otimes V)) \arrow[r, "m_\times"] \arrow[d, "\alpha \times 1"] & (A \times B) \otimes_\infty (U \otimes V) \arrow[ddd, "\alpha"] \\
A \times ((B \otimes_\infty U) \otimes_\infty V) \arrow[d, "(A \otimes_\infty \eta_V) \times 1"] & (A \otimes_\infty V) \times (B \otimes_\infty (U \otimes V)) \arrow[dl, "1 \times \alpha"] \arrow[r, "((A \otimes_\infty \eta_U) \otimes_\infty V) \times 1"] & (A \otimes_\infty U) \otimes_\infty V \times (B \otimes_\infty U \otimes_\infty V) \arrow[d, "1 \times \alpha"] & \\
(A \otimes_\infty V) \times ((B \otimes_\infty U) \otimes_\infty V) \arrow[rr, "(A \otimes_\infty \eta_U) \otimes_\infty V \times 1"] \arrow[d, "m_\times"] & & (A \otimes_\infty U) \otimes_\infty V \times (B \otimes_\infty U) \otimes_\infty V \arrow[d, "m_\times"] & \\
(A \times (B \otimes_\infty U)) \otimes_\infty V \arrow[rr, "((A \otimes_\infty \eta_U) \times 1) \otimes_\infty V"'] & & ((A \otimes_\infty U) \times (B \otimes_\infty U)) \otimes_\infty V \arrow[r, "m_\times"'] & ((A \times B) \otimes_\infty U) \otimes_\infty V
\end{tikzcd}
$$

From the top left to the bottom right, clockwise is $\theta_{U \otimes V} \alpha$ and counterclockwise is $(1 \times \alpha); \theta_V; (\theta_U \otimes_\infty V)$. The right rectangle commutes because of the coherence of the action preserving products. The bottom rectangle commutes because of naturality. The middle trapezoid and left triangle commute because of bifunctoriality. This leaves the upper left trapezoid.

It suffices to prove that

$$(A \otimes_\infty \eta_{U \otimes V}); \alpha = (A \otimes_\infty \eta_V); ((A \otimes_\infty \eta_U) \otimes_\infty V)$$

Note that

$$A \equiv A \otimes_\infty R \equiv (A \otimes_\infty R) \otimes_\infty R \equiv A \otimes_\infty (R \otimes R)$$

and that $A \otimes_\infty (R \otimes R) \xrightarrow{\alpha} (A \otimes_\infty R) \otimes_\infty R$ is the identity. Finally $A \xrightarrow{A \otimes_\infty \eta_{U \otimes V}} A \otimes_\infty (U \otimes V)$ is exactly

$$A \equiv A \otimes_\infty (R \otimes R) \xrightarrow{A \otimes_\infty (\eta_U \otimes \eta_V)} A \otimes_\infty (U \otimes V)$$

and the result follows.

2. As every isomorphism is epic, it suffices to show that

$$(1 \times \psi)\mathsf{ev} = (1 \times \psi)(1 \times \psi^{-1})\theta_U(\mathsf{ev} \otimes_\infty U)$$

which, $(1 \times \psi)\mathsf{ev} = \theta_U(\mathsf{ev} \otimes_\infty U)$ by definition.

□

Proposition 6.6.2. *Let $\mathbb{X}$ be a cartesian closed category with actegory structure $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$ such that $\_ \otimes_\infty V$ preserves products for all $V \in \mathscr{W}_1$. Suppose X is $\mathscr{W}_1$ exponentiable.*

1. *$X \otimes_\infty U$ is $\mathscr{W}_1$ exponentiable for every U.*
2. *If Y is another $\mathscr{W}_1$ exponentiable object, then so is $X \times Y$.*
3. *For every B, $[B, X]$ is $\mathscr{W}_1$ exponentiable.*

Proof. Suppose X is $\mathscr{W}_1$ exponentiable.

1. Let U be a $\mathscr{W}_1$ algebra. Let Y be arbitrary in $\mathbb{X}$ and V be arbitrary in $\mathscr{W}_1$ (or Weil). There is clearly an isomorphism:

$$\begin{aligned}&[X, Y \otimes_\infty U] \otimes_\infty V \simeq ([X, Y] \otimes_\infty U) \otimes_\infty V \simeq [X, Y] \otimes_\infty (U \otimes V)\\ &\simeq [X, Y \otimes_\infty (U \otimes V)] \simeq [X, (Y \otimes_\infty U) \otimes_\infty V]\end{aligned}$$

However, we must show it is the correct isomorphism. Let ψ_U denote $\lambda(\theta_U; (\mathsf{ev} \otimes_\infty U))$. Writing down the isomorphisms used above explicitly, our goal is to show that

$$(\psi^{-1} \otimes_\infty V); \alpha^{-1}; \psi; [X, \alpha] = \lambda(\theta_V; \mathsf{ev} \otimes_\infty V) : [X, Y \otimes_\infty U] \otimes_\infty V \longrightarrow [X, (Y \otimes_\infty U) \otimes_\infty V]$$

Because $g\lambda(f) = \lambda((1 \times g)f)$, we have that

$$(\psi^{-1} \otimes_\infty V); \alpha^{-1}; \psi; [X, \alpha] = \lambda((1 \times (\psi^{-1} \otimes_\infty V)); (1 \times \alpha^{-1}); \theta_{U \otimes V}; (\mathsf{ev} \otimes_\infty (U \otimes V)); \alpha)$$

Hence it suffices to show that

$$(1 \times (\psi^{-1} \otimes_\infty V)); (1 \times \alpha^{-1}); \theta_{U \otimes V}; (\mathsf{ev} \otimes_\infty (U \otimes V)); \alpha = \theta_V; (\mathsf{ev} \otimes_\infty V)$$

Then

$$\begin{aligned}
&(1\times(\psi^{-1}\otimes_\infty V));(1\times\alpha^{-1});\theta_{U\otimes V};(\mathsf{ev}\otimes_\infty(U\otimes V));\alpha\\
&=(1\times(\psi^{-1}\otimes_\infty V));(1\times\alpha^{-1});\theta_{U\otimes V};\alpha;(\mathsf{ev}\otimes_\infty U)\otimes_\infty V\\
&=(1\times(\psi^{-1}\otimes_\infty V));(1\times\alpha^{-1});(1\times\alpha);\theta_V;(\theta_U\otimes_\infty V);(\mathsf{ev}\otimes_\infty U)\otimes_\infty V\\
&=(1\times(\psi^{-1}\otimes_\infty V));\theta_V;(\theta_U\otimes_\infty V);(\mathsf{ev}\otimes_\infty U)\otimes_\infty V\\
&=\theta_V;((1\times\psi^{-1})\otimes_\infty V);(\theta_U\otimes_\infty V);(\mathsf{ev}\otimes_\infty U)\otimes_\infty V\\
&=\theta_V;(\mathsf{ev}\otimes_\infty V)
\end{aligned}$$

The moves are naturality of α, followed by lemma 6.6.1.1, followed by a tautology, followed by naturality of θ_V, followed by lemma 6.6.1.2.

This completes the proof.

2. In this calculation, $m_{\Rightarrow,\times}$ is the isomorphism $[A,B\times C]\longrightarrow[A,B]\times[A,C]$, and $m_\times$ is the isomorphism $(A\otimes_\infty U)\times(B\otimes_\infty U)\longrightarrow(A\times B)\otimes_\infty U$. Then consider the following chain of (left to right directed) isomorphisms:

$$\begin{aligned}
[A,X\times Y]\otimes_\infty U &\simeq([A,X]\times[A,Y])\otimes_\infty U && m^{-1}_{\Rightarrow,\times}\otimes_\infty U\\
&\simeq[A,X]\otimes_\infty U\times[A,Y]\otimes_\infty U && m_\times\\
&\simeq[A,X\otimes_\infty U]\times[A,Y\otimes_\infty U] && \psi\times\psi\\
&\simeq[A,X\otimes_\infty U\times Y\otimes_\infty U] && m_{\Rightarrow,\times}\\
&\simeq[A,(X\times Y)\otimes_\infty U] && [A,m_\times]
\end{aligned}$$

And our goal is to show that this is the right isomorphism, that is the above must equal ψ. In the proof of lemma 5.5.8, we proved that $m_\times;\psi\times\psi;m_{\Rightarrow,\times}=(m^{-1}_{\Rightarrow,\times}\otimes_\infty U);\psi;[A,m^{-1}_\times$. It is the long equational proof demonstrating the commutativity of the top left rectangle of the fourth diagram. Thus

$$m_\times;\psi\times\psi;m_{\Rightarrow,\times}=(m^{-1}_{\Rightarrow,\times}\otimes_\infty U);\psi;[A,m^{-1}_\times]$$

Then

$$\begin{aligned}
&(m^{-1}_{\Rightarrow,\times}\otimes_\infty U);m_\times;\psi\times\psi;m_{\Rightarrow,\times};[A,m_\times]\\
&=(m^{-1}_{\Rightarrow,\times}\otimes_\infty U);(m^{-1}_{\Rightarrow,\times}\otimes_\infty U);\psi;[A,m^{-1}_\times];[A,m_\times]\\
&=\psi
\end{aligned}$$

3. Let $\mathsf{cur} : [A \times B, C] \longrightarrow [B, [A, C]]$ denote the curry isomorphism. Then consider the chain of isomorphisms $(\mathsf{cur}^{-1} \otimes_\infty U); \psi; \mathsf{cur}; [A, \psi^{-1}]$:

$$\begin{aligned} &[A, [B, X]] \otimes_\infty U \simeq [B \times A, X] \otimes_\infty U \simeq [B \times A, X \otimes_\infty U] \\ &\simeq [A, [B, X \otimes_\infty U]] \simeq [A, [B, X] \otimes_\infty U] \end{aligned}$$

   We must show that the above is ψ. As ψ is an exponential strength 5.2.9, it satisfies $\psi \mathsf{cur} = (\mathsf{cur} \otimes_\infty U); \psi; [A, \psi]$ by definition. Thus

$$\begin{aligned} &(\mathsf{cur}^{-1} \otimes_\infty U); \psi; \mathsf{cur}; [A, \psi^{-1}] \\ &= (\mathsf{cur}^{-1} \otimes_\infty U); (\mathsf{cur} \otimes_\infty U); \psi; [A, \psi]; [A, \psi^{-1}] \\ &= \psi \end{aligned}$$

   This completes the proof.

$\square$

Proposition 6.6.3. *Let $\mathbb{X}$ be a cartesian closed category with a product preserving actegory structure $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$. If A is microlinear and exponentiable by $W^1 = R[x]/(x^2)$, then A is exponentiable for all objects in $\mathscr{W}_1$.*

Proof. We prove this by the inductive construction of objects in $\mathscr{W}_1$. For R, the map $\theta = 1$ and $\lambda(\theta; \mathsf{ev}) = \lambda(\mathsf{ev}) = 1$. The base case W^1 is exponentiable by hypothesis. Next we assume that W^n is exponentiable, and show that W^{n+1} is exponentiable. Note, that $[A, \_]$ always preservelimits, the square on the right hand side is the pullback.

$$\begin{tikzcd}
 & {[B,A]\otimes_\infty (W^n \times W^1)} \arrow[rrrr, dotted, "\exists!"] \arrow[dl] \arrow[dr] & & & & {[B, A \otimes_\infty W^{n+1}]} \arrow[dl] \arrow[dr] & \\
 & & {[B,A]\otimes_\infty W^1} \arrow[rrrr, "\psi"] \arrow[dl] & & & & {[B, A \otimes_\infty W^1]} \arrow[dl] \\
{[B,A]\otimes_\infty W^n} \arrow[rrrr, "\psi"] \arrow[dr] & & & & {[B, A \otimes_\infty W^n]} \arrow[dr] & & \\
 & {[B,A]} \arrow[rrrr, equal] & & & & {[B,A]} &
\end{tikzcd}$$

Note, that $\psi : [B, A] \otimes_\infty W^{n+1} \longrightarrow [B, \otimes_\infty W^{n+1}]$ makes the diagram commute. But then ψ^{-1} can be made by considering a similar diagram built with ψ^{-1} on the sides, and these inverses exist by the inductive hypothesis and the hypothesis that A is exponentiable by W^1.

In 6.6.2.1, we showed that if A is exponentiable, then so is $A \otimes_\infty U$; a similar calculation used in the proof of 6.6.2 shows that $[B, A] \otimes_\infty (U \otimes V) \longrightarrow [B, A \otimes_\infty (U \otimes V)]$ is invertible. □

Then we immediately get the following corollary.

Corollary 6.6.4. $\mathbb{X}$ *is a coherently closed tangent category if and only the actegory structure preserves products and every object is* $\mathscr{W}_1$ *exponentiable.*

Theorem 6.6.5. *Let* $\mathbb{X}$ *be a Cartesian closed category with a product preserving actegory structure* $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$.

1. *The microlinear,* $\mathscr{W}_1$ *(or* Weil*) exponentiable objects form a coherently closed tangent category,* Microl-Exp($\mathbb{X}$).
2. *If* L *is a transverse limit in* $\mathscr{W}_1$*, then for any microlinear, exponentiable object* A*,* $A \otimes L$ *is a transverse limit in* Microl-Exp($\mathbb{X}$).
3. *If* $\mathbb{X}$ *is complete, then every transverse limit of microlinear, exponentiable objects is microlinear, exponentiable. That is,* Microl-Exp($\mathbb{X}$) *is closed under transverse limits in* $\mathbb{X}$.

Proof. Suppose $\mathbb{X}$ has a product preserving actegory structure $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$.

1. Because the actegory structure preserves products, the microlinear objects are closed to products 6.5.3. Similarly, by 6.6.2.1, exponentiable objects are closed to product. Thus, the microlinear, exponentiable objects are closed to product.

   By proposition 6.5.2, microlinear objects are closed under the action of Weil algebras; by 6.6.2.2, so are the exponentiable objects. Thus the microlinear, exponentiable objects are a cartesian tangent category by 6.5.3.

   Next, we show that it is a cartesian closed category. Proposition 6.6.2.3 shows that if A is microlinear and exponentiable, then $[B, A]$ is exponentiable for every B. It remains to show that it is microlinear. Let L be a limit in $\mathscr{W}_1$. Then

$$[B, A] \otimes L \simeq [B, A \otimes L]$$

and A is microlinear, so that $A \otimes L$ is a limit, and $[B, \_]$ is continuous so that $[B, A \otimes L]$ is a limit, and hence $[B, A]$ is microlinear, proving that microlinear, exponentiable objects are cartesian closed.

The microlinear, exponentiable objects are a coherently closed tangent category by 6.6.4.

2. Immediate from 6.5.5, as $\mathsf{Microl\text{-}Exp}(\mathbb{X})$ is closed to the action by Weil algebras.

3. By 6.5.6, the transverse limit of microlinear, exponentiable objects is always microlinear. We must show it is exponentiable.

   First, note that if $F \equiv \mathsf{lim}_i X_i$ is a transverse limit, then the functor $[A, \_ \otimes_\infty U]$ preserves this limit for each A, U:

$$[A, F \otimes_\infty U] \equiv [A, \mathsf{lim}_i X_i \otimes_\infty U] \simeq [A, \mathsf{lim}_i (X_i \otimes_\infty U)] \simeq \mathsf{lim}_i [A, X_i \otimes_\infty U]$$

   The first step is transversality and the second step is that hom is continuous in the covariant argument. But $[A, \_ \otimes_\infty U] \simeq [A, \_] \otimes_\infty U$ we have that $[A, \_] \otimes_\infty U$ preserves any limit that $[A, \_ \otimes_\infty U]$ preserves. Thus,

$$\begin{aligned}
&[A, \mathsf{lim}_i X_i] \otimes_\infty U \equiv \mathsf{lim}_i([A, X_i] \otimes_\infty U) \\
&\xrightarrow[\simeq]{\mathsf{lim}_i \psi_i} \mathsf{lim}_i [A, X_i \otimes_\infty U] \\
&\equiv [A, \mathsf{lim}_i (X_i \otimes_\infty U)] \\
&\equiv [A, \mathsf{lim}_i X_i \otimes_\infty U]
\end{aligned}$$

   In the last two steps, the use of $\equiv$ instead of $\simeq$ is justified as limits are only defined up to isomorphism. That $\mathsf{lim}_i \psi_i = \psi$ follows immediately from universality.

$\square$

An immediate corollary is

Corollary 6.6.6. *Suppose $\mathbb{X}$ has all limits. Then if for every $V \in \mathscr{W}_1$ (or Weil), the functor $\_ \otimes_\infty V$ is continuous; that is, every limit is transversal, then $\mathsf{Microl\text{-}Exp}(\mathbb{X})$ has all limits. Further the functor*

$$\mathsf{Microl\text{-}Exp}(\mathbb{X}) \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathsf{Microl\text{-}Exp}(\mathbb{X})$$

is continuous.

6.7 Examples

In this section we produce examples of coherently closed tangent categories.

6.7.1 Representable Weil Actions

A Cartesian closed category $\mathbb{X}$ has a **representable pre-Weil-prolongation** if there is a functor $D(\_) : \mathscr{W}_1^{\mathrm{op}} \longrightarrow \mathbb{X}$ that preserves products up to isomorphism. Thus representable pre-Weil-prolongations arise as models of the theory $\mathscr{W}_1^{\mathrm{op}}$ in a cartesian closed category.

Products in $\mathscr{W}_1^{\mathrm{op}}$ are coproducts in $\mathscr{W}_1$ and R is a zero object; thus, we have that $D(U \otimes V) \simeq D(U) \times D(V)$ and $D(R) \simeq 1$. In this case, we have a bifunctor

$$\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X} \quad ; \quad (M, V) \mapsto [D(V), M]$$

This bifunctor is a monoidal action as

$$[D(U \otimes V), M] \simeq [D(U) \times D(V), M] \simeq [D(U), [D(V), M]]$$

The functor $D(\_)$ is not required to be cocontinuous; thus, colimits in $\mathscr{W}_1^{\mathrm{op}}$ i.e. limits in $\mathscr{W}_1$, do not necessarily get sent to colimits in $\mathbb{X}$. A space M is microlinear if for each limit L in $\mathscr{W}_1$, $[D(L), M]$ is a limit in $\mathbb{X}$. As $[\_, M]$ sends colimits to limits, such M are sometimes said to perceive $D(L)$ as a colimit. From above, the microlinear objects always form a tangent category.

Proposition 6.7.1. *If $\mathbb{X}$ has a representable pre-Weil-prolongation, then every object is Weil exponentiable.*

Proof. We must show that the following map is an isomorphism for every V:

$$A\times[D(V),[A,B]] \xrightarrow{[!,A]\times 1} [D(V),A]\times[D(V),[A,B]] \simeq [D(V),A\times[A,B]] \xrightarrow{[D(V),\mathsf{ev}]} [D(V),B]$$

We intepret the above map in the internal language of a cartesian closed category:

$$(x,\lambda d a.t\ d\ a) \mapsto (\lambda d.x, \lambda d a.t\ d\ a) \mapsto \lambda d.(x,\lambda a.t\ d\ a) \mapsto \lambda d.t\ d\ x$$

Thus currying the above map yields

$$[D(V),[A,B]] \longrightarrow [A,[D(V),B]] \qquad ; \qquad \lambda d a.t\ d\ a \mapsto \lambda a d.t\ d\ a$$

Which is the canonical swap isomorphism $[D(V),[A,B]] \simeq [A,[D(V),B]]$. □

Thus, the microlinear, exponentiable objects are just the microlinear objects. Combining this with 6.6.5:

Corollary 6.7.2. *The microlinear objects of a category with representable pre-Weil-prolongation is a coherently closed tangent category.*

If each $D(V)$ is microlinear then the tangent functor on microlinear objects is again representable. As $[D(V),\_]$ is a continuous functor we obtain, from 6.5.7:

Corollary 6.7.3. *If $\mathbb{X}$ is complete and has a representable pre-Weil-prolongation where each $D(V)$ is microlinear, then* $\mathsf{Microl}(\mathbb{X})$ *is a coherently closed, representable tangent category with all limits.*

6.7.2 Synthetic Differential Geometry

Let $\mathscr{E}$ be topos with a ring $\mathscr{R}$. We may consider $R\text{-}\mathscr{W}_1$, the category of Weil algebras over $\mathscr{R}$. In fact, consider all Weil algebras over $\mathscr{R}$. There is a functor

$$D(\_) : \mathscr{R}\text{-}\mathsf{Weil}^{\mathrm{op}} \longrightarrow \mathscr{E}$$

Suppose a Weil algebra is presented as $\mathscr{R}[x_1,\ldots,x_n]/I$ and we choose a finite set of polynomials $p_1,\ldots,p_k$ that generate I; this is possible as Weil algebras are finitely generated. Then define the *spectrum* by:

$$D(U) := \{(a_1,\ldots,a_n) \in R^n \mid \forall k.p_k(a_1,\ldots,a_n) = 0\}$$

For example

$$D(W^1) = D(\mathscr{R}[x]/(x^2)) := \{d \in \mathscr{R} \mid d^2 = 0\}$$

Given a homomorphism of Weil algebras $U \xrightarrow{f} V$ where U has n generators and V has m generators, let $(b_1, \ldots, b_m) \in D(V)$. For each generator x_i of U, $f(x_i)$ is a polynomial (of V). Let $a_i := f(x_i)(b_1, \ldots, b_m)$.

$$D(f)(b_1, \ldots, b_m) := (a_1, \ldots, a_n)$$

This is well defined. A map $U \xrightarrow{f} V$ on presentations is a map $R[x_1, \ldots, x_n]/I_U \xrightarrow{f} \mathscr{R}[x_1, \ldots, x_m]/I_V$. This must arise from a map $\hat{f} : \mathscr{R}[x_1, \ldots, x_n] \longrightarrow \mathscr{R}[x_1, \ldots, x_m]/I_V$ that sends $p \in I_U$ to $0 \mod I_V$. In turn this means that $\hat{f}(p) \in I_V$. Let $(b_1, \ldots, b_m) \in D(V)$. In particular, $\hat{f}(p)(b_1, \ldots, b_m) = 0$. But

$$\begin{aligned}
&\hat{f}(p)(b_1, \ldots, b_m) \\
&= \hat{f}(\sum_\alpha c_\alpha x^\alpha)(b_1, \ldots, b_m) \\
&= \sum_\alpha c_\alpha (f x_1)^{\alpha_1}(b_1, \ldots, b_m) \cdots (f x_n)^{\alpha_n}(b_1, \ldots, b_m) \\
&= \sum_\alpha c_\alpha (f(x_1)(b_1, \ldots, b_m))^{\alpha_1} \cdots (f(x_n)(b_1, \ldots, b_m))^{\alpha_n} \\
&= \sum_\alpha c_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha^n} = p(a_1, \ldots, a_n)
\end{aligned}$$

Then define an action

$$\mathscr{E} \times \mathscr{R}\text{-}\mathsf{Weil}^{\mathrm{op}} \xrightarrow{\otimes_\infty} \mathscr{E}$$

Where $X \otimes_\infty U := [D(U), X]$ on objects and $f \otimes_\infty g := [D(g), f]$ on arrows. This is clearly a bifunctor as $D(\_)$ is a functor. It is also a monoidal action: note that $D(U \otimes V) \simeq D(U) \times D(V) \in \mathscr{E}$ (see (Lavendhomme, 1996) *3* in 2.1.2). Thus, we have

Proposition 6.7.4. *For any topos $\mathscr{E}$, and ring $\mathscr{R}$ in $\mathscr{E}$, the induced action from above, is a representable pre-Weil-prolongation. Therefore,* $\mathsf{Microl}(\mathscr{E})$ *is a coherently closed, representable tangent category with all limits.*

However, this is not satisfying. The image of D need not be microlinear, nor does $\mathscr{R}$ need to be microlinear, and thus the most basic space for assembling differential geometry may not be microlinear!

A topos $\mathscr{E}$ is called **smooth** (with respect to $\mathscr{R}$) when the following canonical map

$$W \xrightarrow{\alpha} \mathscr{R} \otimes_\infty W := [D(W), \mathscr{R}]$$

is an isomorphism for each Weil algebra W. If W is presented as $W \simeq R[x_1, \ldots, x_n]/I$ then α arises by considering $\hat{\alpha} : R[x_1, \ldots, x_n] \longrightarrow [D(W), R]$ which sends a polynomial p to the function which evaluates p on $D(W)$. This map passes to the quotient, since any $p \in I$ is 0 on any element of $D(W)$.

A consequence of the smoothness assumption is:

$$W_1 \otimes W_2 \simeq \mathscr{R} \otimes_\infty (W_1 \otimes W_2) \simeq (\mathscr{R} \otimes_\infty W_1) \otimes_\infty W_2 \simeq W_1 \otimes_\infty W_2$$

The following is then immediate

Proposition 6.7.5. *The microlinear spaces of a smooth topos $\mathscr{E}$ are a coherently closed, representable tangent category with all limits where $\mathscr{R}$ is microlinear and for each Weil algebra W, $D(W)$ is microlinear.*

In $\mathsf{Microl}(\mathscr{E})$ the differential objects are precisely the Euclidean $\mathscr{R}$-vector space; that is, vector spaces that satisfy the Kock-Lawvere axiom: $[D(W_1), V] \simeq V \times V$. One can formulate the notion of manifold modelled on Euclidean $\mathscr{R}$-vector spaces. Kock showed the following in (Kock, 2009)

Proposition 6.7.6. *Synthetic manifolds are microlinear.*

6.7.3 Functors out

We revisit the functors out example. Let $\mathbb{X}$ be a tangent category presented by some action $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$ and $\mathbb{Y}$ be any category. We put an actegory structure on $[\mathbb{X}, \mathbb{Y}]$ by assigning $F : \mathbb{X} \longrightarrow \mathbb{Y}$ and $U : \mathscr{W}_1$ the functor $(F \otimes_\infty U) : \mathbb{X} \longrightarrow \mathbb{Y}$ which sends $X \mapsto F(X \otimes_\infty U)$. Given a natural transformation $\beta : F \longrightarrow G$, define $\beta \otimes_\infty U$ on components by $(\beta \otimes_\infty U)_X := \beta_{X \otimes_\infty U}$.

The microlinear functors are then the ones for which each limit $L \in \mathscr{W}_1$ has $F(X \otimes_\infty L)$ a limit in $\mathbb{Y}$ for every $X \in \mathbb{X}$.

Observation 6.7.7. *Continuous functors are always microlinear.*

Of particular interest is when $\mathbb{Y} = \mathsf{Set}$. Then $[\mathbb{X}, \mathsf{Set}]$ is a topos, and in particular is cartesian closed and has all limits.

Proposition 6.7.8. *Representable functors* $\mathbb{X} \xrightarrow{\mathbb{X}(A,\_)} \mathsf{Set}$ *are microlinear.*

This follows immediately as representable functors preserve limits in the covariant argument.

Proposition 6.7.9. *Let* $\mathbb{X}$ *be a tangent category, given by* $\mathbb{X} \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathbb{X}$*. Then every limit of microlinear functors in* $[\mathbb{X}, \mathsf{Set}]$ *is transverse.*

Proof. A microlinear functor F has the property that that for each transverse limit of Weil algebras , $\mathsf{lim}_j V_j$, $F \otimes_\infty (\mathsf{lim}_j V_j)$ is a limit. As limits are computed pointwise into Set, we have $F(X \otimes_\infty (\mathsf{lim}_j V_j)) \simeq \mathsf{lim}_j F(X \otimes_\infty V_j)$.

$\mathsf{lim}_i F_i$ is a limit of microlinear functors then let $\mathsf{lim}_j V_j$ be a transverse limit of Weil algebras.

$$(F \otimes_\infty \mathsf{lim}_j V_j)(X) \simeq F(X \otimes_\infty \mathsf{lim}_j V_j) \simeq \mathsf{lim}_i(F_i(X \otimes_\infty \mathsf{lim}_j V_j))$$
$$\simeq \mathsf{lim}_i \mathsf{lim}_j F_i(X \otimes_\infty V_j) \simeq \mathsf{lim}_j(F \otimes_\infty \mathsf{lim}_j V_j)(X) \simeq \mathsf{lim}_j(F \otimes_\infty V_j)(X)$$

Showing that the limit is transverse. $\square$

A similar proof to the above uses that in any Grothendieck topos (e.g. $[\mathbb{X}, \mathsf{Set}]$), filtered colimits commute with finite limits, to show that filtered colimits of microlinear functors are microlinear.

Proposition 6.7.10. *For any tangent category* $\mathbb{X}$*,* $\mathsf{Microl}([\mathbb{X}, \mathsf{Set}])$ *has filtered colimits of microlinear functors.*

Proof. Let $F = \mathsf{colim}_i F_i$ be a filtered colimit where each F_i is a microlinear functor. To show that F is microlinear, let L be a transverse limit $\mathsf{lim}_j V_j$ in $\mathscr{W}_1$. Note this is a finite limit, by the inductive construction of transverse limits in $\mathscr{W}_1$. As colimits and limits are computed pointwise:

$$(F \otimes_\infty \mathsf{lim}_j V_j)(X) \simeq F(X \otimes_\infty \mathsf{lim}_j) \simeq \mathsf{colim}_i F_i(X \otimes_\infty \mathsf{lim}_j V_j)$$
$$\simeq \mathsf{colim}_i \mathsf{lim}_j F_i(X \otimes_\infty V_j) \simeq \mathsf{lim}_j \mathsf{colim}_i F_i(X \otimes_\infty V_j)$$
$$\simeq \mathsf{lim}_j F(X \otimes_\infty V_j) \simeq \mathsf{lim}_j(F \otimes_\infty V_j)(X)$$

is a limit, thus F is microlinear. $\square$

Thus for a tangent category $\mathbb{X}$, the category $\mathsf{Microl}(\mathbb{X}, \mathsf{Set})$ has transfinite composition.

Corollary 6.7.11. *Let $\mathbb{X}$ be a tangent category.* $\mathsf{Microl}(\mathbb{X}, \mathsf{Set})$ *is a complete tangent category, with filtered colimits, where every limit is transverse.*

Immediate from 6.5.6, 6.7.9, 6.7.10, and 6.5.7.

One could look to extract the Weil exponentiable functors. It would be surprising if representable functors were always exponentiable.

6.7.4 Weil spaces

The categories $\mathscr{W}_1$ and Weil are both tangent categories. We can extract the microlinear functors $\mathscr{W}_1 \longrightarrow \mathsf{Set}$ and $\mathsf{Weil} \longrightarrow \mathsf{Set}$; it turns out here that every microlinear functor is exponentiable.

Recall the Yoneda lemma, for $F : \mathbb{X} \longrightarrow \mathsf{Set}$, $F(X) \simeq \mathsf{Nat}(\mathbb{X}(X, \_), F)$. Also recall the hom, $[H, K](X) := \mathsf{Nat}(H \times \mathbb{X}(X, \_), K)$.

Observation 6.7.12. *The Yoneda embedding is product preserving when viewed as a functor* $\mathbb{X}^{op} \xrightarrow{\mathscr{Y}} [\mathbb{X}, \mathsf{Set}]$.

In $\mathscr{W}_1^{\mathrm{op}}$ products are the coproducts of $\mathscr{W}_1$ which are the operation $U \otimes V$. That $\mathscr{Y}$ preserves products, means that $\mathscr{Y}(U \otimes V) \simeq \mathscr{Y}(U) \times \mathscr{Y}(V)$; it also sends the terminal object in $\mathscr{W}_1^{\mathrm{op}}$ which is the zero object R to the terminal object $\mathscr{Y}(R) \simeq 1$.

Thus,

Proposition 6.7.13. *The categories* $[\mathsf{Weil}, \mathsf{Set}]$ *and* $[\mathscr{W}_1, \mathsf{Set}]$ *have representable pre-Weil-prolongations, $D(\_) := \mathscr{Y}(\_)$.*

Corollary 6.7.14. *The categories* $\mathsf{Microl}([\mathsf{Weil}, \mathsf{Set}])$ *and* $\mathsf{Microl}([\mathscr{W}_1, \mathsf{Set}])$ *are complete, coherently closed, representable tangent categories with respect to the action induced by having a representable pre-Weil-prolongation.*

This is immediate from corollary 6.7.3

Proposition 6.7.15. *The actegory structures* $[\mathsf{Weil}, \mathsf{Set}] \times \mathsf{Weil} \longrightarrow [\mathsf{Weil}, \mathsf{Set}]$ *and* $[\mathscr{W}_1, \mathsf{Set}] \times \mathscr{W}_1 \longrightarrow [\mathscr{W}_1, \mathsf{Set}]$ *given by representable pre-Weil-prolongation* $D(\_) := \mathscr{Y}$ *are the standard Weil prolongation on categories of functors into* Set.

Proof. We must show that $(F \otimes_\infty U) := F(\_ \otimes_\infty U)$ and $(F \otimes_\infty U) := [\mathscr{Y}(U), F]$ are the same. The proof is Yoneda's lemma followed by the fact that in $\mathscr{W}_1$ $X \otimes_\infty U = X \otimes U$, and finally followed the product preservation of $\mathscr{Y}$.

$$F(\_\otimes_\infty U) \simeq \mathsf{Nat}(\mathscr{Y}(\_\otimes_\infty U), F) \simeq \mathsf{Nat}(\mathscr{Y}(\_\otimes U), F) \simeq \mathsf{Nat}(\mathscr{Y}(U)\times\mathscr{Y}(\_), F) \simeq [\mathscr{Y}(U), F]$$

□

Thus we can use proposition 6.7.8, to show that R is microlinear, and that more generally every Weil algebra is microlinear, when regarded as representable functors.

Proposition 6.7.16. *The functors* $\mathsf{Weil} \longrightarrow \mathsf{Set}$ *and* $\mathscr{W}_1 \longrightarrow \mathsf{Set}$ *that represent Weil algebras are microlinear (i.e.* $\mathscr{W}_1(U, \_)$ *is microlinear).*

Observation 6.7.17. *A microlinear functor* $\mathsf{Weil} \longrightarrow \mathsf{Set}$ *or* $\mathscr{W}_1 \longrightarrow \mathsf{Set}$ *is precisely a functor that preserves transverse limits of Weil algebras.*

Proof. If a functor F preserves all transverse limits then as $U \otimes L$ is a transverse limit by construction, $F(U \otimes L)$ is a limit.

If $F(U \otimes L)$ is a limit for every transverse limit L, and every U, then set $U = R$. □

Functors $\mathsf{Weil} \longrightarrow \mathsf{Set}$ are called *Weil spaces* by (Bertram, 2014).

Consider R-$\mathscr{W}_1$ for some ring R, and let V be an R-module. This then gives rise to a Weil space.

$$\underline{V} : \mathsf{Weil} \longrightarrow \mathsf{Set}$$

where

$$\underline{V}(U) := V \otimes_R U \qquad \underline{V}(f : U \longrightarrow U') := V \otimes_R f$$

by taking the tensor over R. One recovers V as $\underline{V}(R) \simeq V \otimes R \simeq V$.

Proposition 6.7.18. *The assignment $V \mapsto \underline{V}$ extends to a functor*

$$R\text{-}\mathsf{Mod}_{\mathsf{fp}} \longrightarrow \mathsf{Microl}(\mathsf{Weil}\text{-}\mathsf{S})$$

The proof follows immediately from a theorem in commutative algebra.

Lemma 6.7.19. *Suppose that B is finitely presented, and that A_i are of finite length (or less generally finitely generated and free). Then*

$$\mathsf{lim}_i(A_i \otimes_R B) \simeq \mathsf{lim}_i A_i \otimes_R B$$

There is a simpler proof. The category of R-modules and linear maps are a cartesian differential category with $D[f] = \pi_0 f$; hence they are a tangent category with $TA := A \times A$ and $Tf := f \times f$.

Proof. The product of Weil algebras, $U \oplus V$, is a transverse limit. Then:

$$A \otimes (U \oplus V) \simeq (A \otimes U) \oplus (A \otimes V)$$

as $\otimes$ distributes over $\oplus$.

Next the equalizer of 6.1.13 is transverse. As any Weil algebra is finite dimensional, and $\otimes$ distributes over $\oplus$ that $A \otimes U \simeq A^n$ where n is the dimension of U. Then

$$A \otimes (W^1 \times W^1) \xrightarrow{A \otimes \langle \pi_0 \lambda, \pi_1 \eta \rangle} A \otimes ((W^1 \times W^1) \otimes W^1) \xrightarrow{A \otimes (\sigma \otimes W^1)} A \otimes (W^1 \otimes W^1)$$

is, up to linear isomorphism, the map:

$$A \oplus A \oplus A \xrightarrow{\langle \pi_0, \pi_1, 0, \pi_2 \rangle} A \oplus A \oplus A \oplus A$$

which may be thought of as the map $(a, h, v) \mapsto (a, h, 0, v)$. The map $A \otimes (W^1 \otimes W^1) \xrightarrow{A \otimes (\epsilon \otimes W^1)} A \otimes W^1$ is, up to linear isomorphism, the map:

$$A^4 \xrightarrow{\langle \pi_0, \pi_2 \rangle} A^2$$

which is thought of as the map $(a, h, v, c) \mapsto (a, v)$. Finally, the map $A \otimes (W^1 \otimes W^1) \xrightarrow{A \otimes ((\epsilon \otimes \epsilon)\eta)} A \otimes W^1$ is the map

$$A^4 \xrightarrow{\langle \pi_0, 0 \rangle} A^2$$

which is thought of as the map $(a,h,v,c) \mapsto (a,0)$.

$\langle \pi_0, \pi_1, 0, \pi_2 \rangle$ weakly equalizes $\langle \pi_0, \pi_2 \rangle$ and $\langle \pi_0, 0 \rangle$. Suppose that $H = \langle H_1, H_2, H_3, H_4 \rangle : X \longrightarrow A^4$ weakly equalizes the maps. Then it is immediate that $H_3 = 0$. Then

$$X \xrightarrow{\langle H_1, H_2, H_4 \rangle} A^3$$

has

$$\langle H_1, H_2, H_4 \rangle \langle \pi_0, \pi_1, 0, \pi_2 \rangle = \langle H_1, H_2, H_3, H_4 \rangle$$

and $\langle H_1, H_2, H_4 \rangle$ is the unique map that makes the above an equality, thus $\langle \pi_0, \pi_1, 0, \pi_2 \rangle$ is an equalizer. □

Thus for any finitely presented R-module A, the associated Weil space $\underline{A}$ is microlinear. The idea of this proof is due to (Bertram, 2014), we just extended the proof to all the cases required for microlinearity.

Corollary 6.7.20. *The category of finitely presented modules, regarded as Weil spaces, form a Cartesian differential subcategory of the differential objects of* $\mathsf{Microl}(\mathsf{Weil}\text{-}\mathsf{S})$.

Using the notion of generalized cartesian differential category, we may enlarge finitely presented modules to a generalized cartesian differential subcategory. Before beginning, recall again that for an R-module A we have

$$A \otimes U \simeq A \otimes (R \oplus \dot{U}) \simeq A \oplus (A \otimes \dot{U})$$

where $\dot{U}$ is the ideal of augmentation.

Now, let X be a subset of A where A is a finitely presented R-module. Then define a Weil space:

$$\underline{X}(U) := X \times (A \otimes \dot{U})$$

Lemma 6.7.21. *For subsets X of A, the Weil spaces $\underline{X}$ are microlinear.*

Proof. As any Weil algebra U is augmented, $U \simeq R \oplus \dot{U}$, a limit of Weil algebras always takes the form:

$$\mathsf{lim}_j U_j \simeq \mathsf{lim}_j (R \oplus \dot{U}_j) R \oplus \mathsf{lim}_j \dot{U}_j$$

due to the fact that $\oplus$ is a product and hence commutes with limits. Then let $\lim_j U_j$ be a transverse limit in $\mathscr{W}_1$. We have

$$\begin{aligned}&\underline{X}(\lim_j U_j) = X \times (A \otimes \lim_j \dot{U}_j) \simeq X \times \lim_j (A \otimes U_j)\\&\simeq \lim_j (X \times A \otimes U_j) = \lim_j \underline{X}(U_j)\end{aligned}$$

The first $\simeq$ is again 6.7.19. The second $\simeq$ follows as the product functor preserves limits. □

Corollary 6.7.22. *For subsets X of A, the Weil spaces $\underline{X}$ form a tangent subcategory of* Microl(Weil-S)*, denoted $\mathscr{O}(R$*-Mod-Weil)

Now that we have a generalized cartesian differential category, we may consider manifolds. Bertram call these manifolds *set theoretic manifolds*. We first consider set theoretic manifolds modelled on R-modules, and then show that this can be lifted to Weil spaces.

A **set theoretic manifold** modelled on an R-module A is a set M, together with subsets $X_i \subseteq A$ and transitions $X_i \xrightarrow{\phi_{ij}} X_j$, and $M \simeq \bigcup_i U_i$.

Set up this way, it is clear that a set theoretic manifold is uniquely determined up to isomorphism, by the charts and transitions. Given a set theoretic manifold M, we obtain a Weil space $\underline{M}$. This category of manifolds is a tangent category; below we show that the tangent structure in this tangent category is the tangent structure in Weil-S; that is, these *Weil manifolds* are all microlinear, and form a tangent subcategory of Microl(Weil-S). We briefly summarize Bertram's argument.

Proposition 6.7.23 ((Bertram, 2014))**.** *The Weil spaces that are generated as manifolds modelled on $\mathscr{O}(R$*-Mod-Weil) *are all microlinear in* Weil-S*. Moreover, the tangent structure on them is the tangent structure when regarded as objects of* Weil-S.

Proof. Let $\underline{M}$ be a manifold modelled on $\mathscr{O}(R$-Mod-Weil). Then $\underline{M}$ is a family of Weil spaces $\underline{X_i}$ together with transition functions $\underline{\phi_{ij}}$. Note that

$$X_i \simeq X_i \times 1 \simeq X_i \times (A \otimes \dot{R}) = \underline{X_i}(R)$$

and similarly $\phi_{ij} = \underline{\phi_{ij}}_R$. Thus the manifold $\underline{M}$ is completely determined by a set theoretic manifold M.

Further, $\underline{M} \otimes_\infty U$ is defined by the atlas

$$(\underline{X_i} \otimes_\infty U, \underline{\phi_{ij}} \otimes_\infty U) \equiv (\underline{X_i}(\_ \otimes U), \underline{\phi_{ij}}_{\_\otimes U})$$

Thus, for each U, we obtain a Weil space $\underline{M} \otimes_\infty U$, which is again a Weil manifold.

For any limit $\lim_j V_j$ in $\mathscr{W}_1$, each chart $\underline{X_i}(\lim_j V_j) = \lim_j \underline{X_i}(V_j)$ hence the atlas as a whole preserves the limit. The tangent structure on these manifolds is definitionally the same as the one on Weil-S. □

In the above proof, we constructed the image for each Weil algebra, by defining a new set theoretic manifold for each $\underline{M}(U)$.

Corollary 6.7.24. *The category whose objects are* $\mathsf{Mf}(\mathsf{jn}(\mathscr{O}(R\text{-}\mathsf{Mod}\text{-}\mathsf{Weil})))$ *is precisely the subcategory of functors* $\underline{M} : \mathsf{Weil} \longrightarrow \mathsf{Set}$ *such that*

$$\begin{tikzcd} \mathsf{Weil} \arrow[rr, "\underline{M}"] \arrow[dr, "\underline{M}"'] & & \mathsf{Set} \\ & \mathsf{Set\text{-}Th\text{-}Man} \arrow[ur, hook] & \end{tikzcd}$$

and $\underline{M}$ *has an atlas* $(\underline{X_i}, \phi_{ij})$ *where* $\underline{X_i}$ *is underlied by a subset of an* R*-module* A.

A functor $\underline{M} : \mathsf{Weil} \longrightarrow \mathsf{Set}$ that factors through set theoretic manifolds as above is called a **Weil manifold**. In summary,

Proposition 6.7.25. *The category* $\mathsf{Microl(Weil\text{-}S)}$ *is a coherently closed tangent category with all limits and filtered colimits that contains all the Weil manifolds.*

6.7.5 Convenient vector spaces

(Kock, 1986) introduces the Weil prolongation to formalize the calculus of jets for convenient vector spaces and smooth maps. We formalize the Weil prolongation of convenient vector spaces using the characterization of $\mathscr{R}$-Weil algebras given in section 6.1.2.

In example 2.1.12 we introduced convenient vector spaces as special cases of Frölicher spaces. We need a few facts about Frölicher spaces which we state without proof but refer to the reader for example to (**book:kriegl-frolicher**; **thesis:DG-In-CCC**).

Proposition 6.7.26. *The category of Frölicher spaces:*

1. *is topological over* Set*;*

2. *is complete and cocomplete;*

3. *is Cartesian closed;*

We also will need

Proposition 6.7.27. *The category of convenient vector spaces is an exponential ideal of Frölicher spaces and hence is Cartesian closed.*

Reference. This is proposition 23.4 of (Kriegl and Michor, 1997). $\square$

We use the Cartesian closedness of convenient vector spaces and the characterization of $\mathscr{R}$-weil algebras to give a Weil prolongation for convenient vector spaces.

Let (X, C_X, F_X) be a Frölicher space. Define the action at the level of sets first. $X \otimes_\infty U := \mathsf{Fröl}\,(R^n, X)_{/\sim_I}$ where $g_1 \sim_I g_2$ if

- $g_1(0,\ldots,0) = g_2(0,\ldots,0)$;

- $g_1 f - g_2 f : \mathscr{R}^n \longrightarrow \mathscr{R} \in I$ for every $f \in F_X$.

Let $X \xrightarrow{h} Y$ be a Frölicher map. Then define

$$(h \otimes_\infty U)([g]) := [gh] \qquad h \otimes_\infty U : X \otimes_\infty U \longrightarrow Y \otimes_\infty U$$

Note that this is well defined at the level of sets. If $g_1 \sim_I g_2$ then

- $h(g_1(0,\ldots,0)) = h(g_2(0,\ldots,0))$;

- We must show that $g_1 hf - g_2 hf \in I$ for every $f \in F_Y$. But, $hf \in F_X$ as h is smooth, hence $g_1(hf) - g_2(hf) \in I$.

We must put a Frölicher structure on these actions.

Consider $(\mathscr{R}, C^\infty(\mathscr{R}), C^\infty(\mathscr{R}))$. Then the action by each $U \in \mathsf{Weil}$ on $\mathscr{R}$ inherits the action from the category of convenient vector spaces (or smooth manifolds, if preferred). Then we may equip $(\mathscr{R} \otimes_\infty U, \mathsf{CVS}(R, R \otimes_\infty U), \mathsf{CVS}(R \otimes_\infty U, R))$.

Then the Frölicher structure put on $X \otimes_\infty U$ is the initial Frölicher structure such that $X \otimes_\infty U \xrightarrow{f \otimes_\infty U} \mathscr{R} \otimes_\infty U$ is smooth for all $f \in F_X$.

Lemma 6.7.28. *When the prolongation of a Frölicher space is applied to a convenient vector space, the result is again a convenient vector space. The action is functorial in both arguments*

reference. This is (Kock, 1986), proposition 2.2. □

Proposition 6.7.29. *The action of prolongation by Weil algebras on convenient vector spaces is an actegory.*

reference. This is developed in section 3, and in particular theorem 3.1 of (Kock, 1986). □

At the end of section 5 (the top of page 14) of (Kock, 1986), it is pointed out that $X \otimes_\infty U$ is a power of X, and since products preserve limits, we have that every object is microlinear with respect to this prolongation. Thus,

Proposition 6.7.30. *The category of microlinear convenient vector spaces is the category of all convenient vector spaces.*

Finally:

Theorem 6.7.31. *The category of convenient vector space is a coherently closed tangent category.*

Proof. On page 14 of (Kock, 1986), it is pointed out that $[A,B] \otimes_\infty U \simeq [A, B \otimes_\infty U]$. On the same page shows that the isomorphism takes a certain form, namely

$$[A,B] \otimes_\infty U \xrightarrow{\lambda(\theta;\mathsf{ev})} [A, B \otimes_\infty U]$$

as required. □

6.7.6 Sikorski spaces

Sikorski spaces also known as *differential spaces* have been used to formalize the notions of stratifold and orientifold, and to pursue symplectic reduction (Śniatycki, 2013).

A **classical Sikorski space** (**book:diff-alg-top**) is a

- A topological space X;
- A set of functionals $F \subseteq C_0(X, \mathscr{R})$ that is a subalgebra of the $\mathscr{R}$ algebra of continuous functionals;

such that

- Let $g_i = g_{|V_i}$ with $g \in F$; if $\cup_i V_i = X$ then $\vee_i g_i \in F$;
- If $f_1, \ldots, f_k \in F$ and $g \in C^{\infty}(\mathscr{R}^k, \mathscr{R})$ then $\langle f_1, \ldots, f_k \rangle g \in F$.

Sikorski spaces can be generalized to presheaves $\mathscr{O}(\mathscr{R}) \longrightarrow \mathsf{Set}$ where $\mathscr{O}(\mathscr{R})$ is the generalized cartesian differential category of smooth maps on open subsets of $\mathscr{R}^n$. The category of **abstract Sikorski spaces** is the category of functors $\mathsf{Fun}(\mathscr{O}(\mathscr{R}), \mathsf{Set})$. Note that classical Sikorski spaces embed into abstract Sikorski spaces due to the second condition.

We know that in general we can extract a complete tangent category from this category of abstract Sikorski spaces.

Proposition 6.7.32. *The category of microlinear abstract Sikorksi spaces is a complete tangent category with filtered colimits.*

If one wants the topological properties of a classical Sikorski spaces and microlinearity, one can select those classical Sikorski spaces which as abstract Sikorski spaces are microlinear, and for which all Weil actions are classical Sikorski spaces.

However it is not clear that Sikorski spaces have a nice subcategory of Weil exponential objects.

Note that an abstract Sikorski space $F : \mathscr{O}(\mathscr{R}) \xrightarrow{F} \mathsf{Set}$ naturally gives rise to a functor

$$\mathscr{O}(\mathscr{R}) \times \mathscr{W}_1 \xrightarrow{\otimes_\infty} \mathscr{O}(\mathscr{R}) \xrightarrow{F} \mathsf{Set}$$

that sends transverse limits in the second argument to limits.

A **Weil-Sikorksi** space is a functor

$$\mathcal{O}(\mathcal{R}) \times \mathcal{W}_1 \longrightarrow \mathsf{Set}$$

that sends transverse limits in the second argument to limits.

This can be curried to obtain a map

$$\mathcal{O}(\mathcal{R}) \longrightarrow \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set})$$

As we have seen $\mathsf{Microl}(\mathcal{W}_1, \mathsf{Set})$ is a complete, cartesian closed tangent category. Hence

$$\mathsf{Fun}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$$

is a complete cartesian closed category, that is also a tangent category. However, it need not be coherently closed. In fact, it is not clear that we can even exponentiate representable functors.

Proposition 6.7.33. *The category of microlinear, exponentiable objects in* $\mathsf{Fun}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$ *contains the category of tangent functors* $\mathsf{Tan}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$.

Proof. For any tangent categories, $\mathbb{X}, \mathbb{Y}$, there is an embedding of tangent categories

$$\mathsf{Tan}(\mathbb{X}, \mathbb{Y}) \hookrightarrow \mathsf{Microl}(\mathbb{X}, \mathbb{Y})$$

As $\mathsf{Tan}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$ is representable 6.3.3, every object is exponentiable 6.7.1. Thus,

$$\mathsf{Tan}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set})) \hookrightarrow \mathsf{Microl\text{-}Exp}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$$

□

Proposition 6.7.34.

$$\mathsf{Tan}(\mathcal{O}(\mathcal{R}), \mathsf{Microl}(\mathcal{W}_1, \mathsf{Set}))$$

contains the representable functors $\mathcal{O}(S, \_)(\_)$*; hence, representables are microlinear and exponentiable.*

Proof. Note that by Wood's characterization of actegories as presheaf categories that admit powers by representables:

$$\mathscr{O}(S, TM)(\_) \simeq [\mathscr{Y}(W^1), \mathscr{O}(S, M)(\_)]$$

And in $\mathsf{Microl}(\mathscr{W}_1, \mathsf{Set})$, $TX \simeq [\mathscr{Y}(W^1), X]$ by proposition 6.7.15. Thus the functor preserves tangent structure, and hence is microlinear and exponentiable by proposition 6.7.33. □

Of course, this means, that we obtain a model of the differential λ-calculus in $\mathsf{Microl\text{-}Exp}(\mathscr{O}(\mathscr{R}), \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$.

6.7.7 Weil Diffeological Spaces

Diffeological spaces are the concrete sheaves on the concrete site $\mathscr{O}(\mathscr{R})$ (**preprint:baez-hoffnung-diffeologica** and are hence a quasitopos. Baez also argues that it might be easier to work with all sheaves, which for general reasons are localizations of presheaves, and all sheaves form a topos. From the point of view of tangent categories, sheaves are not well understood yet.

By **abstract diffeological space**, we mean an object in the category of presheaves

$$\mathsf{Fun}(\mathscr{O}(\mathscr{R})^{\mathrm{op}}, \mathsf{Set}).$$

We can extract the microlinear objects from this category if we desire; however, from the tangent categories perspective, this is a harder category to work with than microlinear Sikorski spaces, as the opposite tangent category is less well behaved. Instead we jump straight to *Weil diffeological* spaces.

A **Weil-Diffeological** space is a functor

$$\mathscr{O}(\mathscr{R})^{\mathrm{op}} \times \mathscr{W}_1 \longrightarrow \mathsf{Set}$$

that sends transverse limits in $\mathscr{W}_1$ to limits.

A **module** or **profunctor** $\mathbb{X} \nrightarrow \mathbb{Y}$ is a functor $\mathbb{X}^{\mathrm{op}} \times \mathbb{Y} \longrightarrow \mathsf{Set}$. When $\mathbb{X}$ and $\mathbb{Y}$ are tangent categories, a **tangent module** $\mathbb{X} \nrightarrow \mathbb{Y}$ is a module that is underlied by a tangent preserving functor $\mathbb{X}^{\mathrm{op}} \longrightarrow \mathsf{Microl}(\mathbb{Y}, \mathsf{Set})$.

Garner proved that the tangent modules $\mathbb{X} \nrightarrow \mathscr{W}_1$ for any tangent category $\mathbb{X}$ is a representable tangent category (Garner, 2018) (theorem 28). Thus by proposition 6.7.1 the tangent modules underlie exponentiable, microlinear functors. Hence.

Proposition 6.7.35. *The category of microlinear, exponentiable objects in*

$$\mathsf{Fun}(\mathscr{O}(\mathscr{R})^{op}, \mathsf{Microl}(\mathscr{W}_1, \mathsf{Set}))$$

contains the category of tangent modules $\mathscr{O}(\mathscr{R}) \nrightarrow \mathscr{W}_1$.

And also similarly to Weil-Sikorski spaces

Proposition 6.7.36. *The category of tangent modules* $\mathscr{O}(\mathscr{R}) \nrightarrow \mathscr{W}_1$ *contains the representable presheaves.*

This follows as (enriched) presheaves always give (enriched) profunctors.

Thus from the point of view of Weil actions abstract diffeological spaces can be made very easy to work with. However, one of the reasons people work with diffeological spaces is that they are a category of sheaves. It could be interesting to consider the tangent modules $\mathscr{O}(\mathscr{R}) \nrightarrow \mathscr{W}_1$ that satisfy the sheaf condition in the first argument. However, we require a better understanding of how colimits interact with tangent structure to make this precise.

Chapter 7

Future Work

This thesis considered the differential λ-calculus as a syntax relevant to differential geometry. To make this precise, we considered models of the differential λ-calculus in coherently closed tangent categories.

This brings up an immediate question. The differential λ-calculus only gives the internal logic for the differential objects of a tangent category. More generally, it could be useful to have a syntactic presentation of a tangent category. Much of the work would involve turning the actegory characterization of tangent categories into explicit syntax for tangent categories. The syntax would be similar to the syntax for synthetic ∞-categories (**journal:riehl-shulman-infity-types**). There would be a Weil layer, that formalizes the types and maps of $\mathscr{W}_1$ over $\mathscr{N}$. There would then be a tope layer where one has a "stooped" context that expresses the action of the Weil layer. Alternatively, the type theory could be expressed through the enriched Yoneda lemma, and the presentations of type theories via Yoneda's lemma by (**book:taylor-practical-foundations**).

More substantially to obtain closed models, we essentially looked at localizations of presheaf categories (locally presentable categories), and enriched functor categories. We showed that looking at these kinds of categories gave the right behaviour with respect to limits: namely the enriched Yoneda embedding preserves transverse limits.

However, the behaviour with respect to colimits is a bit mysterious. We get colimits abstractly from theorems about locally presentable categories, and enriched functor categories into cocomplete enriched categories. However, obtaining colimits in a tangent category is a bit, admittedly, strange. For example, take the pushout of two copies of $\mathscr{R}$ along the origin 0; there is no reason to think this is a good smooth space.

Orbifolds, orientifolds, stratifolds, and sheaves on $\mathscr{O}(\mathscr{R})$ present progressively more general ways to obtain controlled and well behaved colimits or quotients. But it is not known how colimits interact with tangent structure. The desire to understand colimits, could lead to a better notion of sheaves for tangent categories; diffeological spaces certainly do not work – in this thesis we resorted to a generalization of diffeological spaces to get around this. However, understanding sheaves in a tangent category sense could lead to a better notion of diffeological space.

We also believe that the category of microlinear Weil spaces is regular. This would be significant because it would allow the development of regular tangent categories. Regular categories admit categories of internal relations. Regular tangent categories may have application to classically subtle topics, like the Weinstein, symplectic, and Fukaya categories, as these are all categories of relations built out of certain kinds of subobjects, for example, Lagrangian submanifolds (**chapter:symplectic-category**; **journal:lagrangian-subman**; **preprint:symplectic-cats**; **preprint:orbi-symplectic**).

This thesis then seems to be the start of a long-term research project to understand colimits and regularity in tangent categories and more broadly in the context of differential geometry.

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