UNIVERSITY OF CALGARY

Turing Categories and Realizability

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A THESIS SUBMITTED TO THE FACULTY OF GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

GRADUATE PROGRAM IN COMPUTER SCIENCE

CALGARY, ALBERTA

February, 2017

Abstract

We present a realizability tripos construction in which the usual partial combinatory algebra is replaced with a Turing category, and the category of partial functions on sets is replaced with a discrete cartesian closed restriction category. As an intermediate step we construct in this setting a restriction category of assemblies. Our constructions generalize existing constructions in the field.

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1 Introduction

In recent years, the field of category theory has found wide application in computer science, particularly in modelling the semantics of programming languages. The three way correspondence between computable functions, proofs in a logical system, and arrows in some category plays a central role. While the relationship of category theory to logic and of logic to the computable functions is fairly well understood, the relationship between category theory and computability theory has not been so thoroughly investigated. A recent development is that of Turing categories [8], which present the general machinery of computability theory (in the form of partial combinatory algebras) in categorical terms. Computability theory has classically taken place in the category of sets and partial functions, and this new, more general presentation leads to questions about how constructions involving classical computability work in the new setting.

A specific source of motivation for avoiding the category of sets and partial functions when studying models of computation comes from complexity theory. It is possible to construct a Turing category whose total maps are precisely the PTIME maps, but this Turing category has no faithful embedding into sets and partial functions [20]. In order to carry out computability-theoretic constructions with this Turing category, we must work more generally.

One such computability-theoretic construcion is the realizability topos of a partial combinatory algebra. The idea of realizability originated with Kleene [24], who constructed a model of the logic of intuitionistic first-order arithmetic out of the partial recursive functions. In the realizability model, such functions play the role of mathematical proof, and a proposition is provable ("realizable") in case there is a partial recursive function which witnesses this provability in a constructive sense. Some time later, realizability toposes – categorical models of higher-order intuitionistic logic – were constructed [17] [19], with the intermediate notion of a tripos [31] playing a central role. Every tripos defines a topos, and the realizability topos is the topos associated with a specific tripos (the "realizability tripos"). The work on realizability toposes also generalized the role of the partial recursive functions, with the elements of an arbitrary partial combinatory algebra acting as constructive evidence instead.

The study of these realizability toposes led to the development of the category of assemblies [1]. Originally used to construct models of polymorphism using the realizability machinery, assemblies have been used to study the semantics of programming languages [29]. In this approach, partial combinatory algebras are treated as computing machines. Assemblies are then sets equipped with an "implementation", relating each element of the set to one or more elements of the partial combinatory algebra. A map between assemblies is now "implementable" in case there is some element of the partial combinatory algebra that mimics the action of the map between sets when applied.

In this thesis, we introduce the category of assemblies for a restriction functor with a cartesian restriction category as its codomain. If the domain is a Turing category, the codomain is a discrete cartesian closed restriction category, and the functor preserves partial products, then the resulting category of assemblies is also a discrete cartesian closed restriction category. Relating this to the classical case, the computable maps of every set-based partial combinatory algebra are a Turing category, and this Turing category embeds into sets and partial functions, which is a discrete cartesian closed restriction category. The total maps of the category of assemblies for this embedding is the classical category of assemblies for the partial combinatory algebra in question. Our construction results in a restriction category of assemblies whose total maps correspond to the classical category of assemblies.

In addition to capturing the classical category of assemblies and its natural extension to a category of partial maps, our construction allows us to talk about a restriction category of assemblies in the presence of very little structure. Any restriction functor into a cartesian restriction category will do. We consider a variety of (partial) categorical structures, showing what the domain and codomain of the functor must satisfy in order for a given structure to be present in the category of assemblies. For example, if the domain is weakly cartesian closed and the codomain is a cartesian closed restriction category, then the category of assemblies is also a cartesian closed restriction category. The ability to vary the domain, and especially the codomain, of this functor makes it possible to construct categories of assemblies far more generally than the classical construction, which limits the codomain to the category of sets and partial functions.

The category of assemblies for a restriction functor defines a forgetful restriction functor into its codomain, and it is from this functor that we construct the realizability tripos. A tripos is a fibration, but our forgetful functor is not fibration. Instead, it is a *latent fibration*, the restriction categorical analogue of a fibration. We develop some elementary theory of latent fibrations, and use it to construct a fibration (in the usual sense) from the forgetful functor. When the functor we used to construct the category of assemblies has a Turing category as its domain and a discrete cartesian closed restriction category as its codomain, this fibration is a tripos. As with the category of assemblies, this construction generalizes the classical construction of the realizability tripos.

Having obtained a tripos in our more general setting, we define the realizability topos of a cartesian restriction functor with suitable domain and codomain to be the topos obtained from this tripos via the tripos-to-topos construction. We also show that the tripos-to-topos construction can be modified to give a partial topos in the sense of [13], the total maps of which are the associated topos.

1.1 Overview

In chapter 2 we review the existing theory of restriction categories that we will need going forward. We begin with the definition of a restriction category, and introduce restriction functors and the extension ordering, as well as relating restriction categories to the category of partial maps given by a stable system of monics in a category. We then deal with the restriction categorical version of products and terminal objects, introduce meets in a restriction category, and introduce the related concept of discrete cartesian restriction categories as cartesian restriction categories with meets. Next, we introduce latent pullbacks, joins, and interleaving in a restriction category. We end the chapter by introducing existential and universal quantification in a restriction category, and finally covering cartesian closed restriction categories: the restriction categorical analogue of cartesian closed categories.

Chapter 3 is all about discrete cartesian closed restriction categories. That is, discrete cartesian restriction categories that are also cartesian closed restriction categories. We show that they posess universal and existential quantification as defined in chapter 2, and that for every object X of such a restriction category the collection of restriction idempotents on X forms a Heyting algebra under the extension ordering. We also show that the total maps of our category give Heyting algebra morphisms between these preorders of restriction idempotents.

In chapter 4, we introduce Turing categories, and compare them to partial combinatory algebras in a cartesian restriction category. We also mention that every Turing category is weakly cartesian closed, and has weak coproducts, which we will need later.

This allows us to move on to chapter 5, in which we introduce the category of assemblies on a restriction functor whose codomain is a cartesian restriction category. We assume progressively more structure in the domain and codomain, showing the effect of this on the resulting category of assemblies. Specifically, If both the domain and codomain of the restriction functor are cartesian restriction categories and the functor itself preserves these products, then the category of assemblies is also a cartesian restriction category. If in addition the codomain is discrete, then the category of assemblies is discrete. If the domain is weakly cartesian closed and the codomain is a cartesian closed restriction category, then the category of assemblies is a cartesian closed restriction category, meaning in particular that if the domain is a Turing category and the codomain is a discrete cartesian closed restriction category then the resulting category of assemblies is a cartesian closed restriction category. Finally, if the domain has finite interleaving, the codomain has finite joins, and the functor preserves joins, then the category of assemblies has finite joins.

In chapter 6 we begin by introducing latent fibrations and investigating their elementary properties. This done, we give two examples of latent fibrations. The first, defined on any restriction category, is the domain latent fibration. The second arises from the category of assemblies we constructed above, and we call it the realizability latent fibration. We also show that pulling these latent fibrations back (as restriction functors) along the inclusion of the total map category into the base gives a fibration in the traditional sense.

We begin chapter 7 with the definition of a tripos, and immediately observe that the domain fibration obtained in chapter 6 is a tripos when the associated restriction category is a discrete cartesian closed restriction category. Next, we show how the well known construction of a topos from a tripos can be modified to yield a partial topos instead.

In chapter 8 we show that the fibration corresponding to the realizability latent fibration is a tripos when the components have enough structure. Specifically, given a cartesian restriction functor whose domain is a Turing category and whose codomain is a discrete cartesian closed restriction category, the fibration corresponding to the realizability latent fibration is a tripos.

1.2 Contributions

The main novelty of this thesis is our definition of the category of assemblies for a restriction functor. By presenting assemblies as restriction idempotents and giving an equational characterization of tracking maps, we are able to construct categories of assemblies in the presence of very little categorical structure, and capture existing categories of assemblies as special cases of our more general construction. Specifically, **Definition 5.1**, **Proposition 5.2, 5.3, 5.7, 5.8, 5.10**, **Lemma 5.4, 5.5**, and **Theorem 5.9** are novel. We would like to draw special attention to **Proposition 5.10**, which shows how the presence of interleaving in the category of realizers is related to the presence of joins in the category of assemblies. This had not been noticed before, as this sort of join is much more easily expressed in a restriction category, and previous constructions of the category of assemblies have resulted only in categories of total maps.

The chapter on latent fibrations is the first published exposition of this material, although the existence of latent fibrations is alluded to in [10]. Specifically, **Definition 6.1, 6.4, 6.5, 6.6, 6.7**, **Lemma 6.2, 6.3**, and **Proposition 6.8, 6.10, 6.11, 6.12, 6.13, 6.14**, which encompass the definition and elementary properties of latent fibrations, are novel. **Proposition 6.13** is a more explicit version of a remark in [10], in which the definition of a latent fibration was not available. **Proposition 6.14**, which deals with the realizability latent fibration, is of course novel.

The chapter on discrete cartesian closed restriction categories, in which we show that they posess significant logical structure, is essentially a recapitulation of the material in [6]. The related observation that the domain fibration of a discrete cartesian closed restriction category is a tripos (**Proposition 7.2**) follows immediately from this material, and so while the observation is technically a new one, it is not terribly original. This is, however, the first published exposition of the structure of discrete cartesian closed restriction categories from the perspective of fibered categorical logic, which may be of interest.

The contents of the final chapter, in which we show that the realizability fibration of a cartesian restriction functor with a Turing category as its domain and a discrete catesian closed restriction category as its codomain is a tripos, is adapted from [4]. In fact, the idea to define assemblies for a restriction functor is appropriated from the cited work as well. Our take on the concept is novel in two ways. First, the constructions we present are different from those in the cited work. We first construct the category of assemblies for a suitable restriction functor, and from this category of assemblies construct a tripos. In the cited work, realizability triposes are constructed first, and used to define the category of assemblies. Second, our constructions are more generally applicable. The setting of the cited work is roughly equivalent to fixing the codomain of our restriction functor as the category of sets and partial functions. Specifically, **Proposistion 8.1, 8.2, 8.4, 8.5, 8.8, 8.9**, **Lemma 8.6, 8.10**, and **Theorem 8.6, 8.7** are novel.

Finally, the definition of the partial topos of a tripos (**Definition 7.6**), as well as **Proposition 7.7** and the chain of reasoning following it that culminates in the observation that the total maps of the partial topos of a tripos are exactly the topos of that tripos, is novel. It is not, however, terribly original, since the modification to the tripos-to-topos construction one makes to obtain a partial topos is trivial, and the proof that the result is in fact a partial topos is straightforward.

One way to understand the contribution of this thesis is in terms of the functor $F : \mathbb{A} \to \mathbb{X}$ from which we construct the category of assemblies. The original realizability topos construction [19] can be recovered by insisting that F is the embedding of a set-based partial combinatory algebra into partial functions on sets, later work on generalized realizability [3] allows the domain of the functor to vary, and the present work allows the codomain to vary as well. Additionally, our approach leads to a restriction category of assemblies (as opposed to one consisting of only total maps), which allows us to define the associated realizability fibration with relatively little effort.

2 Restriction Categories

Restriction categories [10] [11] [12] are categories whose maps can be thought of as *partial*, where this partiality is captured by an idempotent on the domain of the map. Formally:

Definition 2.1. A *restriction category* is a category $\mathbb{X}$ such that for every map $f : A \to B$ in $\mathbb{X}$ there is a map $\overline{f} : A \to A$ in $\mathbb{X}$ satisfying

[R.1] $\overline{f} f = f$

[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$

[R.3] $\overline{\overline{f} g} = \overline{f}\,\overline{g}$

[R.4] $f\overline{g} = \overline{fg} f$

We sometimes call $\overline{f}$ the *domain of definition* of f, the idea being that every map in a restriction category is a partial map, with $\overline{f}$ defined precisely where f is defined. We also call maps $e = \overline{f}$ for some f *restriction idempotents*, and denote the collection of all restriction idempotents $e : A \to A$ on A by $\mathcal{O}(A)$ for each A in $\mathbb{X}$.

We establish some elementary properties of the restriction structure:

Lemma 2.2. *In a restriction category:*

(i) $\overline{f}\,\overline{f} = \overline{f}$ *(justifying the term "restriction* idempotent*")*

(ii) $\overline{fg}\,\overline{f} = \overline{fg}$

(iii) $\overline{f\overline{g}} = \overline{fg}$

(iv) $\overline{\overline{f}} = \overline{f}$

Proof. (i) $\overline{f}\,\overline{f} = \overline{\overline{f} f} = \overline{f}$

(ii) $\overline{fg}\,\overline{f} = \overline{f}\,\overline{fg} = \overline{\overline{f} fg} = \overline{fg}$

(iii) $\overline{f\overline{g}} = \overline{\overline{fg}f} = \overline{fg}\,\overline{f} = \overline{fg}$

(iv) $\overline{\overline{f}} = \overline{1\overline{f}} = \overline{1f} = \overline{f}$

□

We also introduce restriction functors, which are functors between restriction categories that preserve the restriction structure.

Definition 2.3. If $\mathbb{C}$ and $\mathbb{X}$ are restriction categories, we say that a functor $F : \mathbb{C} \to \mathbb{X}$ is a *restriction functor* in case it preserves the domain of definition in the following sense

$$\overline{F(f)} = F(\overline{f})$$

When working with partial maps, we may wish to say of two parallel maps $f, g : A \to B$ that g is an *extension* of the f in the sense that where both maps are defined they agree, and that the domain of definition of f is subsumed by the domain of definition of g. Formally, in any restriction category we define the *extension ordering* on parallel maps by

Definition 2.4. Given $f, g : A \to B$ in a restriction category $\mathbb{C}$, we say that g *extends* f, written $f \leq g$, in case $\overline{f}g = f$. That is,

$$\frac{f \leq g}{\overline{f}g = f}$$

Proposition 2.5. *The extension ordering on maps in a restriction category gives a partial order. That is, $\leq$ is reflexive, transitive, and antisymmetric.*

Proof. For reflexivity, immediately we have $\overline{f}f = f$, so $f \leq f$. For transitivity, suppose we have $f \leq g$ and $g \leq h$. Then $\overline{f}h = \overline{\overline{f}g}h = \overline{f}\,\overline{g}h = \overline{f}g = f$ as required. Finally, if $f \leq g$ and $g \leq f$ then $f = \overline{f}g = \overline{f}\,\overline{g}f = \overline{g}\,\overline{f}f = \overline{g}f = g$, which is antisymmetry. □

This ordering is preserved by restriction functors. If $f \leq g$ in the domain then we have

$$\overline{F(f)}F(g) = F(\overline{f})F(g) = F(\overline{f}g) = F(f)$$

and so $F(f) \leq F(g)$ in the codomain.

We make heavy use of the extension ordering in what follows. Note also that as an immediate consequence we obtain:

Proposition 2.6. *For any object A of a restriction category $\mathbb{C}$, the collection $\mathcal{O}(A)$ of restriction idempotents on A is a meet semilattice with respect to the extension ordering.*

Proof. The top element is 1_A, as for any $e \in \mathcal{O}(A)$ we have $\overline{e}1_A = e$ and thus $e \leq 1_A$. The meet of two idempotents $e, h \in \mathcal{O}(A)$ is the composite eh. Immediately $\overline{eh}e = eh$ and $\overline{eh}h = eh$, so $eh \leq e$ and $eh \leq h$. If for $k \in \mathcal{O}(A)$ we have $k \leq e$ and $k \leq h$, then $\overline{k}eh = kh = \overline{k}h = k$, so $k \leq eh$, and eh is indeed the meet with respect to the extension ordering. □

We also observe that if $f \leq f'$ and $g \leq g'$ then

$$\begin{aligned} \overline{fg}f'g' &= \overline{\overline{f}\, f'\overline{g}\, g'}\, f'g' = \overline{\overline{f}\, f'\overline{g}\, g'}\, \overline{f}\, f'g' = \overline{f\overline{g}\, g'}\, fg' \\ &= \overline{\overline{fg}\, fg'}\, \overline{fg}\, fg' = \overline{fg}\, fg' = f\overline{g}\, g' = fg \end{aligned}$$

which gives $fg \leq f'g'$, meaning that every restriction category is preorder enriched.

We say that a map $f : A \to B$ is *total* in case $\overline{f} = 1_A$. Thus, total maps are defined on the entire domain. Note that if $f : A \to B$ and $g : B \to C$ are total, then

$$\overline{fg} = \overline{f\overline{g}} = \overline{f} = 1_A$$

so their composite is also total. Identity maps are necessarily total as

$$\overline{1_A} = \overline{1_A} 1_A = 1_A$$

Thus, the total maps of a restriction category $\mathbb{C}$ form a subcategory, which we write $\mathsf{total}(\mathbb{C})$.

There are other ways to get at the idea of a category of partial maps. Here we introduce a common one – indeed this is what many authors mean by a "category of partial maps" – and relate it to restriction categories. This will be useful when we talk about partial toposes, which are defined in these terms in [13].

Definition 2.7. A *stable system of monics* in a category $\mathbb{X}$ is a collection $\mathcal{M}$ of maps in $\mathbb{X}$ that is closed to composition, contains all isomorphisms, and has the property that if $m : A \to B$ is a map in $\mathcal{M}$ and $f : C \to B$ is any map of $\mathbb{X}$, the pullback

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "m'"'] & A \arrow[d, "m"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

exists in $\mathbb{X}$, and further m' is in $\mathcal{M}$.

For example, the collection of all isomorphisms in any category is a stable system of monics, and if the category has pullbacks then so is the collection of monics.

Given a category with a stable system $\mathcal{M}$ of monics, we can define a category of partial maps, where the domain of each map is captured by one of the maps in $\mathcal{M}$.

Definition 2.8. If $\mathbb{X}$ is a restriction category and $\mathcal{M}$ is a stable system of monics in $\mathbb{X}$, we define a restriction category $\mathsf{Par}(\mathbb{X}, \mathcal{M})$ by

objects are objects of $\mathbb{X}$.

maps $(m, f) : A \to B$ are given by equivalence classes of spans

$$\begin{tikzcd}
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

where m is in $\mathcal{M}$, and where two maps $(m, f) : A \to B$ and $(m', f') : A \to B$ are equivalent in case there exists an isomorphism α making

$$\begin{tikzcd}
 & A' \arrow[r, "\alpha"] \arrow[dl, "m"'] \arrow[drr, "f"] & A'' \arrow[dll, "m'"'] \arrow[dr, "f'"] & \\
A & & & B
\end{tikzcd}$$

commute.

composition is by pullback. That is, the composite of $(m, f) : A \to B$ and $(m', f') : B \to C$ is given by $(m''m, f''f') : A \to C$, as in

$$\begin{tikzcd}
 & & A'' \arrow[dl, "m''"'] \arrow[dr, "f''"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & & B' \arrow[dl, "m'"'] \arrow[dr, "f'"] & \\
A & & B & & C
\end{tikzcd}$$

where

$$\begin{tikzcd}
A'' \arrow[r, "f''"] \arrow[d, "m''"'] & B' \arrow[d, "m'"] \\
A' \arrow[r, "f"'] & B
\end{tikzcd}$$

is the pullback of m' along f.

identities The identity on an object A is $(1_A, 1_A) : A \to A$.

restriction The domain of definition is given by $\overline{(m, f)} = (m, m)$.

For the proof that this is a well-defined restriction category, we refer the reader to [10] or [5].

Having shown that a stable system of monics gives rise to a restriction category, a natural question is whether or not every restriction category arises this way, and if not which ones do. The answer to this question is given in terms of *split restriction categories*, which we introduce now:

Definition 2.9. An idempotent $e : X \to X$ in a category is called *split* in case there are maps $s : X' \to X$ and $r : X \to X'$ for some object X' such that $sr = 1_{X'}$ and $rs = e$. Notice that if e splits as $e = rs$, then s is necessarily a section, and r is correspondingly a retraction. We often refer to the components of a split idempotent as "the section" or "the retraction" accordingly.

Definition 2.10. A *split restriction category* is a restriction category in which all restriction idempotents split.

We give the sections of split restriction idempotents a name:

Definition 2.11. A *restriction monic* in a restriction category $\mathbb{X}$ is a map $m : X \to Y$ that is the section in the splitting of some restriction idempotent $e \in \mathcal{O}(Y)$.

We then observe that the restriction monics of a split restriction category give a stable system of monics in its total map category. In fact, the partial map category formed by these maps is isomorphic to the original restriction category.

Proposition 2.12. *If $\mathbb{X}$ is a split restriction category then the collection of all restriction monics in $\mathbb{X}$, which we will write $\mathcal{M}_{\mathbb{X}}$, forms a stable system of monics in* $\mathsf{total}(\mathbb{X})$*. Further, there is an isomorphism*

$$\mathbb{X} \simeq \mathsf{Par}(\mathsf{total}(\mathbb{X}), \mathcal{M}_{\mathbb{X}})$$

We omit the proof, which can be found in [5] and [10].

Thus, every split restriction category is isomorphic to a partial map category. Next, we consider the relationship between split restriction categories and restriction categories in general. This relationship is captured by the *idempotent splitting* construction.

Definition 2.13. If $\mathbb{X}$ is a category, and E is a collection of idempotent maps in $\mathbb{X}$, define the category $\mathsf{split}_E(\mathbb{X})$ as follows:

objects idempotents in E.

maps between objects $e : X \to X$ and $e' : X' \to X'$ of $\mathsf{split}_E(\mathbb{X})$ are maps $f : X \to X'$ of $\mathbb{X}$ such that $efe' = f$.

composition is as in $\mathbb{X}$. That is, if $f : e \to e'$ and $g : e' \to e''$, then in $\mathbb{X}$ we have $efge'' = efe'e'ge'' = fg$, so $fg : e \to e''$ in $\mathsf{split}_E(\mathbb{X})$.

The **identity map** on e is given by e itself. This is well-defined as clearly $eee = e$, and if $f = efe'$, then $eefe' = efe' = f$ and $efe'e' = efe' = f$, making e the identity.

This construction allows us to formally split a collection of idempotents in the following sense:

Proposition 2.14. *If $\mathbb{X}$ is a restriction category and E includes the identity maps of $\mathbb{X}$, then $\mathsf{split}_E(\mathbb{X})$ is a restriction category, and there is an embedding*

$$K_E : \mathbb{X} \to \mathsf{split}_E(\mathbb{X})$$

Further, K_E is a restriction functor, and for every e in E, $K_E(e)$ is split.

Proof. We begin by showing that $\mathsf{split}_E(\mathbb{X})$ is a restriction category. Define the domain of defintion $\overline{f} : e \to e$ of a map $f : e \to e'$ to be the map given by $e\overline{f}$ in $\mathbb{X}$. This is well-defined, since $ee\overline{f}e = e\overline{f}e = \overline{ef}ee = \overline{ef}e = e\overline{f}$. We show that the restriction category axioms hold:

[R.1] $e\overline{f}f = ef = eefe' = f$

[R.2] $e\overline{f}e\overline{g} = \overline{ef}\,\overline{eg}e = \overline{eg}\,\overline{ef}e = e\overline{g}e\overline{f}$

[R.3] $\overline{ee\overline{f}g} = \overline{ee\overline{f}ge} = \overline{\overline{ef}ege} = \overline{ef}\,\overline{ege} = e\overline{f}e\overline{g}$

[R.4] $fe'\overline{g} = efe'\overline{g} = ef\overline{g} = e\overline{fg}f$

as required. The embedding $K_E : \mathbb{X} \to \mathsf{split}_E(\mathbb{X})$ is the functor mapping $f : X \to Y$ in $\mathbb{X}$ to $f : 1_X \to 1_Y$ in $\mathsf{split}_E(\mathbb{X})$, and if $e : X \to X$ is a member of E, then $K_E(e) : 1_X \to 1_X$ splits into $e : e \to 1_X$ and $e : 1_X \to e$. $\square$

This means that in the case where E is precisely the collection of restriction idempotents in $\mathbb{X}$, $\mathsf{split}_E(\mathbb{X})$ is a split restriction category, into which $\mathbb{X}$ embeds. We also have immediately that if E is the collection of restriction idempotents then there is an embedding

$$\mathbb{X} \longrightarrow \mathsf{Par}(\mathsf{total}(\mathsf{split}_E(\mathbb{X})), \mathcal{M}_{\mathsf{split}_E(\mathbb{X})})$$

The relationship between restriction categories and categories $\mathsf{Par}(\mathbb{X}, \mathcal{M})$ is now clear. The latter are restriction categories, and every restriction category embeds into one of them via the category obtained by splitting the restriction idempotents.

2.1 Products in Restriction Categories

We motivate the definition of products in a restriction category by considering the usual situation in which a category $\mathbb{C}$ has finite products.

Recall that a *product* of two objects A and B of $\mathbb{C}$ is a diagram

$$A \xleftarrow[\pi_0]{} A \times B \xrightarrow[\pi_1]{} B$$

in $\mathbb{C}$ with the property that for any pair of maps $f : C \to A$, $g : C \to B$ there is a unique map $\langle f, g \rangle : C \to A \times B$ for which

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd}$$

commutes. This definition still makes sense if $\mathbb{C}$ is a restriction category, since every restriction category is a category in the usual sense. Supposing $\mathbb{C}$ is a restriction category with this sort of binary products, consider the following commutative diagram

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "\langle f,1\rangle"] \arrow[dr, equal] & \\ A & A \times C \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & C \end{tikzcd}$$

We want our partial product to correspond to a product diagram in $\mathsf{total}(\mathbb{C})$, and for this to be the case the projections π_0 and π_1 must be total maps. If, however, we assume π_0 and π_1 are total, we have

$$\overline{f} = \overline{\langle f,1\rangle\pi_0} = \overline{\langle f,1\rangle\overline{\pi_0}} = \overline{\langle f,1\rangle\overline{\pi_1}} = \overline{1} = 1$$

meaning that every map in $\mathbb{C}$ is total! In this way, the usual definition of products collapses the restriction structure in a restriction category. However, we would very much like to have a notion of products in our restriction categories, and to keep our restriction structure intact. This is possible, and we will use the following definition of products in restriction categories in the sequel:

Definition 2.15. A *restriction product* of two objects A, B in a restriction category $\mathbb{C}$ is an object $A \times B$ of $\mathbb{C}$, with total projection maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that for any maps $f : C \to A$, $g : C \to B$ there is a unique map $\langle f, g\rangle : C \to A \times B$ such that

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] \arrow[dl, phantom, "\geq", shift left=4] \arrow[dr, phantom, "\leq", shift right=4] & \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd}$$

commutes, with $\langle f, g\rangle\pi_0 = \overline{g}f$ and $\langle f, g\rangle\pi_1 = \overline{f}g$.

An immediate consequence of this definition is

$$\overline{\langle f,g\rangle} = \overline{\langle f,g\rangle\pi_0} = \overline{\overline{g}f} = \overline{g}\,\overline{f}$$

which means that our earlier problem with $\langle f, 1\rangle$ does not occur! For more details on limits in restriction categories, and why *this* definition of products in a restriction category is the correct one, the interested reader is encouraged to consult [10] [12].

Similarly, we require a special notion of terminal object to go along with our special notion of products for a restriction category. Intuitively, this makes sense, as while there is certainly only one *total* map into the terminal object from any other in say, **Ptl**, there may be many partial maps. The definition is

Definition 2.16. A *restriction terminal object* in a restriction category $\mathbb{C}$ is an object 1 with the property that for any object A of $\mathbb{C}$, there is a unique total map $!_A : A \to 1$ such that for any map $f : A \to B$ of $\mathbb{C}$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "!_A"] & \\ B \arrow[r, "!_B"'] \arrow[ur, phantom, "\leq"] & 1 \end{tikzcd}$$

Important to note is that the above diagram expresses an *inequality*. That is, it does not commute in the usual sense, expressing that $f!_B \leq !_A$, not that the two maps are equal. The sequel will contain many such diagrams, which can be identified by the presence of $\leq$.

Finally, we arrive at the definition of products in a restriction category [10] that we will be using:

Definition 2.17. A restriction category $\mathbb{C}$ *has finite restriction products* in case there is a distinguished restriction terminal object 1 and for every two objects A, B of $\mathbb{C}$ there is a distinguished restriction product $A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1}$ of A and B in $\mathbb{C}$. In this case we call $\mathbb{C}$ a *cartesian restriction category*

Note that we insist on distinguished, or "chosen" products in our cartesian restriction categories. This simplifies things significantly in many of our calculations.

Lemma 2.18. *In a cartesian restriction category:*

(i) $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$

(ii) If $e = \overline{e}$ *then* $e\langle f, g\rangle = \langle ef, g\rangle = \langle f, eg\rangle$

Proof. (i) $\overline{\langle f, g\rangle} = \overline{\langle f, g\rangle \pi_0} = \overline{\overline{g} f} = \overline{f}\,\overline{g}$

(ii) First,

$$\overline{e\langle f, g\rangle}\langle ef, g\rangle = \langle \overline{e\langle f, g\rangle} f, \overline{e\langle f, g\rangle} g\rangle = \overline{e\langle f, g\rangle}\langle f, g\rangle = e\langle f, g\rangle$$

gives $e\langle f, g\rangle \leq \langle ef, g\rangle$, and similarly $e\langle f, g\rangle \leq \langle f, eg\rangle$. Next,

$$\overline{\langle ef, g\rangle} e\langle f, g\rangle = \langle \overline{ef}\,\overline{g} ef, \overline{ef}\,\overline{g} eg\rangle = \overline{ef}\,\overline{g}\langle ef, g\rangle = \langle ef, g\rangle$$

gives $\langle ef, g\rangle \leq e\langle f, g\rangle$. Similarly, $\langle f, eg\rangle \leq e\langle f, g\rangle$.

□

For example, **Ptl** – the category of sets and partial functions – has finite restriction products. The restriction terminal object is given by any one element set $\{*\}$, where the unique total map from a set A to $\{*\}$ is the partial function that sends each $a \in A$ to $*$. If A and B are sets, the restriction product is the cartesian product

$$A \times B := \{(a, b) \mid a \in A, b \in B\}$$

with total projection maps defined by

$$\pi_0 : A \times B \to A \quad \pi_0((a, b)) := a$$
$$\pi_1 : A \times B \to B \quad \pi_1((a, b)) := b$$

If $f : C \to A$ and $g : C \to B$ are arrows in **Ptl**, then the mediating map is

$$\langle f, g\rangle : C \to A \times B \quad \langle f, g\rangle(c) = (f(c), g(c))$$

That the required identities hold is straightforward to verify, once one realizes that $(f(x), g(x))$ is defined only when both $f(x)$ and $g(x)$ are defined.

Restriction products lie over products in the usual sense in the total map category:

Proposition 2.19. *If $\mathbb{C}$ is a cartesian restriction category then* $\mathsf{total}(\mathbb{C})$ *has finite products.*

Proof. The restriction terminal object becomes a terminal object in $\mathsf{total}(\mathbb{C})$ since for any total map $f : A \to 1$, we have $f \leq !_A$, meaning $!_A = \overline{f}!_A = f$, and so there is exactly one map, $!_A$ from A into 1 in $\mathsf{total}(\mathbb{C})$.

For restriction products, if

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

is a restriction product in $\mathbb{C}$, then since π_0 and π_1 are total it is also a diagram in $\mathsf{total}(\mathbb{C})$. In fact, it is a product diagram as follows: If $f : C \to A$, $g : C \to B$ are maps in $\mathsf{total}(\mathbb{C})$ then since they are also maps of $\mathbb{C}$ and the diagram is a restriction product in $\mathbb{C}$ we have a mediating map $\langle f, g\rangle$, which is also in $\mathsf{total}(\mathbb{C})$ since $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$, making

$$\begin{tikzcd}
 & C \arrow[dl, "g"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

commute in $\mathsf{total}(\mathbb{C})$. The uniqueness condition for the restriction product gives the uniqueness condition here, and we are done: $\mathsf{total}(\mathbb{C})$ has finite products. $\square$

This pattern of partial structure lying over more familiar structure in the total map category will be a major theme in the latter part of the sequel.

Often we ask that some functor preserves certain limit diagrams, and of course there is a version of this for restriction functors, and in particular for restriction products:

Definition 2.20. If $\mathbb{C}$ and $\mathbb{X}$ are cartesian restriction categories, we define a *cartesian restriction functor* between them to be a restriction functor $F : \mathbb{C} \to \mathbb{X}$ that preserves the distinguished products. That is, $F(A \times B) = F(A) \times F(B)$, $F(\pi_0) = \pi_0$, $F(\pi_1) = \pi_1$, and the diagram

$$F(A) \xleftarrow[\pi_0]{} F(A) \times F(B) \xrightarrow[\pi_1]{} F(B)$$

is the distinguished restriction product in $\mathbb{X}$.

Notice that we then have $F(\Delta_X) = \Delta_{F(X)}$, which will be useful.

2.2 Meets and Discrete Cartesian Restriction Categories

We know that for any restriction category $\mathbb{X}$, each homset is a partial order with respect to the extension ordering. Assuming the existence of a meet in each homset is very powerful, and we explore this now:

Definition 2.21. A *meet restriction category* is a restriction category $\mathbb{X}$ such that for every pair of parallel maps $f, g : A \to B$ in $\mathbb{X}$, there is a map $f \cap g$ in $\mathbb{X}$, called the *meet* of f and g, satisfying:

(i) $f \cap f = f$

(ii) $f \cap g \leq f$ and $f \cap g \leq g$

(iii) $h(f \cap g) \leq (hf \cap hg)$

The definition implies various useful properties of the meet:

Lemma 2.22. *In a meet restriction category:*

(i) if $h \leq f$ and $h \leq g$ then $h \leq (f \cap g)$ (justifying the term "meet")

(ii) $f \cap g = g \cap f$

(iii) if $f \leq f'$ then $(f \cap g) \leq (f' \cap g)$

(iv) $\overline{h}(f \cap g) = (\overline{h}f \cap g) = (f \cap \overline{h}g)$

(v) $(f \cap g)\overline{h} = (f\overline{h} \cap g\overline{h})$

(vi) $\overline{f \cap g}f = \overline{f \cap g}g$

(vii) $(f \cap g\overline{h}) = (f\overline{h} \cap g)$

Proof. (i) $\overline{f}(f \cap g) = \overline{h}f \cap \overline{h}g = h \cap h = h$, and so $h \leq f \cap g$.

(ii) Immediate from (i), $f \cap g \leq g$ and $f \cap g \leq g$ give $(f \cap g) \leq (g \cap f)$ and vice-versa.

(iii) $f \cap g \leq f \leq f'$ and $f \cap g \leq g$ gives $f \cap g \leq f' \cap g$.

(iv) $(\overline{h}f \cap g) \leq g$ and $(\overline{h}f \cap g) \leq \overline{h}f$ give

$$\overline{(\overline{h}f \cap g)}\,\overline{h}g = \overline{\overline{(\overline{h}f \cap g)}\,\overline{h}f}\,\overline{h}g = \overline{\overline{(\overline{h}f \cap g)}\,\overline{h}\,\overline{h}f}\,g$$
$$= \overline{\overline{(\overline{h}f \cap g)}\,\overline{h}f}\,g = \overline{(\overline{h}f \cap g)}\,g = \overline{h}f \cap g$$

so $(\overline{h}f \cap g) \leq \overline{h}g$, meaning $(\overline{h}f \cap g) \leq (\overline{h}f \cap \overline{h}g) = \overline{h}(f \cap g)$. $\overline{h}g \leq g$, so we also have $\overline{h}(f \cap g) = (\overline{h}f \cap \overline{h}g) \leq (\overline{h}f \cap g)$. Thus, $(\overline{h}f \cap h) = \overline{h}(f \cap g)$. The symmetric case is similar.

(v) $(f\overline{h} \cap g\overline{h}) \leq (f \cap g) \leq (f \cap g)\overline{h}$, and

$$(f \cap g)\overline{h} = \overline{(f \cap g)h}(f \cap g) = \overline{(f \cap g)h}f \cap \overline{(f \cap g)h}g$$
$$\leq \overline{fh}f \cap \overline{gh}g = f\overline{h} \cap g\overline{h}$$

(vi) $f \cap g \leq f$ and $f \cap g \leq g$ give $\overline{f \cap g}f = f \cap g = \overline{f \cap g}g$.

(vii)

$$(f \cap g\overline{h}) = (f \cap \overline{gh}g) = \overline{gh}(f \cap g) = \overline{(f \cap g)}\,\overline{gh}(f \cap g)$$
$$= \overline{\overline{(f \cap g)}gh}(f \cap g) = \overline{\overline{(f \cap g)}fh}(f \cap g) = \overline{fh}(f \cap g)$$

$$= (f\overline{h} \cap g)$$

$\square$

There is another way to state that a cartesian restriction category has meets, in terms of partial inverses.

Definition 2.23. A map $f : A \to B$ of a restriction category $\mathbb{X}$ is said to have a *partial inverse* if there is a map $f^{(-1)} : B \to A$ of $\mathbb{X}$ such that $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$.

We note that the partial inverse of a map f is unique when it exists. If some other map g has $fg = \overline{f}$ and $gf = \overline{g}$, then

$$\overline{g}f^{(-1)} = gff^{(-1)} = g\overline{f} = gfg = \overline{g}g = g$$

which means $g \leq f^{(-1)}$. Similarly $f^{(-1)} \leq g$, and so $f^{(-1)} = g$.

A map with a partial inverse is called a *partial isomorphism*. We are particularly interested in the case where the diagonal map is a partial isomorphism:

Definition 2.24. A *discrete cartesian restriction category* is a cartesian restriction category in which $\Delta : A \to A \times A$ (defined by $\Delta = \langle 1_A, 1_A \rangle$) has a partial inverse for every object A of $\mathbb{X}$. That is, there exists a map $\Delta^{(-1)} : A \times A \to A$ such that $\Delta\Delta^{(-1)} = 1$, and $\Delta^{(-1)}\Delta = \overline{\Delta^{(-1)}}$.

This captures precisely when a cartesian restriction category has meets!

Proposition 2.25. *A cartesian restriction category $\mathbb{X}$ has meets if and only if it is discrete.*

Proof. Let $\mathbb{X}$ be a cartesian restriction category. If $\mathbb{X}$ has meets, we define $\Delta^{(-1)} :$ $X \times X \to X$ by

$$\Delta^{(-1)} := \pi_0 \cap \pi_1$$

We show that this is partial inverse to Δ as follows:

$$\Delta\Delta^{(-1)} = \langle 1,1\rangle(\pi_0 \cap \pi_1) = (\langle 1,1\rangle\pi_0 \cap \langle 1,1\rangle\pi_1)$$
$$= 1 \cap 1 = 1 = \overline{\Delta}$$

and

$$\Delta^{(-1)}\Delta = (\pi_0 \cap \pi_1)\langle 1,1\rangle = \langle(\pi_0 \cap \pi_1),(\pi_0 \cap \pi_1)\rangle$$
$$= \langle\overline{(\pi_0 \cap \pi_1)}\pi_0 \cap \overline{(\pi_0 \cap \pi_1)}\pi_1\rangle = \overline{(\pi_0 \cap \pi_1)}\langle\pi_0,\pi_1\rangle$$
$$= \overline{(\pi_0 \cap \pi_1)} = \overline{\Delta^{(-1)}}$$

as required, meaning that $\mathbb{X}$ is discrete.

For the converse, suppose that $\mathbb{X}$ is discrete, and defined $f \cap g$ by

$$f \cap g := \langle f,g\rangle\Delta^{(-1)}$$

We show that this is the meet. For $f \cap f = f$, we have

$$f \cap f = \langle f,f\rangle\Delta^{(-1)} = f\Delta\Delta^{(-1)} = f$$

Next, for $f \cap g \leq f$:

$$\overline{(f \cap g)}\, f = \overline{\langle f,g\rangle\Delta^{(-1)}}\, f = \overline{\langle f,g\rangle\overline{\Delta^{(-1)}}}\, f$$

$$
\begin{aligned}
&= \overline{\overline{\langle f, g\rangle\Delta^{(-1)}}\langle f, g\rangle}\, f = \overline{\langle f, g\rangle\Delta^{(-1)}}\ \overline{\langle f, g\rangle\pi_0}\, f \\
&= \overline{\langle f, g\rangle\Delta^{(-1)}}\ \overline{g} f = \overline{\langle f, g, \rangle\Delta^{(-1)}}\, \langle f, g\rangle\pi_0 \\
&= \langle f, g\rangle\overline{\Delta^{(-1)}}\pi_0 = \langle f, g\rangle\Delta^{(-1)}\Delta\pi_0 \\
&= \langle f, g\rangle\Delta^{(-1)} = f \cap g
\end{aligned}
$$

and similarly we have $f \cap g \leq g$. Finally, $h(f \cap g) = hf \cap hg$ since

$$
h(f \cap g) = h\langle f, g\rangle\Delta^{(-1)} = \langle hf, hg\rangle\Delta^{(-1)} = hf \cap hg
$$

so $\mathbb{X}$ has meets. $\square$

2.3 Latent Pullbacks

A restriction category is said to have *latent limits* of a diagram in case the idempotent splitting of that category has limits of that diagram [23]. In this thesis we will need to use latent pullbacks, which we introduce now:

Definition 2.26. A *latent pullback* in a restriction category $\mathbb{C}$ is a commuting square

$$
\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
$$

with $\overline{h} = \overline{hf} = \overline{kg} = \overline{k}$ such that for any two maps $v : X \to A, w : X \to B$ such that $vf = wg$, there is a unique map $\alpha : X \to D$ satisfying

(i) $\alpha h \leq v$, $\alpha k \leq w$

(ii) $\overline{\alpha} = \overline{vf} = \overline{wg}$

(iii) $\alpha = \alpha\overline{k}$

This definition may seem contrived, but is justified by the more general existence of latent limits [23]. Note that latent limits are unique up to partial isomorphism (and so, up to isomorphism in the splitting). Now, we have

Lemma 2.27. *Every discrete cartesian restriction category has canonical latent pullbacks of arbitrary maps.*

Proof. We show that for any f, g the square

$$\begin{tikzcd}
A \times B \arrow[r, "\overline{\pi_0 f \cap \pi_1 g}\pi_0"] \arrow[d, "\overline{\pi_0 f \cap \pi_1 g}\pi_1"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

is a latent pullback. Obviously the square commutes, and we have

$$\overline{\overline{\pi_0 f \cap \pi_1 g}\pi_0 f} = \overline{\overline{\pi_0 f \cap \pi_1 g}\pi_1 g} = \overline{\pi_0 f \cap \pi_1 g}$$
$$= \overline{\overline{\pi_0 f \cap \pi_1 g}\pi_0} = \overline{\overline{\pi_0 f \cap \pi_1 g}\pi_1}$$

so the square also commutes in the splitting, as required. Now, suppose $v : X \to A$ and $w : X \to B$ are maps with $vf = wg$. Define the mediating map to be

$$\alpha := \overline{vf \cap wg}\langle v, w\rangle = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g}$$

we then have

(i)

$$\alpha\overline{\pi_0 f \cap \pi_1 g}\pi_0 = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g}\;\overline{\pi_0 f \cap \pi_1 g}\pi_0 = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g}\pi_0$$
$$= \overline{vf \cap wg}\langle v, w\rangle\pi_0 = \overline{w}\,\overline{vf \cap wg}\,v = \overline{vf \cap wg}\,v \leq v$$

and

$$\alpha\overline{\pi_0 f \cap \pi_1 g}\pi_1 = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g}\;\overline{\pi_0 f \cap \pi_1 g}\pi_1 = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g}\pi_1$$
$$= \overline{vf \cap wg}\langle v, w\rangle\pi_1 = \overline{v}\,\overline{vf \cap wg}\,w = \overline{vf \cap wg}\,w \leq w$$

(ii) $\overline{vf} = \overline{vf \cap vfvf \cap wg} = \overline{wg \cap wg} = \overline{wg}$ gives

$$\overline{\alpha} = \overline{\overline{vf \cap wg}\langle v, w\rangle} = \overline{v}\,\overline{w}\,\overline{vf \cap wg} = \overline{\overline{v}vf \cap \overline{w}wg} = \overline{vf \cap wg}$$

(iii)

$$\alpha\overline{\overline{\pi_0 f \cap \pi_1 g}\pi_1} = \langle v, w\rangle\overline{\pi_0 f \cap \pi_1 g\pi_0 f \cap \pi_1 g} = \alpha$$

as required. For uniqueness, suppose $\beta : X \to A \times B$ is another map satisfying the definition. Then

$$\begin{aligned}
\alpha &= \overline{vf \cap wg}\langle v, w\rangle = \overline{vf}\langle v, w\rangle = \overline{\beta}\langle v, w\rangle = \langle\overline{\beta}v, \overline{\beta}w\rangle \\
&= \langle\overline{\beta\overline{\pi_0 f \cap \pi_1 g}\pi_1}\,v, \overline{\beta\overline{\pi_0 f \cap \pi_1 g}\pi_1}\,w\rangle = \langle\overline{\beta\overline{\pi_0 f \cap \pi_1 g}\pi_0}\,v, \overline{\beta\overline{\pi_0 f \cap \pi_1 g}\pi_1}\,w\rangle \\
&= \langle\beta\overline{\pi_0 f \cap \pi_1 g}\pi_0, \beta\overline{\pi_0 f \cap \pi_1 g}\pi_1\rangle = \beta\overline{\pi_0 f \cap \pi_1 g}\langle\pi_0, \pi_1\rangle \\
&= \beta\overline{\pi_0 f \cap \pi_1 g} = \beta
\end{aligned}$$

as required. □

This is analogous to the way that any category with products and equalizers must also have pullbacks. Restriction products play the role of products, meets the role of equalizers, and latent pullbacks the role of pullbacks.

2.4 Joins and Interleaving

In addition to meets, restriction categories are capable of having joins of certain parallel maps, or more generally, of having an interleaving of any two parallel maps.

Definition 2.28. We say that two parallel maps $f, g : A \to B$ in a restriction category $\mathbb{X}$ are *compatible*, written $f \smile g$, in case $\overline{f}g = \overline{g}f$.

Definition 2.29. A restriction category $\mathbb{X}$ is said to *have finite joins* in case

(a) For each pair A, B of objects of $\mathbb{X}$ there is a *zero map* $0_{AB} : A \to B$ such that for any $f : A \to B$, $0_{AB} \leq f$, and for $g : C \to D$, $f 0_{BC} g = 0_{AD}$. We omit the subscripts when they are clear from the context.

(b) For each pair $f, g : A \to B$ of maps of $\mathbb{X}$ such that $f \smile g$, there is a map $f \vee g : A \to B$ such that $f \vee g$ is the join of f and g (that is, $f \leq f \vee g$, $g \leq f \vee g$, and if $f \leq h$ and $g \leq h$, $f \vee g \leq h$), and $h(f \vee g) = hf \vee hg$.

Restriction categories with (finite) joins are called (finite) *join restriction categories.*

We collect some elementary properties of finite joins:

Lemma 2.30. *In a restriction category with finite joins:*

(i) $\overline{0_{AB}} = 0_{AA}$

(ii) $\overline{f \vee g} = \overline{f} \vee \overline{g}$

(iii) $(f \vee g)h = fh \vee gh$

Proof. (i) $\overline{0_{AB}} 0_{AA} = \overline{\overline{0_{AB}}\,\overline{0_{AA}}} = \overline{\overline{0_{AB}} 0_{AA} 0_{AB}} = \overline{0_{AB}}$ gives $\overline{0_{AB}} \leq 0_{AA}$, and we already have $0_{AA} \leq \overline{0_{AB}}$, so $\overline{0_{AB}} = 0_{AA}$.

(ii) Immediately $\overline{f} \vee \overline{g} \leq \overline{f \vee g}$. This also tells us $\overline{f} \vee \overline{g}$ is a restriction idempotent. We then have

$$
\begin{aligned}
&\overline{f \vee g}(\overline{f} \vee \overline{g}) = \overline{\overline{f \vee g}(\overline{f} \vee \overline{g})} \\
&= \overline{(\overline{f} \vee \overline{g})\overline{f \vee g}} = \overline{(\overline{f} \vee \overline{g})(f \vee g)} \\
&= \overline{(\overline{f} \vee \overline{g})f \vee (\overline{f} \vee \overline{g})g} = \overline{(\overline{f} \vee \overline{g})\overline{f}f \vee (\overline{f} \vee \overline{g})\overline{g}g} \\
&= \overline{(\overline{f}f \vee \overline{g}g)} = \overline{f \vee g}
\end{aligned}
$$

so $\overline{f \vee g} \leq \overline{f} \vee \overline{g}$, and therefore $\overline{f \vee g} = \overline{f} \vee \overline{g}$.

(iii) Immediately $fh \vee gh \leq (f \vee g)h$, and we have

$$
\begin{aligned}
(f \vee g)\overline{h} &= \overline{(f \vee g)h}(f \vee g) = \overline{(f \vee g)h}\, f \vee \overline{(f \vee g)h}\, g \\
&= \overline{(f \vee g)h}\,\overline{f}f \vee \overline{(f \vee g)h}\,\overline{g}g = \overline{\overline{f}(f \vee g)f}\, f \vee \overline{\overline{g}(f \vee g)h}\, g \\
&= \overline{(f \vee \overline{f}g)h}\, f \vee \overline{(\overline{g}f \vee g)h}\, g = \overline{(f \vee \overline{g}f)h}\, f \vee \overline{(\overline{f}g \vee g)h}\, g \\
&= \overline{fh}\, f \vee \overline{gh}\, g = f\overline{h} \vee g\overline{h}
\end{aligned}
$$

which gives

$$
\overline{fh \vee gh} = \overline{f\overline{h}} \vee \overline{g\overline{h}} = \overline{(f \vee g)\overline{h}} = \overline{(f \vee g)h}
$$

and so

$$
(fh \vee gh) = \overline{fh \vee gh}\,(f \vee g)h = \overline{(f \vee g)h}\,(f \vee g)h = (f \vee g)h
$$

$\square$

If a restriction category with joins also has meets, then the meet distributes over the join:

Lemma 2.31. *In a restriction category with meets and joins*

$$
h \cap (f \vee g) = (h \cap f) \vee (h \cap g)
$$

Proof.

$$
\begin{aligned}
h \cap (f \vee g) &= \overline{(f \vee g)}(h \cap (f \vee g)) \\
&= (\overline{f} \vee \overline{g})(h \cap (f \vee g)) = \overline{f}(h \cap (f \vee g)) \vee \overline{g}(h \cap (f \vee g)) \\
&= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g)) = (h \cap f) \vee (h \cap g)
\end{aligned}
$$

$\square$

In a cartesian restriction category with joins, the join interacts with the product functor in an interesting way:

Lemma 2.32. *In a cartesian restriction category $\mathbb{X}$ with finite joins, if $f, g : A \to B$ and $h, k : C \to D$ are pairs of parallel maps with $f \smile g$ and $h \smile k$, then $(f \times h) \smile (g \times k)$ and*

$$(f \vee g) \times (h \vee k) = (f \times h) \vee (g \times k)$$

Proof. For $(f \times h) \smile (g \times k)$ we have $\overline{(f \times h)}(g \times k) = \overline{f}g \times \overline{h}k = \overline{g}f \times \overline{k}h = \overline{(g \times k)}(f \times h)$ as required.

We show the main result by establishing both inequalities. For $(f \times h) \vee (g \times k) \leq (f \vee g) \times (h \vee k)$ it suffices to show both $(f \times h) \leq (f \vee g) \times (h \vee k)$ and $(g \times k) \leq (f \vee g) \times (h \vee k)$, but this is immediate.

For $(f \vee g) \times (h \vee k) \leq (f \times h) \vee (g \times k)$ we have

$$\begin{aligned}
&\overline{(f \vee g) \times (h \vee k)}(f \times h) \vee (g \times k) \\
&= ((\overline{f} \vee \overline{g}) \times (\overline{h} \vee \overline{k}))(f \times h) \vee ((\overline{f} \vee \overline{g}) \times (\overline{h} \vee \overline{k}))(g \times k) \\
&= ((\overline{f} \vee \overline{g})f \times (\overline{h} \vee \overline{k})h) \vee ((\overline{f} \vee \overline{g})g \times (\overline{h} \vee \overline{k})k) \\
&= ((f \vee \overline{g}f) \times (h \vee \overline{k}h)) \vee ((\overline{f}g \vee g) \times (\overline{h}k \vee k)) \\
&= (f \times h) \vee (g \times k)
\end{aligned}$$

and we are done. $\square$

We say that a restriction functor $F : \mathbb{X} \to \mathbb{Y}$ between join restriction categories $\mathbb{X}$ and $\mathbb{Y}$ *preserves joins* if $F(0) = 0$ and $F(f \vee g) = F(f) \vee F(g)$

In classical computability theory, the interleaving of two partial computable functions is an important construction. By alternating which of two functions is being computed until one of them terminates, we can construct a partial computable function that is defined on some input when at least one of the two original functions is. It is possible to express this essential property of the interleaving in any restriction category with meets and joins:

Definition 2.33. In a restriction category $\mathbb{X}$ with meets and joins, an *interleaving* of a pair of parallel maps $f, g : A \to B$ is a map $h : A \to B$ with $\overline{f} \leq \overline{h}$, $\overline{g} \leq \overline{h}$, and $h = (h \cap f) \vee (h \cap g)$.

If all pairs of parallel maps have an interleaving, then $\mathbb{X}$ is said to *have finite interleaving*.

While two maps need not be compatible for an interleaving of them to exist, an interleaving of two compatible maps is the join of those two maps.

Lemma 2.34. *In a restriction category with meets and joins, any interleaving of two compatible maps is the join of those maps.*

Proof. Suppose $f \smile g$ and that h is an interleaving of f and g. Right away we have $(h \cap f) \leq f \leq f \vee g$ and $(h \cap g) \leq g \leq f \vee g$, which gives $h = (h \cap f) \vee (h \cap g) \leq f \vee g$. For the reverse inequality, we have

$$\begin{aligned}
\overline{f}h &= \overline{f}((h \cap f) \vee (h \cap g)) \\
&= \overline{f}(h \cap (f \vee g)) = h \cap (f \vee \overline{f}g) \\
&= h \cap (f \vee \overline{g}f) = h \cap f \leq f
\end{aligned}$$

and then $f = \overline{f}f = \overline{f}\,\overline{h}f = \overline{\overline{f}h}\,f = \overline{f}h$, so $f \leq h$. Similarly, we have $g \leq h$, which gives $f \vee g \leq h$ as required.

$\square$

2.5 Existential Quantification

In categorical logic we are often concerned with existential quantification in a fibration, where it takes the form of a series of adjunctions between the fibers [18] [27]. In this section, we introduce existential quantification in a restriction category. These two notions are related, but not identical. Specifically, existential quantification in a restriction category is *intended* to correspond to the presence of existential quantification in $\mathcal{O}$, the domain latent fibration (see chapter 6). If a restriction category has existential quantification, then the domain fibration $\mathsf{total}(\mathcal{O})$ has existential quantification in the usual fibrational sense, but the converse does not hold. The presense of existential quantification in the fibration $\mathsf{total}(\mathcal{O})$ only deals with the total maps of the base category, where our notion of existential quantification in a restriction category encompasses all the maps of the base category. While there is almost certainly a definition of existential quantification for a latent fibration that ties this all together nicely, the details have not been worked out, and so we content ourselves with what follows.

Definition 2.35. A restriction category $\mathbb{X}$ has *existential quantification* in case for each map $f : X \to Y$ of $\mathbb{X}$, the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)/\overline{f}$ defined by $f^*(e) = \overline{fe}$ has a left Frobenius adjoint $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$, and the Beck-Chevalley condition holds. That is, there is a preorder morphism $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$ for which, for every $e \leq \overline{f}$, the two way inference

$$\frac{\exists_f(e) \leq e'}{\overline{\overline{e \leq \overline{fe'}}}}$$

holds, the Frobenius condition

$$\exists_f(e) \wedge e' \leq \exists_f(e \wedge \overline{fe'})$$

holds, and for any latent pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

the Beck-Chevalley condition holds in the sense that

$$\overline{g\exists_f(e)} = \exists_k(\overline{he})$$

We also make explicit an alternate formulation of adjoint functors. This will simplify some future calculations.

Lemma 2.36. *Suppose that for $f : X \to Y$ a map of a restriction category $\mathbb{X}$, $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$ is right adjoint to $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)/\overline{f}$. Then*

(i) $e \leq \overline{f\exists_f(e)}$ *for* $e \in \mathcal{O}(X)/\overline{f}$

(ii) $\exists_f(\overline{fe'}) \leq e'$ *for* $e' \in \mathcal{O}(Y)$

(iii) $\exists_f$ *is a preorder morphism*

Proof. (i) If $e \in \mathcal{O}(X)/\overline{f}$, then $\exists_f(e) \leq \exists_f(e)$, and so $e \leq \overline{f\exists_f(e)}$.

(ii) If $e' \in \mathcal{O}(Y)$, then $\overline{fe'} \leq \overline{fe'}$ and so $\exists_f(\overline{fe'}) \leq e'$.

(iii) Suppose $e_1, e_2 \in \mathcal{O}(X)/\overline{f}$ such that $e_1 \leq e_2$. Then $e_1 \leq e_2 \leq \overline{f\exists_f(e_2)}$, which gives $\exists_f(e_1) \leq \exists_f(e_2)$ as required.

□

Lemma 2.37. *If $f : X \to Y$ is a map in a restriction category $\mathbb{X}$ and $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$ is a preorder morphism satisfying*

(i) $\overline{f\exists_f(e)} \leq e$ *for* $e \in \mathcal{O}(X)/\overline{f}$

(ii) $e' \leq \exists_f(\overline{fe'})$ *for* $e' \in \mathcal{O}(Y)$

then $\exists_f$ is left adjoint to $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)/\overline{f}$.

Proof. Let $e \in \mathcal{O}(X)$, $e' \in \mathcal{O}(Y)$. Suppose $\exists_f(e) \leq e'$. Then since f^* is a preorder morphism, we have $\overline{f\exists_f(e)} \leq \overline{fe'}$, which gives

$$e \leq \overline{f\exists_f(e)} \leq \overline{fe'}$$

For the converse, suppose $e \leq \overline{fe'}$. Then since $\exists_f$ is a preorder morphism, we have $\exists_f(e) \leq \exists_f(\overline{fe'})$, which gives

$$\exists_f(e) \leq \exists_f(\overline{fe'}) \leq e'$$

as required. $\square$

While the above definition of existential quantification in a restriction category is motivated by existential quantification in a fibration, another way to understand existential quantification in a restriction category is in terms of *ranges* [22] [23]. In the same way that the domain of definition of a map in a restriction category tells us which part of the domain it is defined on, the range of a map tells us which part of the codomain is the image.

Definition 2.38. A *range category* is a restriction category $\mathbb{X}$ such that for every map $f : X \to Y$ of $\mathbb{X}$ there is a map $\widehat{f} : Y \to Y$ in $\mathbb{X}$, called the *range* of f, satisfying:

[RR.1] $\overline{\widehat{f}} = \widehat{f}$

[RR.2] $f\widehat{f} = f$

[RR.3] $\widehat{f\overline{g}} = \widehat{f}\,\overline{g}$

[RR.4] $\widehat{\widehat{f}g} = \widehat{fg}$

In order to relate this to existential quantification, we say that a map $f : X \to Y$ in a restriction category $\mathbb{X}$ is *open* in case the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)/\overline{f}$

has a left Frobenius adjoint $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$. It turns out that a map is open precisely when it has a range.

Lemma 2.39. *A map $f : X \to Y$ in a restriction category $\mathbb{X}$ is open if and only if it has a range.*

Proof. If $f : X \to Y$ has a range $\widehat{f}$, define $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$ by $\exists_f(e) := \widehat{ef}$. We show that this gives a left Frobenius adjoint to f^*. Suppose that for $e \leq \overline{f}$, $e' \in \mathcal{O}(Y)$, we have $\exists_f(e) \leq e'$. Then we have:

$$e\overline{fe'} = \overline{efe'} = \overline{ef\widehat{ef}e'} = \overline{ef\exists_f(e)e'}$$
$$= \overline{ef\exists_f(e)} = \overline{ef\widehat{ef}} = \overline{ef} = e\overline{f} = e$$

which means $e \leq \overline{fe'}$. For the other inequality, suppose that $e \leq \overline{fe'}$. Then we have:

$$\exists_f(e)e' = \widehat{ef}\,\overline{e'} = \widehat{ef\overline{e'}} = \widehat{e\overline{fe'}f}$$
$$= \widehat{ef} = \exists_f(e)$$

meaning $\exists_f(e) \leq e'$. Having shown that $\exists_f$ is left adjoint to f^*, only the Frobenius identity remains. Immediately:

$$(\exists_f(e) \wedge e') = \widehat{ef}\overline{e'} = \widehat{ef\overline{e'}} = \widehat{e\overline{fe'}f} = \exists_f(e \wedge \overline{fe'})$$

Thus f is open.

For the converse, if $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)/\overline{f}$ has a left Frobenius adjoint $\exists_f : \mathcal{O}(X)/\overline{f} \to \mathcal{O}(Y)$, define $\widehat{f} := \exists_f(\overline{f})$. We show that $\widehat{f}$ satisfies the range axioms:

[RR.1] $\exists_f(\overline{f})$ is a restriction idempotent, so $\overline{\widehat{f}} = \overline{\exists_f(\overline{f})} = \exists_f(\overline{f}) = \widehat{f}$.

[RR.2] We have $\overline{f} \le \overline{f\exists_f(\overline{f})}$, so

$$f\widehat{f} = f\exists_f(\overline{f}) = f\overline{\exists_f(\overline{f})} = \overline{f\exists_f(\overline{f})}\, f = \overline{f}\ \overline{f\exists_f(\overline{f})}\, f = \overline{f} f = f$$

[RR.3] Again we know $\overline{f} \le \overline{f\exists_f(\overline{f})}$, which means $\overline{f\exists_f(\overline{f})} = \overline{f}\,\overline{f\exists_f(\overline{f})} = \overline{f}$, but then:

$$\begin{array}{c} \widehat{f\overline{g}} \le \widehat{f}\,\overline{g} \\ \hline\hline \exists_{f\overline{g}}(\overline{f\overline{g}}) \le \exists_f(\overline{f})\overline{g} \\ \hline\hline \overline{f\overline{g}} \le \overline{f\overline{g}\exists_f(\overline{f})\overline{g}} \end{array}$$

and then since

$$\begin{aligned} \overline{f\overline{g}\exists_f(\overline{f})\overline{g}} &= \overline{f\overline{g}}\ \overline{\overline{f\exists_f(\overline{f})}\ \overline{f\overline{g}}} \\ &= \overline{\overline{f\overline{g}}\ \overline{f\exists_f(\overline{f})}} = \overline{\overline{f\overline{g}}\,\overline{f}} = \overline{f\overline{g}} \end{aligned}$$

we know $\widehat{f\overline{g}} \le \widehat{f}\,\overline{g}$.

For the other inequality, the Frobenius identity gives

$$\exists_f(\overline{f})\overline{g} = \exists_f(\overline{f}) \wedge \overline{g} = \exists_f(\overline{f} \wedge \overline{f\overline{g}}) = \exists_f(\overline{f\overline{g}})$$

which means

$$\begin{array}{c} \widehat{f}\,\overline{g} \le \widehat{f\overline{g}} \\ \hline\hline \exists_f(\overline{f})\overline{g} \le \exists_{f\overline{g}}(\overline{f\overline{g}}) \\ \hline\hline \exists_f(\overline{f\overline{g}}) \le \exists_{f\overline{g}}(\overline{f\overline{g}})) \\ \hline\hline \overline{f\overline{g}} \le \overline{f\exists_{f\overline{g}}(\overline{f\overline{g}})} \end{array}$$

but $\overline{f\overline{g}} \le \overline{f\overline{g}\exists_{f\overline{g}}(\overline{f\overline{g}})}$, so

$$\overline{f\overline{g}}\ \overline{f\exists_{f\overline{g}}(\overline{f\overline{g}})} = \overline{f\overline{g}}\ \overline{f\overline{g}}\ \overline{f\exists_{f\overline{g}}(\overline{f\overline{g}})}$$

$$
\begin{aligned}
&= \overline{f\overline{g}}\ \overline{\overline{f\overline{g}}\, f\exists_{f\overline{g}}(\overline{f\overline{g}})} = \overline{f\overline{g}}\ \overline{f\overline{g}\exists_{f\overline{g}}(\overline{f\overline{g}})} \\
&= \overline{f\overline{g}}
\end{aligned}
$$

which means, $\overline{f\overline{g}} \le \overline{f\exists_{f\overline{g}}(\overline{f\overline{g}})}$, and then we have $\widehat{f}\,\overline{g} \le \widehat{f\overline{g}}$.

Having established both inequalities, we now know that $\widehat{f}\,\overline{g} = \widehat{f\overline{g}}$, as required.

[RR.4] We know that

$$
\begin{array}{c}
\widehat{\widehat{f}g} \le \widehat{fg} \\
\hline\hline
\exists_{\widehat{f}g}(\overline{\widehat{f}g}) \le \exists_{fg}(\overline{fg}) \\
\hline\hline
\overline{\widehat{f}g} \le \overline{\widehat{f}g\exists_{fg}(\overline{fg})} \\
\hline\hline
\widehat{f\overline{g}} \le \overline{\widehat{f}g\exists_{fg}(\overline{fg})} \\
\hline\hline
\exists_{f\overline{g}}(\overline{f\overline{g}}) \le \overline{\widehat{f}g\exists_{fg}(\overline{fg})} \\
\hline\hline
\overline{f\overline{g}} \le \overline{f\overline{g}\exists_f(\overline{f})g\exists_{fg}(\overline{fg})}
\end{array}
$$

Now, we observe that

$$
\begin{aligned}
&\overline{f\overline{f}\exists_f(\overline{f})g\exists_{fg}(\overline{fg})} = \overline{f\exists_f(\overline{f})\overline{g}g\exists_{fg}(\overline{fg})} \\
&= \overline{\overline{f\exists_f(\overline{f})}\, fg\exists_{fg}(\overline{fg})} = \overline{f\exists_f(\overline{f})}\ \overline{fg\exists_{fg}(\overline{fg})}
\end{aligned}
$$

and further that since $\overline{f} \le \overline{f\exists_f(\overline{f})}$ and $\overline{fg} \le \overline{fg\exists_{fg}(\overline{fg})}$ we have

$$
\overline{fg}\ \overline{f\exists_f(\overline{f})}\ \overline{fg\exists_{fg}(\overline{fg})} = \overline{f}\ \overline{f\exists_f(\overline{f})}\ \overline{fg}\ \overline{fg\exists_{fg}(\overline{fg})} = \overline{f}\ \overline{fg} = \overline{fg}
$$

which together give $\widehat{\widehat{f}g} \le \widehat{fg}$.

For the other inequality, we have

$$
\begin{array}{c}
\widehat{fg} \leq \widehat{\widehat{f}g} \\
\hline\hline
\exists_{fg}(\overline{fg}) \leq \exists_{\widehat{f}g}(\overline{\widehat{f}g}) \\
\hline\hline
\overline{fg} \leq \overline{fg\exists_{\widehat{f}g}(\overline{\widehat{f}g})}
\end{array}
$$

and also

$$
\overline{fg} = \overline{f\widehat{f}g} = \overline{f\overline{\widehat{f}g}} \leq \overline{f\overline{\widehat{f}g\exists\widehat{f}g(\overline{\widehat{f}g})}} = \overline{fg\exists_{\widehat{f}g}(\overline{\widehat{f}g})}
$$

which together give $\widehat{fg} \leq \widehat{\widehat{f}g}$.

The two inequalities then give $\widehat{\widehat{f}g} = \widehat{fg}$ as required.

Therefore $\mathbb{X}$ is a range restriction category. $\square$

Thus, the only difference between a range category and a restriction category with existential quantification is that in the latter the Beck-Chevalley condition must be satisfied by the $\exists_f$ adjoints. If the range category is also a cartesian restriction category, then it was shown in [23] (Lemma 3.8) that the $\exists_f$ adjoints satisfy the Beck-Chevalley condition if and only if $\widehat{f \times g} = \widehat{f} \times \widehat{g}$ for all maps f, g. Ranges will not play a major role in the sequel, but it may be helpful to think of existential quantification in a restriction category in these terms.

2.6 Universal Quantification

As with existential quantification, universal quantification in a restriction category is intended to correspond to some notion of universal quantification in a latent fibration, applied to the domain latent fibration. The difference is again that while the presence of universal quantification in a restriction category implies the presence of the fibrational kind of universal quantification in $\mathsf{total}(\mathcal{O})$, the converse does not hold because the latter universal quantification does not account for the partial maps of the base category. Again, there ought to be a notion of universal quantification in a latent fibration tying all this together, but we use the more specific notion here.

Definition 2.40. A restriction category $\mathbb{X}$ is said to have *universal quantification* in case for every map $f : X \to Y$ of $\mathbb{X}$ the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ has a right adjoint $\forall_f : \mathcal{O}(X) \to \mathcal{O}(Y)$, and the Beck-Chevalley condition holds. That is, there is a preorder morphism $\forall_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ for which the two way inference

$$\frac{\overline{fe} \leq e'}{\overline{\overline{\quad e \leq \forall_f(e') \quad}}}$$

holds, and for any latent pullback square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

the Beck-Chevalley condition holds in the sense that for $e \leq \overline{f}$,

$$\overline{g\forall_f(e)} = \overline{g}\forall_k(\overline{he})$$

Again, we state explicitly an alternate formulation of this adjunction.

Lemma 2.41. *If $f : X \to Y$ is a map in a restriction category $\mathbb{X}$ and $\forall_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ is right adjoint to $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, then*

(i) $\overline{f\forall_f(e)} \leq e$ for $e \in \mathcal{O}(X)$

(ii) $e' \leq \forall_f(\overline{fe'})$ for $e' \in \mathcal{O}(Y)$

(iii) $\forall_f$ is a preorder morphism

Proof. (i) $\forall_f(e) \leq \forall_f(e)$ gives $\overline{f\forall_f(e) \leq e}$.

(ii) $\overline{fe'} \leq \overline{fe'}$ gives $e' \leq \forall_f(\overline{fe'})$.

(iii) Suppose $e_1, e_2 \in \mathcal{O}(X)$ such that $e_1 \leq e_2$. Then $\overline{f\forall_f(e_1)} \leq e_1 \leq e_2$, which gives $\forall_f(e_1) \leq \forall_f(e_2)$.

$\square$

Lemma 2.42. *If $f : X \to Y$ is a map in a restriction category $\mathbb{X}$ and $\forall_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ is a preorder morphism satisfying*

(i) $\overline{f\forall_f(e)} \leq e$ for $e \in \mathcal{O}(X)$

(ii) $e' \leq \forall_f(\overline{fe'})$ for $e' \in \mathcal{O}(Y)$

then $\forall_f$ is left adjoint to $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Proof. Let $e \in \mathcal{O}(X)$ and $e' \in \mathcal{O}(Y)$. Suppose $\overline{fe'} \leq e$. Then since $\forall_f$ is a preorder morphism $\forall_f(\overline{fe'}) \leq \forall_f(e)$, and we have

$$e' \leq \forall_f(\overline{fe'}) \leq \forall_f(e)$$

For the converse, suppose $e' \leq \forall_f(e)$. Then since f^* is a preorder morphism $\overline{fe'} \leq \overline{f\forall_f(e)}$, and we have

$$\overline{fe'} \leq \overline{f\forall_f(e)} \leq e$$

as required. $\square$

2.7 Cartesian Closed Restriction Categories

It will come as no surprise that there is a notion of cartesian closedness for restriction categories [7]. The main difference from the total case is that in order to ensure that the partial cartesian closed structure survives the idempotent splitting construction, we require the existence of an exponential transpose for each restriction idempotent satisfying a condition:

Definition 2.43. A *cartesian closed restriction category* is a cartesian restriction category $\mathbb{X}$ such that for every pair A, B of objects in $\mathbb{X}$ there is an object B^A and a map $\mathsf{ev}_{A,B} : A \times B^A \to B$ in $\mathbb{X}$ such that for every map $f : A \times C \to B$ and every restriction idempotent $e \in \mathcal{O}(C)$ satisfying $(1 \times e)f = f$, there is a unique map $\lambda_e(f) : C \to B^A$ with $\overline{\lambda_e(f)} = e$ which makes

$$\begin{tikzcd} A \times B^A \arrow[r, "\mathsf{ev}_{A,B}"] & B \\ A \times C \arrow[u, "1\times\lambda_e(f)"] \arrow[ur, "f"'] & \end{tikzcd}$$

commute.

We call $\lambda_e(f)$ the *transpose of f relative to e*, or simply the *transpose of f*. We often make special use of $\lambda_1(f)$ because it is total, and denote it $\lambda(f)$. It is equivalent to ask there is a unique total map $\lambda(f) : C \to B^A$ making the diagram commute, and that for each f' such that $(1 \times f')\mathsf{ev} = f$ we have $f' \leq \lambda(f)$. This alternate presentation looks more like the other structure in restriction categories, but makes the connection to the idempotent splitting less obvious.

Lemma 2.44. *In a cartesian closed restriction category $\mathbb{X}$, if $f : A \times B \to C$ and $g : D \to B$, then $g\lambda(f) = \overline{g}\lambda((1 \times g)f)$.*

Proof. $\lambda((1 \times g)f)$ is total, so $g\lambda(f) \leq \lambda((1 \times g)f)$, which gives

$$g\lambda(f) = \overline{g\lambda(f)}\lambda((1 \times g)f) = \overline{g}\lambda((1 \times g)f)$$

as required. $\square$

We can also manipulate the transpose as follows: Observe that

$$\lambda(\pi_0 h)\lambda((g \times 1)\mathsf{ev} f) = \lambda(\pi_0 g h f)$$

since

$$(1 \times \lambda(\pi_0 h)\lambda((g \times 1)\mathsf{ev} f))\mathsf{ev} = (1 \times \lambda(\pi_0 h))(g \times 1)\mathsf{ev} f$$
$$= (g \times 1)(1 \times \lambda(\pi_0 h))\mathsf{ev} f = (g \times 1)\pi_0 h f = \pi_0 g h f$$

and $\lambda(\pi_0 g h f)$ is unique with that property. π_0 is an isomorphism, so we can suppress it to obtain a total map $[g,h]$ with $\lambda(f)[g,h] = \lambda(fgh)$. We make use of this briefly in the sequel.

Another common piece of notation in a cartesian closed restriction category is as follows: For a map $f : X \to Y$, we define the total map $\ulcorner f \urcorner : 1 \to Y^X$ by $\ulcorner f \urcorner := \lambda(\pi_0 f)$, as in

$$\begin{tikzcd}
X \times Y^X \arrow[r, "\mathsf{ev}"] & Y \\
X \times 1 \arrow[u, "1\times\lambda(\pi_0 f)"] \arrow[ur, "\pi_0 f"'] &
\end{tikzcd}$$

We think of $\ulcorner f \urcorner$ as the "element" of Y^X corresponding to $f : X \to Y$, and recover f from $\ulcorner f \urcorner$ as follows:

$$\langle 1, !\ulcorner f \urcorner \rangle \mathsf{ev} = \langle 1, ! \rangle (1 \times \lambda(\pi_0 f))\mathsf{ev} = f$$

In the sequel, we will want to compare the domain of definition of maps in a restriction category internally. The following structure makes this possible:

Definition 2.45. A *subobject classifier* in a cartesian restriction category $\mathbb{X}$ is an object Ω together with a map $\mathsf{ev} : \Omega \to 1$ such that for each map $f : X \to 1$ of $\mathbb{X}$ there is a unique total map $\chi_f : A \to \Omega$ making

$$\begin{tikzcd}
A \arrow[r, "\chi_f"] \arrow[dr, "f"'] & \Omega \arrow[d, "\mathsf{ev}"] \\
& 1
\end{tikzcd}$$

commute, and further, for any $h : A \to \Omega$, if $h\mathsf{ev} = f$ then $h \leq \chi_f$.

If $e \in \mathcal{O}(X)$ is a restriction idempotent, we use the following notation

$$\chi_e := \chi_{e!}$$

We mention these subobject classifiers in conjunction with cartesian closed restriction categories because every cartesian closed restriction category has one.

Proposition 2.46. *If $\mathbb{X}$ is a cartesian closed restriction category, then $\mathbb{X}$ has a subobject classifier. Further, for any $f : X \to Y$ of $\mathbb{X}$, $e \in \mathcal{O}(Y)$, the equation $f\chi_e = \overline{f}\chi_{fe}$ holds*

Proof. We define $\Omega := 1^1$ and $\mathsf{ev} := \langle !, 1\rangle\mathsf{ev} : 1^1 \to 1$. For a map $f : A \to 1$ of $\mathbb{X}$, define $\chi_f := \lambda(\pi_1 f)$, as in

$$\begin{tikzcd}
1 \times 1^1 \arrow[r, "\mathsf{ev}"] & 1 \\
1 \times A \arrow[u, "1\times\lambda(\pi_1 f)"] \arrow[ur, "\pi_1 f"'] &
\end{tikzcd}$$

Clearly χ_f is total, and

$$\chi_f\mathsf{ev} = \lambda(\pi_1 f)\langle !, 1\rangle\mathsf{ev} = \langle !, \chi_f\rangle\mathsf{ev}$$
$$= \langle !, 1\rangle(1 \times \lambda(\pi_1 f))\mathsf{ev} = \langle !, 1\rangle\pi_1 f = f$$

Now, suppose $h : A \to 1^1$ is such that $h\mathsf{ev} = f$. Then

$$\overline{h}\chi_f = \overline{h}\lambda(\pi_1 f) = \overline{h}\lambda(\pi_1 h\mathsf{ev}) = \overline{h}\lambda((1 \times h)\pi_1\mathsf{ev})$$
$$= h\lambda(\pi_1\mathsf{ev}) = h\chi_{\overline{\mathsf{ev}}} = h$$

and so $h \le \chi_f$, as required.

We show $f\chi_e = \overline{f}\chi_{\overline{fe}}$ as follows

$$f\chi_e = f\lambda(\pi_1 e!) = \overline{f}\lambda((1 \times f)\pi_1 e!)$$
$$= \overline{f}\lambda(\pi_1 f e!) = \overline{f}\chi_{\overline{fe}}$$

□

3 Discrete Cartesian Closed Restriction Categories

We have already seen that every discrete cartesian closed restriction category has a subobject classifier. We proceed by showing that they have both universal and existential quantification, and that each $\mathcal{O}(X)$ for X an object in a discrete cartesian closed restriciton category is a Heyting algebra. We also consider the action of the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ for some map $f : X \to Y$ of a discrete cartesian closed restriction category. While this is not in general a Heyting algebra morphism, we will see that it is when the map f is total.

3.1 Universal Quantification

Proposition 3.1. *If $\mathbb{X}$ is a discrete cartesian closed restriction category, then $\mathbb{X}$ has universal quantification.*

Proof. It suffices to define a mapping $\forall_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ for each $f : X \to Y$ of $\mathbb{X}$ satisfying, for $e \in \mathcal{O}(X)$, $e' \in \mathcal{O}(Y)$

$$\frac{\overline{fe'} \leq e}{e' \leq \forall_f(e)}$$

and also satisfying the Beck-Chevalley condition. To that end, let $f : X \to Y$ be a map of $\mathbb{X}$, and define $\forall_f(e) \in \mathcal{O}(Y)$ for $e \in \mathcal{O}(X)$ by

$$\forall_f(e) := \overline{\lambda(\pi_0 f \cap \pi_1) \cap \lambda(\pi_0 e f \cap \pi_1)}$$

with the types as in $(\pi_0 f \cap \pi_1) : X \times Y \to Y$.

Now, let $e : \mathcal{O}(X)$, $e' : \mathcal{O}(Y)$, $f : X \to Y$, and supose $\overline{fe'} \leq e$. Then $efe' = e\overline{fe'}f = \overline{fe'}f = fe'$, and we have

$$e'\lambda(\pi_0 f \cap \pi_1) = e'\lambda((1 \times e')(\pi_0 f \cap \pi_1)) = e'\lambda(\pi_0 f \cap (1 \times e')\pi_1)$$

$$
= e'\lambda(\pi_0 f \cap \pi_1 e') = e'\lambda(\pi_0 f e' \cap \pi_1) = e'\lambda(\pi_0 e f e' \cap \pi_1)
$$

$$
= e'\lambda(\pi_0 e f \cap \pi_1 e') = e'\lambda((1 \times e')(\pi_0 e f \cap \pi_1)) = e'\lambda(\pi_0 e f \cap \pi_1)
$$

and so

$$
\begin{aligned}
& e'\forall_f(e) = e'\overline{\lambda(\pi_0 f \cap \pi_1) \cap \lambda(\pi_0 e f \cap \pi_1)} \\
& = \overline{e'\lambda(\pi_0 f \cap \pi_1) \cap e'\lambda(\pi_0 e f \cap \pi_1)} = \overline{e'\lambda(\pi_0 f \cap \pi_1) \cap e'\lambda(\pi_0 f \cap \pi_1)} \\
& = e'\overline{\lambda(\pi_0 f \cap \pi_1) \cap \lambda(\pi_0 f \cap \pi_1)} = e'\overline{\lambda(\pi_0 f \cap \pi_1)} = e'
\end{aligned}
$$

which means $e' \leq \forall_f(e)$, as required.

Conversely, suppose $e' \leq \forall_f(e)$. We then have $\overline{f\forall_f(e)} \leq e$ by

$$
\begin{aligned}
& \overline{f\forall_f(e)} = \overline{f\forall_f(e) \cap f\forall_f(e)} = \overline{\overline{f\forall_f(e)} f \cap f\forall_f(e)} \\
& = \overline{\langle 1, f\forall_f(e)\rangle \pi_0 f \cap \langle 1, f\forall_f(e)\rangle \pi_1} = \overline{\langle 1, f\forall_f(e)\rangle (\pi_0 f \cap \pi_1)} \\
& = \overline{\langle 1, f\rangle (1 \times \forall_f(e))(1 \times \lambda(\pi_0 f \cap \pi_1))\mathsf{ev}} = \overline{\langle 1, f\rangle (1 \times \forall_f(e)\lambda(\pi_0 f \cap \pi_1))\mathsf{ev}} \\
& = \overline{\langle 1, f\rangle (1 \times \overline{\lambda(\pi_0 f \cap \pi_1) \cap \lambda(\pi_0 e f \cap \pi_1)}\, \lambda(\pi_0 f \cap \pi_1))\mathsf{ev}} \\
& = \overline{\langle 1, f\rangle (1 \times \overline{\lambda(\pi_0 f \cap \pi_1) \cap \lambda(\pi_0 e f \cap \pi_1)}\, \lambda(\pi_0 e f \cap \pi_1))\mathsf{ev}} \\
& = \overline{\langle 1, f\rangle (1 \times \forall_f(e))(\pi_0 e f \cap \pi_1)} \\
& \leq \overline{\langle 1, f\rangle (\pi_0 e f \cap \pi_1)} = \overline{\overline{f} e f \cap f} = \overline{e(f \cap f)} \\
& = \overline{ef} = \overline{e\overline{f}} \leq e
\end{aligned}
$$

and we use this to obtain

$$
\overline{fe'} \leq \overline{f\forall_f(e)} \leq e
$$

as required.

For the Beck-Chevalley condition, we must show that for any latent pullback in $\mathbb{X}$

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

we have, for $e \leq \overline{f}$,

$$\overline{g\forall_f(e)} = \overline{g}\forall_k(\overline{he})$$

We proceed by showing each inequality to establish the promised equality. For *any* commuting square

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

in a discrete cartesian closed restriction category, we have, for $e \leq \overline{f}$,

$$\overline{g\forall_f(e)} \leq \forall_k(\overline{he})$$

as follows. From $\overline{f\forall_f(e)} \leq e$ we obtain

$$\overline{k\overline{g\forall_f(e)}} = \overline{kg\forall_f(e)} = \overline{hf\forall_f(e)} = \overline{h\overline{f\forall_f(e)}} \leq \overline{he}$$

and then since $\forall_f$ the right adjoint we have

$$\overline{g\forall_f(e)} \leq \forall_k(\overline{he})$$

For the reverse inequality, we require the square

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

to be a latent pullback. In this case, we first show that the inequality holds for the canonical latent pullback in a discrete cartesian closed restriction category.

$$\begin{tikzcd}
X \times Z \arrow[r, "\overline{\pi_0 f \cap \pi_1 g}\,\pi_0"] \arrow[d, "\overline{\pi_0 f \cap \pi_1 g}\,\pi_1"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

From there, it is easy to show that it holds for any latent pullback. We begin by observing that

$$\overline{\overline{\pi_0 f \cap \pi_1 g}\,\pi_0 e} = \overline{\pi_0 e}\;\overline{\pi_0 f \cap \pi_1 g}$$

$$= \overline{\overline{\pi_0 e}(\pi_0 f \cap \pi_1 g)} = \overline{\overline{\pi_0 e}\pi_0 f \cap \pi_1 g}$$

$$= \overline{\pi_0 e f \cap \pi_1 g}$$

and so

$$\langle 1, \pi_1 \rangle (\pi_0 \overline{\overline{\pi_0 f \cap \pi_1 g}\,\pi_0 e}\,\pi_1 \cap \pi_1) g = \langle 1, \pi_1 \rangle (\pi_0 \overline{\pi_0 e f \cap \pi_1 g}\,\pi_1 \cap \pi_1) g$$

$$= (\overline{\pi_0 e f \cap \pi_1 g}\,\pi_1 \cap \pi_1) g = \overline{\pi_0 e f \cap \pi_1 g}(\pi_1 \cap \pi_1) g = \overline{\pi_0 e f \cap \pi_1 g}\,\pi_1 g$$

$$= \pi_0 e f \cap \pi_1 g$$

and, similarly

$$\langle 1, \pi_1 \rangle (\pi_0 \overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1) g = \overline{\pi_0 f \cap \pi_1 g}(\pi_1 \cap \pi_1) g = \overline{\pi_0 f \cap \pi_1 g}\,\pi_1 g$$

$$= \pi_0 f \cap \pi_1 g$$

Now, we use that $[\langle 1, \pi_1\rangle, g]$ is total to obtain

$$
\begin{aligned}
&\overline{g}\forall_{\overline{\pi_0 f \cap \pi_1 g}\,\pi_1}(\overline{\overline{\pi_0 f \cap \pi_1 g}\,\pi_0 e}) \\
&= \overline{g}\,\overline{\lambda(\pi_0\overline{\overline{\overline{\pi_0 f \cap \pi_1 g}}\,\pi_0 e}\,\overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1) \cap \lambda(\pi_0\overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1)} \\
&= \overline{g}\,\overline{\lambda(\pi_0\overline{\overline{\overline{\pi_0 f \cap \pi_1 g}}\,\pi_0 e}\,\pi_1 \cap \pi_1) \cap \lambda(\pi_0\overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1)} \\
&= \overline{g}\,\overline{(\lambda(\pi_0\overline{\overline{\overline{\pi_0 f \cap \pi_1 g}}\,\pi_0 e}\,\pi_1 \cap \pi_1) \cap \lambda(\pi_0\overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1))[\langle 1, \pi_1\rangle, g]} \\
&\leq \overline{g}\,\overline{\lambda(\pi_0\overline{\overline{\overline{\pi_0 f \cap \pi_1 g}}\,\pi_0 e}\,\pi_1 \cap \pi_1)[\langle 1, \pi_1\rangle] \cap \lambda(\pi_0\overline{\pi_0 f \cap \pi_1 g}\,\pi_1 \cap \pi_1)[\langle 1, \pi_1\rangle, g]} \\
&= \overline{g}\,\overline{\lambda(\pi_0 e f \cap \pi_1 g) \cap \lambda(\pi_0 f \cap \pi_1 g)} = \overline{\overline{g}\lambda(\pi_0 e f \cap \pi_1 g) \cap \overline{g}\lambda(\pi_0 f \cap \pi_1 g)} \\
&= \overline{\overline{g}\lambda((1 \times g)(\pi_0 e f \cap \pi_1)) \cap \overline{g}\lambda((1 \times g)(\pi_0 f \cap \pi_1))} \\
&= \overline{g\lambda(\pi_0 e f \cap \pi_1) \cap g\lambda(\pi_0 f \cap \pi_1)} = \overline{g(\lambda(\pi_0 e f \cap \pi_1) \cap \lambda(\pi_0 f \cap \pi_1))} \\
&= \overline{g\forall_f(e)}
\end{aligned}
$$

which is the desired inequality for the canonical latent pullback. Now, let $\alpha : W \to X \times Z$ be the induced partial isomorphism between the two latent pullbacks, as in

$$
\begin{tikzcd}
W \arrow[dr, "\alpha"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
{} \arrow[r, phantom, "\geq"] & X \times Z \arrow[r] \arrow[d] \arrow[u, phantom, "\leq"] & X \arrow[d, "f"] \\
& Z \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Z \arrow[dr, "\alpha^{(-1)}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
{} \arrow[r, phantom, "\geq"] & W \arrow[r, "h"] \arrow[d, "k"'] \arrow[u, phantom, "\leq"] & X \arrow[d, "f"] \\
& Z \arrow[r, "g"'] & Y
\end{tikzcd}
$$

Observe that together these inequalities ensure the two diagrams commute on the nose. Then we have

$$
\begin{aligned}
&\forall_{\overline{\pi_0 f \cap \pi_1 g}\,\pi_1}(\overline{\overline{\pi_0 f \cap \pi_1 g}\,\pi_0 e}) = \forall_{\alpha^{(-1)}k}(\overline{\alpha^{(-1)}he}) \\
&= \forall_{\alpha^{(-1)}k}(\overline{\alpha^{(-1)}\overline{he}}) = \forall_k(\forall_{\alpha^{(-1)}}(\overline{\alpha^{(-1)}\overline{he}}))
\end{aligned}
$$

$$\geq \forall_k(\overline{he})$$

and so finally, we have

$$\overline{g}\forall_k(\overline{he}) \leq \overline{g}\forall_{\overline{\pi_0 f \cap \pi_1 g}\,\pi_1}(\overline{\overline{\pi_0 f \cap \pi_1 g}\,\pi_0 e}) \leq \overline{g\forall_f(e)}$$

and the Beck-Chevalley condition is satisfied.

$\square$

3.2 $\mathcal{O}(X)$ is a Heyting Algebra

We have already seen that for any object X of a restriction category $\mathbb{X}$, the restriction idempotents $\mathcal{O}(X)$ form a bounded meet semilattice with $\leq$ the extension ordering, $\top$ the identity on X, and $e \wedge e'$ given by composition. In this section, we show that if $\mathbb{X}$ is a discrete cartesian closed restriction category, then each $\mathcal{O}(X)$ is a Heyting algebra. We also consider the action of the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ for an arbitrary map $f : X \to Y$ of $\mathbb{X}$.

Proposition 3.2. *If $\mathbb{X}$ is a discrete cartesian closed restriction category, then the partial order $\mathcal{O}(X)$ has a Heyting implication for every object X of $\mathbb{X}$. Further, for every map $f : X \to Y$ of $\mathbb{X}$ the pullback functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ has $f^*(e_1 \Rightarrow e_2) = \overline{f}(f^*(e_1) \Rightarrow f^*(e_2))$.*

Proof. The meet and top element are defined as above, and if $e_1 \leq e_2 \in \mathcal{O}(Y)$ then certainly $\overline{fe_1} \leq \overline{fe_2}$, so f^* preserves the meet. For $\top$, we have $\overline{fe} \leq \overline{f} = \overline{f\top}$, and f^* preserves the top element.

Heyting implication is defined using the meet and subobject classifier. If e_1 and e_2 are elements of $\mathcal{O}(X)$, we define $(e_1 \Rightarrow e_2) : \mathcal{O}(X)$ by

$$(e_1 \Rightarrow e_2) := \overline{\chi_{e_1} \cap \chi_{e_1 e_2}}$$

The idea being that $e_1 \leq e_2$ precisely when $e_1 = e_1 e_2$, so we express that with the structure of $\mathbb{X}$. To show that this is indeed the Heyting implication, we must establish

$$\frac{(e \wedge e_1) \leq e_2}{e \leq (e_1 \Rightarrow e_2)}$$

To that end, suppose $ee_1 = e \wedge e_1 \leq e_2$, meaning that $ee_1e_2 = ee_1$. Then we have

$$\begin{aligned}
&e(e_1 \Rightarrow e_2) = e\overline{\chi_{e_1} \cap \chi_{e_1e_2}} = \overline{e\chi_{e_1} \cap e\chi_{e_1e_2}} \\
&= \overline{e\chi_{ee_1} \cap e\chi_{ee_1e_2}} = e\overline{\chi_{ee_1} \cap \chi_{ee_1}} = e\overline{\chi_{ee_1}} \\
&= e
\end{aligned}$$

so $e \leq (e_1 \Rightarrow e_2)$.

For the converse, suppose $e \leq (e_1 \Rightarrow e_2)$, meaning $e(e_1 \Rightarrow e_2) = e$. Then

$$\begin{aligned}
&(e \wedge e_1)e_2 = ee_1e_2 = e(e_1 \Rightarrow e_2)e_1e_2 = e\overline{\chi_{e_1} \cap \chi_{e_1e_2}}\, e_1e_2 \\
&= e\overline{\chi_{e_1} \cap \chi_{e_1e_2}}\, \chi_{e_1e_2}\mathsf{ev} = e\overline{\chi_{e_1} \cap \chi_{e_1e_2}}\, \chi_{e_1}\mathsf{ev} \\
&= e(e_1 \Rightarrow e_2)e_1 = ee_1 = (e \wedge e_1)
\end{aligned}$$

so $(e \wedge e_1) \leq e_2$, and our implication is in fact a Heyting implication. We show that $\overline{f(\varphi \Rightarrow \psi)} = \overline{f}(\varphi \Rightarrow \psi)$ as follows:

$$\begin{aligned}
&\overline{f(e_1 \Rightarrow e_2)} = \overline{f\chi_{e_1} \cap f\chi_{e_1e_2}} \\
&= \overline{\overline{f}\chi_{\overline{fe_1}} \cap \overline{f}\chi_{\overline{fe_1e_2}}} = \overline{f}\,\overline{\chi_{\overline{fe_1}} \cap \chi_{\overline{fe_1}\,\overline{fe_2}}} \\
&= \overline{f}(\overline{fe_1} \Rightarrow \overline{fe_2})
\end{aligned}$$

$\square$

Proposition 3.3. *If $\mathbb{X}$ is a discrete cartesian closed restriction category, then the partial order $\mathcal{O}(X)$ has joins and a bottom element for every object X of $\mathbb{X}$. Further, for each $f : X \to Y$ of $\mathbb{X}$ the reindexing functor $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ preserves the join and bottom element.*

Proof. The bottom element of $\mathcal{O}(X)$ is defined to be

$$\bot_X := \forall_{\pi_0}(\overline{\mathsf{ev}})$$

with the typing as in $\mathsf{ev} : X \times 1^X \to 1$. If $e \in \mathcal{O}(X)$, then

$$\begin{aligned}
&\bot_X = \forall_{\pi_0}(\overline{\mathsf{ev}}) = \overline{\forall_{\pi_0}(\overline{\mathsf{ev}})} = \overline{\forall_{\pi_0}(\overline{\mathsf{ev}})\langle 1, \ulcorner e \urcorner\rangle} \\
&= \overline{\langle 1, \ulcorner e \urcorner\rangle(\forall_{\pi_0}(\overline{\mathsf{ev}}) \times 1)} = \overline{\langle 1, \ulcorner e \urcorner\rangle \pi_0 \forall_{\pi_0}(\overline{\mathsf{ev}})} \\
&\leq \overline{\langle 1, \ulcorner e \urcorner\rangle \overline{\mathsf{ev}}} = \overline{\langle 1, \ulcorner e \urcorner\rangle \mathsf{ev}} = \overline{e} \\
&= e
\end{aligned}$$

which means $\bot_X$ is indeed the bottom element of our lattice. Also, if $f : X \to Y$ then $\bot_Y \leq \forall_f(\bot_X)$ gives by way of the adjunction that $\overline{f\bot_Y} \leq \bot_X$. We know that $\bot_X \leq \overline{f\bot_Y}$, so $\bot_X = \overline{f\bot_Y}$, and the pullback functor f^* preserves the bottom element.

The join of e_1 and e_2 in $\mathcal{O}(X)$ is defined to be

$$(e_1 \vee e_2) := \forall_{\pi_0}((\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})$$

We verify that this is in fact the join. For $e_1 \leq (e_1 \vee e_2)$, we have

$$\overline{\pi_0 e_1} \wedge (\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) = \overline{\pi_0 e_1}(\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}})$$

$$\leq \overline{\pi_0 e_1}(\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}}) = \overline{\pi_0 e_1} \wedge (\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}}) \leq \mathsf{ev}$$

which, via properties of Heyting implication, gives

$$\overline{\pi_0 e_1} \leq ((\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})$$

and the adjunction then gives

$$e_1 \leq \forall_{\pi_0}((\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}}) = (e_1 \vee e_2)$$

and we have $e_2 \leq (e_1 \vee e_2)$ similarly. Next, suppose $e_1 \leq e$ and $e_2 \leq e$ for some $e \in \mathcal{O}(X)$. Then we have

$$\begin{aligned}
(e_1 \vee e_2) &= \overline{\langle (e_1 \vee e_2), 1\rangle \pi_0 \ulcorner e \urcorner} \\
&= \overline{\langle 1, \ulcorner e \urcorner\rangle((e_1 \vee e_2) \times 1)} = \overline{\langle 1, \ulcorner e \urcorner\rangle \pi_0 (e_1 \vee e_2)} \\
&= \overline{\langle 1, \ulcorner e \urcorner\rangle \pi_0 \forall_{\pi_0}((\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})} \\
&\leq \overline{\langle 1, \ulcorner e \urcorner\rangle((\overline{\pi_0 e_1} \Rightarrow \overline{\mathsf{ev}})(\overline{\pi_0 e_2} \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})} \\
&= ((\overline{\langle 1, \ulcorner e \urcorner\rangle \pi_0 e_1} \Rightarrow \overline{\langle 1, \ulcorner e \urcorner\rangle \mathsf{ev}})(\overline{\langle 1, \ulcorner e \urcorner\rangle \pi_0 e_2} \Rightarrow \overline{\langle 1, \ulcorner e \urcorner\rangle \mathsf{ev}}) \Rightarrow \overline{\langle 1, \ulcorner e \urcorner\rangle \mathsf{ev}}) \\
&= (e_1 \Rightarrow e)(e_2 \Rightarrow e) \Rightarrow e \\
&= ((e_1 \Rightarrow e) \wedge (e_2 \Rightarrow e)) \Rightarrow e
\end{aligned}$$

but since $e_1 \leq e$ and $e_2 \leq e$ by assumption, $((e_1 \Rightarrow e) \wedge (e_2 \Rightarrow e))$ is $\top$, so we have

$$(e_1 \vee e_2) \leq (\top \Rightarrow e) = e$$

Thus, our $(e_1 \vee e_2)$ is indeed the join. We show that f^* preserves the join as follows: Suppose $f : Z \to X$. Then we have $\overline{fe_1} \leq \overline{f(e_1 \vee e_2)}$ by

$$\overline{fe_1}\ \overline{f(e_1 \vee e_2)} = \overline{\overline{fe_1}\, f(e_1 \vee e_2)}$$
$$= \overline{fe_1(e_1 \vee e_2)} = \overline{fe_1}$$

and similarly we have $\overline{fe_2} \leq \overline{f(e_1 \vee e_2)}$, so we know $\overline{fe_1} \vee \overline{fe_2} \leq \overline{f(e_1 \vee e_2)}$. Now, suppose that for some e, we know $\overline{fe_1} \leq e$ and $\overline{fe_2} \leq e$. The adjunction gives $e_1 \leq \forall_f(e)$ and $e_2 \leq \forall_f(e)$, so we have $f(e_1 \vee e_2) \leq e$. Certainly $\overline{fe_1} \vee \overline{fe_2}$ has this property, and therefore we have shown $\overline{f(e_1 \vee e_2)} \leq \overline{fe_1} \vee \overline{fe_2}$. By asymmetry we now have $\overline{f(e_1 \vee e_2)} = \overline{fe_1} \vee \overline{fe_2}$. Thus, the join is preserved by reindexing over f^*.

□

We have now established that $\mathcal{O}(X)$ for X an object of a discrete cartesian closed restriction category $\mathbb{X}$ is a Heyting algebra. Notice also that if the map $f : X \to Y$ in $\mathbb{X}$ is total, then we have shown that f^* is a Heyting algebra morphism. This will be used in the sequel.

3.3 Existential Quantification

Proposition 3.4. *Every discrete cartesian closed restriction category has existential quantification.*

Proof. For $f : X \to Y$ and $e \leq f$, we define $\exists_f(e) \in \mathcal{O}(Y)$ by

$$\exists_f(e) := \forall_{\pi_0}(\forall_{f \times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})$$

We begin by showing that this is a preorder morphism. First, we observe that Heyting implication has the following property:

$$\frac{a \leq b}{(b \Rightarrow c) \leq (a \Rightarrow c)}$$

which we use in conjunction with the fact that $\forall_f$ is a preorder morphism to obtain that, if $e \leq e'$ for $e, e' \leq \overline{f}$, then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{e \leq e'}{(e \times 1) \leq (e' \times 1)}}{((e' \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \leq ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}})}}{\forall_{f\times 1}((e' \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \leq \forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}))}}{(\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}}) \leq (\forall_{f\times 1}((e' \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})}}{\exists_f(e) \leq \exists_f(e')}$$

as required.

Now, to show that $\exists_f$ is left adjoint to the reindexing functor f^*, it suffices to show that for any $e \leq \overline{f}$, $e' \in \mathcal{O}(Y)$, we have $\exists_f(\overline{fe'}) \leq e'$ and $e \leq \overline{f\exists_f(e)}$.

To show $\exists_f(\overline{fe'}) \leq e'$, we first observe that the $\forall$ adjunction gives

$$((e' \times 1) \Rightarrow \overline{\mathsf{ev}}) \leq \forall_{f\times 1}(\overline{((f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})})$$

but then we have

$$\forall_{f\times 1}(\overline{(f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})}) \leq \forall_{f\times 1}(\overline{(f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})})((e' \times 1) \Rightarrow \overline{\mathsf{ev}})$$
$$= ((e' \times 1) \Rightarrow \overline{\mathsf{ev}})\forall_{f\times 1}(\overline{(f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})}) = ((e' \times 1) \Rightarrow \overline{\mathsf{ev}})$$

and so by antisymmetry

$$\forall_{f\times 1}(\overline{(f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})}) = ((e' \times 1) \Rightarrow \overline{\mathsf{ev}})$$

This allows us to obtain the required inequality as follows

$$\exists_f(\overline{fe'}) = \overline{\langle 1, \ulcorner e' \urcorner \rangle(\exists_f(\overline{fe'}) \times 1)} = \overline{\langle 1, \ulcorner e' \urcorner \rangle\overline{\pi_0 \exists_f(\overline{fe'})}}$$
$$= \overline{\langle 1, \ulcorner e' \urcorner \rangle\overline{\pi_0 \forall_{\pi_0}(\forall_{f\times 1}((\overline{fe'} \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})}}$$

$$
\begin{aligned}
&\leq \overline{\langle 1, \ulcorner e' \urcorner \rangle (\forall_{f\times 1}((\overline{fe'} \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})} \\
&= \overline{\langle 1, \ulcorner e' \urcorner \rangle (\forall_{f\times 1}(\overline{(f \times 1)(e' \times 1)} \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})} \\
&= \overline{\langle 1, \ulcorner e' \urcorner \rangle (\forall_{f\times 1}(\overline{(f \times 1)((e' \times 1) \Rightarrow \overline{\mathsf{ev}})}) \Rightarrow \overline{\mathsf{ev}})} \\
&= \overline{\langle 1, \ulcorner e' \urcorner \rangle ((e' \times 1) \Rightarrow \overline{\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}}} \\
&= ((\overline{\langle 1, \ulcorner e' \urcorner \rangle (e' \times 1)} \Rightarrow \overline{\langle 1, \ulcorner e' \urcorner \rangle \mathsf{ev}}) \Rightarrow \overline{\langle 1, \ulcorner e' \urcorner \rangle \mathsf{ev}}) \\
&= ((\overline{\langle 1, \ulcorner e' \urcorner \rangle \pi_0 e'} \Rightarrow e') \Rightarrow e') = ((e' \Rightarrow e') \Rightarrow e') \\
&= (\top \Rightarrow e') = e'
\end{aligned}
$$

For $e \leq \overline{f\exists_f(e')}$, we have

$$e \leq \forall_{\pi_0}(\overline{\pi_0 e}) = \forall_{\pi_0}(e \times 1)$$

and clearly

$$(e \times 1) \wedge ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \leq \overline{(f \times 1)\mathsf{ev}}$$

so by property of Heyting implication we have

$$(e \times 1) \leq ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}$$

and then the definition of universal quantification gives

$$\overline{(f \times 1)\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}})} \leq ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}$$

and so

$$
\begin{aligned}
&(((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \wedge \overline{(f \times 1)\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}})} \\
&\leq (((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \wedge ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \\
&= ((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \wedge (e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \\
&\leq \overline{(f \times 1)\mathsf{ev}}
\end{aligned}
$$

and the definition of Heyting implication now gives

$$
\begin{aligned}
&(((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \\
&\leq \overline{(f \times 1)\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}))} \Rightarrow \overline{(f \times 1)\mathsf{ev}}
\end{aligned}
$$

putting these things together, we have that

$$
\begin{aligned}
&(e \times 1) = (e \times 1)(e \times 1) \leq (\overline{f} \times 1)(e \times 1) \\
&\leq (\overline{f} \times 1)(((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \\
&\leq (\overline{f} \times 1)(\overline{(f \times 1)\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}})} \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \\
&= \overline{(f \times 1)(\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(f \times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})}
\end{aligned}
$$

Now, since $\forall_{\pi_0}$ is a preorder morphism, we can use this inequality to show

$$
\begin{aligned}
&e = ee \leq \overline{f}\,\forall_{\pi_0}(e \times 1) \\
&\leq \overline{f}\,\forall_{\pi_0}(\overline{(f \times 1)(\forall_{f\times 1}((e \times 1) \Rightarrow \overline{(1 \times f)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})})
\end{aligned}
$$

Next, we observe that since

$$\begin{tikzcd} X \times 1^Y \arrow[r, "f\times 1"] \arrow[d, "\pi_0"'] & Y \times 1^Y \arrow[d, "\pi_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a latent pullback, the Beck-Chevalley condition gives

$$\overline{f\forall_{\pi_0}(a)} = \overline{f}\forall_{\pi_0}(\overline{(f\times 1)a})$$

for any $a \in \mathcal{O}(Y \times 1^Y)$.

Applying this to our situation yields

$$\begin{aligned} e &\leq \overline{f}\,\forall_{\pi_0}(\overline{(f\times 1)(\forall_{f\times 1}((e\times 1) \Rightarrow \overline{(f\times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})}) \\ &= \overline{f\forall_{\pi_0}(\forall_{f\times 1}((e\times 1) \Rightarrow \overline{(f\times 1)\mathsf{ev}}) \Rightarrow \overline{\mathsf{ev}})} \\ &= \overline{f\exists_f(e)} \end{aligned}$$

For the Frobenius identity

$$\exists_f(e) \wedge e' \leq \exists_f(e \wedge \overline{fe'})$$

we use that $\overline{f(\varphi \Rightarrow \psi)} = (\overline{f\varphi} \Rightarrow \overline{f\psi})$, noting that

$$\begin{array}{c} \exists_f(e) \wedge \psi \leq \exists_f(\varphi \wedge \overline{f\psi}) \\ \hline\hline \exists_f(e) \leq \psi \Rightarrow \exists_f(\varphi \wedge \overline{f\psi}) \\ \hline\hline \varphi \leq \overline{f(\psi \Rightarrow \exists_f(\varphi \wedge \overline{f\psi})} \\ \hline\hline \varphi \leq \overline{f}(\overline{f\psi} \Rightarrow \overline{f\exists_f(\varphi \wedge \overline{f\psi})}) \end{array}$$

Since $e \leq \overline{f}$, it suffices to show $\varphi \leq \overline{f\psi} \Rightarrow \overline{f\exists_f(\varphi \wedge \overline{f\psi})}$, and we have

$$\begin{array}{c} \varphi \leq \overline{f\psi} \Rightarrow \overline{f\exists_f(\varphi \wedge \overline{f\psi})} \\ \hline\hline \varphi \wedge \overline{f\psi} \leq \overline{f\exists_f(\varphi \wedge \overline{f\psi})} \end{array}$$

but this is immediate since $\exists_f$ is left adjoint to f^*.

For the Beck-Chevalley condition, we show that if

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

is a latent pullback in $\mathbb{X}$, then for $e \leq \overline{f}$ we have

$$\overline{g\exists_f(e)} = \exists_k(\overline{he})$$

We begin by showing that for *any* commutative square and restriction idempotent e as above, we have $\exists_k(\overline{he}) \leq \overline{g\exists_f(e)}$. For this, it suffices to show

$$\exists_k(\overline{he}) \leq \overline{g\exists_f(e)}$$

which is the case precisely when

$$\overline{he} \leq \overline{k\overline{g\exists_f(e)}}$$

and we have

$$\overline{k\overline{g\exists_f(e)}} = \overline{kg\exists_f(e)} = \overline{hf\exists_f(e)} = \overline{h\overline{f\exists_f(e)}} \geq \overline{he}$$

as required. We procced by showing that when the square in question is a latent pullback, we have $g\exists_f(e) \leq \exists_k(\overline{he})$ as follows: We know $\overline{he} \leq \overline{k\exists_k(\overline{he})}$, which gives

$$e \leq \forall_h(\overline{k\exists_k(\overline{he})})$$

which in turn gives

$$e \leq \overline{f}\forall_h(\overline{k\exists_k(\overline{he})})$$

and we can now use the Beck-Chevalley condition for universal quantification to obtain

$$e \leq \overline{f}\forall_h(\overline{k\exists_k(\overline{he})}) = \overline{f\forall_g(\exists_k(\overline{he}))}$$

which gives

$$\exists_f(e) \leq \forall_g(\exists_k(\overline{he}))$$

which, finally, gives

$$\overline{g\exists_f(e)} \leq \exists_k(\overline{he})$$

as required. Thus, the Beck-Chevalley condition is satisfied, and we are done.

$\square$

4 Turing Categories

Classically [14], a partial combinatory algebra consists of a set A together with a partial binary operation $\bullet : A \times A \to A$ which is *combinatory complete*. That is, for any term t over the relevant signature in variables $x_1, \ldots, x_n$ there is an element $a \in A$ such that for all elements $b_1, \ldots, b_n$ of A, if both $t[b_1/x_1] \ldots [b_n/x_n]$ and $(a \bullet b_1) \bullet \ldots b_n$ are defined, then they are equal. This allows us to use a convenient syntax to denote elements of a partial combinatory algebra, with $\langle x \rangle t$ denoting the element of A for which $(\langle x \rangle t) \bullet b_1$ is equal to $t[b_1]/x$ when both are defined. It is now a small step to connect this syntax with abstraction in the λ-calculus, and thus to computability theory. Indeed, the prototypical example of a partial combinatory algebra is given by the partial recursive functions. The set is $\mathbb{N}$, and we define $\bullet : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\bullet(x, y) := \{y\}(x)$, the result of applying the yth partial recursive function to $x \in \mathbb{N}$. Computability theory contains many other examples [28], to the point that it is reasonable to characterize computability theory as the study of partial combinatory algebras.

Turing categories [8] are a general, categorical characterization of partial combinatory algebras. In the same sense that restriction categories are a generalization of the structure present in the category of partial functions on sets, Turing categories generalize the structure present in the computable maps of a partial combinatory algebra.

4.1 Partial Combinatory Algebras in Context

We begin by giving more abstract characterizations of the important ideas in the study of partial combinatory algebras.

Definition 4.1. A *partial applicative system* $(A, \bullet)$ in a cartesian restriction category $\mathbb{X}$ consists of an object A and a map $\bullet : A \times A \to A$.

For example, let Ptl be the category of sets and partial functions. There is a partial applicative system $(\mathbb{N}, \bullet)$ in Ptl, where $\bullet : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $\bullet(m, n) = \{n\}(m)$, the result of applying the n-th partial recursive function to m. This partial applicative system is rather famous, and is called *Kleene's first model*, written $\mathcal{K}_1$ [28].

In any partial applicative system there is also an "n-fold application" function, allowing us to consider something like functions of more than one argument:

Definition 4.2. If $(A, \bullet)$ is a partial applicative system in $\mathbb{X}$, we construct a family of *iterated application morphisms* $\bullet^{(n)} : A^n \times A \to A$ inductively. $\bullet^{(1)} = \bullet$, and using $\bullet^{(n)}$, we define $\bullet^{(n+1)}$ to be the composite

$$A^n \times A \times A \xrightarrow{(1\times\bullet)} A^n \times A \xrightarrow{\bullet^{(n)}} A$$

For completeness, we define $\bullet^{(0)} : 1 \times A \simeq A \to A$ by $\bullet^{(0)} := \Delta\bullet$.

Next, given a partial applicative system $(A, \bullet)$ in partial functions on sets, we can consider the maps represented by the elements A, which we call *computable*. More generally, the computable maps are those represented by a global section, that is, by a total map $1 \to A$ in the ambient category.

Definition 4.3. If $(A, \bullet)$ is a partial applicative system in $\mathbb{X}$

(i) A map $f : A^n \to A$ of $\mathbb{X}$ for some $n \in \mathbb{N}$ is A-*computable* when there exists a total map $p : 1 \to A$ of $\mathbb{X}$ such that

$$\begin{tikzcd} A^n \times A \arrow[r, "\bullet"] & A \\ A^n \times 1 \simeq A^n \arrow[u, "1\times p"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, and when $n \geq 1$, the composite

$$A^{n-1} \times 1 \xrightarrow{(1\times p)} A^{n-1} \times A \xrightarrow{\bullet^{(n-1)}} A$$

is total.

(ii) A map $A^n \to A^m$ for $m \geq 1$ is A-*computable* when all of its components are A-computable as maps $A^n \to A$.

(iii) A map $A^n \to 1$ is A-*computable* when its domain of definition is A-computable as a map $A^n \to A^n$.

when the partial applicative system is clear from the context, we refer to the A-computable maps simply as the *computable* maps.

For example, in $\mathcal{K}_1$, the computable maps are precisely the partial recursive functions.

Definition 4.4. A partial applicative system $(A, \bullet)$ in a cartesian restriction category $\mathbb{X}$ is *combinatory complete* in case the category $\mathsf{comp}(A)$ whose objects are formal powers $1, A, A \times A, A^3, \ldots$ of A, and whose maps are the A-computable maps in $\mathbb{X}$ is well-defined, and is a cartesian restriction subcategory of $\mathbb{X}$.

To connect this to the classical definition of combinatory completeness, recall the representation of terms with free variables in categorical logic. A term with, say, three free variables x_1, x_2, x_3 of type X_1, X_2, X_3 respectively is interpreted as a map with domain $X_1 \times X_2 \times X_3$ in the categorical model. If we interpret a collection of A-computable maps similarly as functions between terms with "holes", then the category of A-computable maps being a cartesian restriction category is equivalent to any rearrangement of these holes being computable. As in the classical case, we have:

Definition 4.5. A *partial combinatory algebra* is a combinatory complete partial applicative system.

For example, $\mathcal{K}_1$ is combinatory complete, and therefore a partial combinatory algebra.

4.2 Turing Categories

Partial combinatory algebras are defined in relation to an ambient cartesian restriction category. Turing categories are a presentation of very similar structure that, in contrast, are not defined in terms of some other category. This independence of setting allows us to study notions of computation independently from an underlying logic (e.g. ZFC). We proceed to define Turing categories, and show how they relate to partial combinatory algebras.

Definition 4.6. Let $\mathbb{A}$ be a cartesian restriction category.

(i) Let $\tau_{X,Y} : X \times A \to Y$ be a map in $\mathbb{A}$. A map $f : X \times Z \to Y$ is said to *admit a $\tau_{X,Y}$-index* in case there exists a total map $h : Z \to A$ such that

$$\begin{tikzcd} X \times A \arrow[r, "\tau_{X,Y}"] & Y \\ X \times Z \arrow[u, "1\times h"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. In this case we call h a *$\tau_{X,Y}$-index for f*.

(ii) A map $\tau_{X,Y} : X \times A \to Y$ is called a *universal application* in case for every object Z, every map $f : X \times Z \to Y$ admits a $\tau_{X,Y}$-index.

(iii) A *Turing object* in $\mathbb{A}$ is an object A of $\mathbb{A}$ such that for each X, Y of $\mathbb{A}$ there is a universal application map $\tau_{X,Y} : X \times A \to Y$.

Definition 4.7. A *Turing category* is a cartesian restriction category with a Turing object.

Our next aim will be to make the relationship between partial combinatory algebras in some category and Turing categories precise. In the process, we introduce an alternate characterization of Turing categories, allowing us to define them in terms of a universal object with a "Turing morphism". This presentation is often more convenient.

Definition 4.8. Given objects X, Y of a category $\mathbb{X}$, we say X *is a retract of* Y in case there are maps $s : X \to Y$, $r : Y \to X$ such that $sr = 1_X$. We call s the *section*, and t the *retraction*, and write $(s, r) : X \triangleleft Y$.

Definition 4.9. An object A of a category $\mathbb{X}$ is called a *universal object* in case every object X of $\mathbb{X}$ is a retract of A.

Lemma 4.10. *If A is the Turing object of a Turing category $\mathbb{A}$, then A is a universal object.*

Proof. Let X be an object of $\mathbb{A}$ and let $h : X \to A$ be a $\tau_{1,X}$-index for $\pi_1 : 1 \times X \to X$. Observe that

$$\begin{tikzcd}
1 \times A \arrow[r, "\tau_{1,X}"] & X \\
1 \times X \arrow[u, "1\times h"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

commutes. Now, we use $\langle !_X, 1\rangle(1 \times h)\pi_1 : X \to A$ as the section and $\langle !_A, 1\rangle\tau_{1,X} : A \to X$ as the retraction, and it is easy to see that

$$\begin{aligned}
&\langle !_X, 1\rangle(1 \times h)\pi_1\langle !_A, 1\rangle\tau_{1,X} = \langle !_X, 1\rangle\pi_1 h\langle !_A, 1\rangle\tau_{1,X} = h\langle !_A, 1\rangle\tau_{1,X} \\
&= \langle !_A, h\rangle\tau_{1,X} = \langle !_X, 1\rangle(1 \times h)\tau_{1,X} = \langle !_X, 1\rangle\pi_1 = 1
\end{aligned}$$

as required. $\square$

Definition 4.11. A *Turing morphism* for an object A in a cartesian restriction category $\mathbb{A}$ is a universal application map $\bullet : A \times A \to A$.

If A is a Turing object, clearly $\tau_{A,A}$ is a Turing morphism for A, so the Turing object of every Turing category posesses a Turing morphism. The converse does not hold. While an object with a Turing morphism is a Turing object in its computable map category, it need not be a Turing object in the ambient category. However, if the object in question is also a universal object, then it *is* a Turing object in the ambient category:

Lemma 4.12. *Let $\mathbb{A}$ be a cartesian restriction category, A be a universal object in $\mathbb{A}$, and $\bullet : A \times A \to A$ be a Turing morphism for A. Then A is a Turing object in $\mathbb{A}$.*

Proof. It suffices define a universal application map $\tau_{X,Y} : X \times A \to Y$ for every two objects X, Y of $\mathbb{A}$. To that end, Define $\tau_{X,Y}$ by

$$X \times A \xrightarrow{s_X \times 1} A \times A \xrightarrow{\bullet} A \xrightarrow{r_Y} Y$$

where $(s_X, r_X) : X \triangleleft A$, $(s_Y, r_Y) : Y \triangleleft A$. Now, let $f : X \times Z \to Y$, and let $h : Z \to A$ be a $\bullet$-index for $(r_X \times 1) f s_Y : A \times Z \to A$ as in

$$\begin{tikzcd}
A \times A \arrow[r, "\bullet"] & A \\
A \times Z \arrow[u, "1 \times h"] \arrow[ur, "(r_X \times 1) f s_Y"'] &
\end{tikzcd}$$

Then we have

$$\begin{tikzcd}
X \times A \arrow[r, "s_X \times 1"] & A \times A \arrow[r, "\bullet"] & A \arrow[r, "r_Y"] & Y \\
X \times Z \arrow[u, "1 \times h"] \arrow[r, "s_X \times 1"'] & A \times Z \arrow[u, "1 \times h"] \arrow[r, "r_X \times 1"'] & X \times Z \arrow[u, "f s_Y"] \arrow[ur, "f"'] &
\end{tikzcd}$$

but $\tau_{X,Y} = (s_X \times 1) \bullet r_Y$ and $f = (s_X \times 1)(r_X \times 1) f$ means we then have

$$\begin{tikzcd}
X \times A \arrow[r, "\tau_{X,Y}"] & Y \\
X \times Z \arrow[u, "1 \times h"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and so $h : Z \to A$ is a $\tau_{X,Y}$-index for f. Since $f : X \times Z \to Y$ was arbitrary, we have shown that $\tau_{X,Y}$ is a universal application map, and then since X and Y were arbitrary we have that A is a Turing object in $\mathbb{A}$.

□

We have now shown

Theorem 4.13. *The following are equivalent for a cartesian restriction category $\mathbb{A}$:*

(i) $\mathbb{A}$ *is a Turing category.*

(ii) There is a universal object A of $\mathbb{A}$ with a Turing morphism $\bullet : A \times A \to A$.

The relationship to partial combinatory algebras is now clear. The computable map category of a partial combinatory algebra is a Turing category, and Turing object and Turing morphism form a partial combinatory algebra in any Turing category.

The appropriate notion of a morphism between Turing categories would appear to be an *applicative morphism* in the sense of Longley [29], the details of which have been worked out for very general settings, including Turing categories, in [9]. While one could certainly ask about the relationship between these generalized applicative morphisms and the constructions on Turing categories we present in the sequel, we leave this for future work.

Note also that the idempotent splitting of every Turing category is also a Turing category [8].

While Turing categories need not be cartesian closed restriction categories due to the presence of multiple potential exponential transposes of a given map (like the way each partial recursive function is indexed by many elements of $\mathbb{N}$), they do have the rest of the cartesian closed structure. This is captured as follows:

Definition 4.14. A cartesian restriction category $\mathbb{X}$ is *weakly cartesian closed* in case for each pair A, B of objects of $\mathbb{X}$ there is an object $T_{A,B}$ and a map $\tau_{A,B} : A \times T_{A,B} \to B$ such that for every map $f : A \times C \to B$ of $\mathbb{X}$ there is a (not necessarily unique) total map $h : C \to T_{A,B}$ for which

$$\begin{tikzcd} A \times T_{A,B} \arrow[r, "\tau_{A,B}"] & B \\ A \times C \arrow[u, "1\times h"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. We call such an h a *weak exponential transpose* of f.

Lemma 4.15. *Every Turing category is weakly cartesian closed.*

Proof. Let T be the Turing object. Then $T_{A,B} := T$ for every pair of objects A, B, and we are done. $\square$

Similarly, while Turing categories need not have coproducts, they do admit the following, weaker structure:

Definition 4.16. A restriction category $\mathbb{X}$ is said to have *weak binary coproducts* in case for every pair A, B of objects of $\mathbb{X}$, there is an object $A\#B$ of $\mathbb{X}$ with total maps $\kappa : A \to A\#B$ and $\kappa' : B \to A\#B$ such that if $f : A \to C$ and $g : B \to C$ are maps in $\mathbb{X}$, there exists a map $\nu : A\#B \to C$ making

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[dr, "f"'] & A\#B \arrow[d, "\nu"] & B \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commute.

Lemma 4.17. *Every Turing category has weak binary coproducts.*

Proof. Let T be the Turing object, and let $\tau : (T \times T) \times T \to T$ be a universal application map. Define total maps $p_0, p_1 : T \to T$ as follows

$$\begin{tikzcd} (T \times T) \times T \arrow[r, "\tau"] & T \\ (T \times T) \times T \arrow[u, "1\times p_0"] \arrow[ur, "\pi_0\pi_0"'] & \end{tikzcd} \qquad \begin{tikzcd} (T \times T) \times T \arrow[r, "\tau"] & T \\ (T \times T) \times T \arrow[u, "1\times p_1"] \arrow[ur, "\pi_0\pi_1"'] & \end{tikzcd}$$

And define the weak coproduct of A and B to be

$$\begin{tikzcd} A \arrow[r, "\kappa"] & A\#B & B \arrow[l, "\kappa'"'] \end{tikzcd}$$

where $A\#B := T \times T$, $\kappa := s_A\langle 1, p_0\rangle$, and $\kappa' := s_B\langle 1, p_1\rangle$.

Now, suppose $f : A \to C$ and $g : B \to C$. Define total maps $c_f : 1 \to T$ and $c_g : 1 \to T$ as in

$$\begin{tikzcd} T \times T \arrow[r, "\tau"] & T \\ T \times 1 \arrow[u, "1\times c_f"] \arrow[ur, "\pi_0 r_A f s_C"'] \end{tikzcd} \qquad \begin{tikzcd} T \times T \arrow[r, "\tau"] & T \\ T \times 1 \arrow[u, "1\times c_g"] \arrow[ur, "\pi_0 r_B g s_C"'] \end{tikzcd}$$

and define $\nu : T \times T \to C$ by

$$\nu := (1 \times \langle\langle !c_f, !c_g\rangle, 1\rangle\tau)\tau r_C$$

We must show that

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[dr, "f"'] & T \times T \arrow[d, "\nu"] & B \arrow[l, "\kappa'"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes, which we do now:

$$\kappa\nu = s_A\langle 1, p_0\rangle(1 \times \langle\langle !c_f, !c_g\rangle, 1\rangle\tau)\tau r_C$$

$$= s_A\langle 1, p_0\langle\langle !c_f, !c_g\rangle, 1\rangle\tau\rangle\tau r_C$$

$$= s_A\langle 1, \langle\langle !c_f, !c_g\rangle, 1\rangle(1 \times p_0)\tau\rangle\tau r_C$$

$$= s_A\langle 1, \langle\langle !c_f, !c_g\rangle, 1\rangle\pi_0\pi_0\rangle\tau r_C$$

$$= s_A\langle 1, !c_f\rangle\tau r_C = \langle s_A, !\rangle(1 \times c_f)\tau r_C$$

$$= \langle s_A, !\rangle\pi_0 r_A f s_C r_C = f$$

and

$$\kappa'\nu = s_B\langle 1, p_1\rangle(1 \times \langle\langle !c_f, !c_g\rangle, 1\rangle\tau)\tau r_C$$

$$= s_B\langle 1, p_1\langle\langle !c_f, !c_g\rangle, 1\rangle\tau\rangle\tau r_C$$

$$= s_B\langle 1, \langle\langle !c_f, !c_g\rangle, 1\rangle(1 \times p_1)\tau\rangle\tau r_C$$

$$= s_B\langle 1, \langle\langle !c_f, !c_g\rangle, 1\rangle\pi_0\pi_1\rangle\tau r_C$$

$$= s_B\langle 1, !c_g\rangle\tau r_C = \langle s_B, !\rangle(1 \times c_g)\tau r_C$$

$$= \langle s_B, !\rangle\pi_0 r_B g s_C r_C = g$$

This weak structure will be useful later. When we construct the category of assemblies, strong structure in the base category will be present in the category of assemblies only when the category of realizers, from which tracking maps are drawn, has the corresponding weak structure.

$\square$

5 Assemblies

Given a partial combinatory algebra $A = (A, \bullet)$ in the category of sets and partial functions, we can construct the category of assemblies on A [36]. This category can be understood as a "universe of computable functions" generated by A. Its objects are the "datatypes" that can be represented with A, and its morphisms can be seen as the "A-computable" maps between these datatypes.

In this chapter, we introduce a generalization of the category of assemblies. We begin by recalling the classical construction. For a partial combinatory algebra $(A, \bullet)$ in the category of sets and partial functions, the category of assemblies is defined as follows:

objects are pairs (X, φ) where X is a set and $\varphi : X \to \mathcal{P}^*(A)$ is a function assigning a nonempty subset of A to each element of X.

arrows $f : (X, \varphi) \to (Y, \psi)$ are (total) maps $f : X \to Y$ in the category of (total) functions on sets that are *tracked* by some element of A. That is, there exists an element $a \in A$ satisfying

$$\forall x \in X. \forall b \in A. b \in \varphi(x) \Rightarrow ((a \bullet b) \downarrow \wedge (a \bullet b) \in \psi(f(x)))$$

In this case we say f *is tracked by* a.

composition and **identities** are given by composition and identites in the category of functions on sets. 1_X is tracked by $\langle x \rangle x$, giving $1_{(X,\varphi)} : (X, \varphi) \to (X, \varphi)$, and if $f : (X, \varphi) \to (Y, \psi)$ and $g : (Y, \psi) \to (Z, \chi)$ are given by functions $f : X \to Y$, $g : Y \to Z$ and tracked by $a, b \in A$ respectively, then the composite fg is tracked by $\langle x \rangle b(ax)$, giving the composite $fg : (X, \varphi) \to (Z, \chi)$.

For example, for $n \in \mathbb{N}$, let $\overline{n} \in A$ be the nth Church numeral in A, and consider the assembly $(\mathbb{N}, \omega)$ where $\omega : \mathbb{N} \to \mathcal{P}^*(A)$ is defined by $\omega(n) = \{\overline{n}\}$. A map

$f : (\mathbb{N}, \omega) \to (\mathbb{N}, \omega)$ is then a function $f : \mathbb{N} \to \mathbb{N}$ such that there exists some $a \in A$ with

$$\forall x \in \mathbb{N}.(a \bullet \overline{x}) \downarrow \wedge (a \bullet \overline{x}) = \overline{f(n)}$$

which says that there is an A-computable map that does what f does to elements of $\mathbb{N}$ to the elements of A that represent them.

This category of assemblies is regular, cartesian closed, and has finite colimits. [29] [36].

Birkedal [4] gave a more general construction of the category of assemblies, in which the partial combinatory algebra is replaced by a weakly cartesian closed restriction category. He used a cartesian restriction functor from this category to partial functions on sets to construct a realizability tripos and category of assemblies that capture the traditional category of assemblies and realizability tripos as special cases.

We take this approach further, replacing both the weakly cartesian closed restriction category and category of partial functions on sets by arbitrary cartesian restriction categories, and consider a cartesian restriction functor between them.

5.1 Assemblies for a Restriction Functor

Definition 5.1. Let $\mathbb{A}$ be a restriction category, $\mathbb{X}$ be a cartesian restriction category, and $F : \mathbb{A} \to \mathbb{X}$ be a restriction functor. The *restriction category of assemblies on F*, written $\mathsf{asm}(F)$, is given by

objects are restriction idempotents $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times Y)$, and so on for A, B objects of $\mathbb{A}$, X, Y objects of $\mathbb{X}$. We refer to objects of $\mathsf{asm}(F)$ as *assemblies.*

arrows $\varphi \xrightarrow{f} \psi$ between assemblies $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$ are maps $f : X \to Y$ of $\mathbb{X}$ for which there exists a *tracking map* $\gamma : A \to B$ in $\mathbb{A}$. That is, a map $\gamma : A \to B$ satisfying

[Tk.1] $\varphi(F(\gamma) \times f) = \varphi(F(\gamma) \times f)\psi$

[Tk.2] $\overline{\varphi(1 \times f)} = \overline{\varphi(F(\gamma) \times f)}$

composition of arrows is given by composition of the corresponding arrows in $\mathbb{X}$.

identities The identity map for $\varphi \in \mathcal{O}(F(A) \times X)$ is given by 1_X in $\mathbb{X}$.

restriction is as in $\mathbb{X}$. That is, the domain of definition of $\varphi \xrightarrow{f} \psi$, corresponding to $f : X \to Y$ in $\mathbb{X}$, is $\varphi \xrightarrow{\overline{f}} \varphi$, corresponding to $\overline{f} : X \to X$ in $\mathbb{X}$.

When talking about $\mathsf{asm}(F)$ for a restriction functor $F : \mathbb{A} \to \mathbb{X}$, we will refer to $\mathbb{X}$ as the *base category* and $\mathbb{A}$ as the *category of realizers*. To understand the tracking identities, we can think of **[Tk.1]** as insisting that for the part of $F(A) \times X$ on which φ is defined, the image of $F(\gamma) \times f$ is in the part of $F(B) \times Y$ on which ψ is defined, and we can think of **[Tk.2]** as insisting that for the part of $F(A) \times X$ on which φ is defined, if f is defined, then so is $F(\gamma)$.

Proposition 5.2. $\mathsf{asm}(F)$ *is a restriction category.*

Proof. To establish this, we must show that the composition, identities, and restriction above are well-defined in the following sense. For composition, if f, g are composable maps of $\mathsf{asm}(F)$ corresponding to (necessarily composable) maps f, g of $\mathbb{X}$, then the composite fg in $\mathbb{X}$ must give a map in $\mathsf{asm}(F)$. That is, it must be tracked by some map γ of $\mathbb{A}$. Similarly, for identities and restriction we must show that 1_X gives a map $\varphi \to \varphi$ for any assembly $\varphi \in \mathcal{O}(F(A) \times X)$, and that if $\varphi \xrightarrow{f} \psi$ is a map of $\mathsf{asm}(F)$ corresponding to $f : X \to Y$ in $\mathbb{X}$, then $\overline{f} : X \to X$ gives a map $\varphi \xrightarrow{\overline{f}} \varphi$ of $\mathsf{asm}(F)$. Once we have shown this, we are done. That composition is associative, identities are left and right identites, and that the restriction operator satisfies the restriction axioms follows from the fact that this is the case in $\mathbb{X}$.

We begin with composition. If $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times Y)$, $\chi \in \mathcal{O}(F(C) \times Z)$ are assemblies, with $\varphi \xrightarrow{f} \psi$ and $\psi \xrightarrow{g} \chi$ arrows of $\mathsf{asm}(F)$, then there exist tracking maps $\gamma : A \to B$, $\delta : B \to C$ for f and g respectively. Using the fact that the tracking identities are satisfied for these pairs of maps, we show that $\gamma\delta$ tracks fg, giving an arrow of assemblies $\varphi \xrightarrow{fg} \chi$ by checking the tracking identities

[Tk.1]

$$\begin{aligned}
&\varphi(F(\gamma\delta) \times fg) = \varphi(F(\gamma) \times f)(F(\delta) \times g) = \varphi(F(\gamma) \times f)\psi(F(\delta) \times g) \\
&= \varphi(F(\gamma) \times f)\psi(F(\delta) \times g)\chi = \varphi(F(\gamma) \times f)(F(\delta) \times g)\chi \\
&= \varphi(F(\gamma\delta) \times fg)\chi
\end{aligned}$$

[Tk.2]

$$\begin{aligned}
&\overline{\varphi(F(\gamma\delta) \times fg)} = \overline{\varphi(F(\gamma) \times f)\psi(F(\delta) \times g)} = \overline{\varphi(F(\gamma) \times f)\overline{\psi(F(\delta \times g))}} \\
&= \overline{\varphi(F(\gamma) \times f)\psi(1 \times g)} = \overline{\varphi(F(\gamma) \times fg)} = \overline{\varphi(1 \times fg)(F(\gamma) \times 1)} \\
&= \overline{\varphi(1 \times \overline{f})(1 \times fg)(\overline{F(\gamma)} \times 1)} = \overline{\varphi(\overline{F(\gamma)} \times \overline{f})(1 \times fg)} = \overline{\varphi\,\overline{(F(\gamma) \times f)}(1 \times fg)} \\
&= \overline{\overline{\varphi(F(\gamma) \times f)}\varphi(1 \times fg)} = \overline{\overline{\varphi(1 \times f)}\varphi(1 \times f)(1 \times g)} = \overline{\varphi(1 \times fg)}
\end{aligned}$$

and so fg is again an arrow in $\mathsf{asm}(F)$. Composition is associative because composition in $\mathbb{X}$ is associative.

For identity maps, $\varphi \xrightarrow{1_\varphi} \varphi$ is given by 1_X in $\mathbb{X}$, tracked by 1_A in $\mathbb{A}$ since

[Tk.1]

$$\varphi(F(1) \times 1) = \varphi = \varphi\varphi = \varphi(F(1) \times 1)\varphi$$

[Tk.2]

$$\overline{\varphi(F(1) \times 1)} = \overline{\varphi(1 \times 1)}$$

The left and right identity axioms hold because they do in $\mathbb{X}$.

For the restriction structure, if $f\varphi \to \psi$ is a map in $\mathsf{asm}(F)$ corresponding to $f : X \to Y$ in $\mathbb{X}$, $\overline{f} : X \to X$ is tracked by the identity map as

[Tk.1]

$$\varphi(F(1_A) \times \overline{f})\varphi = \varphi\overline{(1 \times f)}\varphi = \overline{\varphi(1 \times f)}\varphi\varphi$$
$$= \overline{\varphi(1 \times f)}\varphi = \varphi\overline{(1 \times f)} = \varphi(F(1_A) \times \overline{f})$$

[Tk.2]

$$\overline{\varphi(F(1_A) \times \overline{f})} = \overline{\varphi(1 \times \overline{F})}$$

and thus gives a map $\varphi \xrightarrow{\overline{f}} \varphi$ of $\mathsf{asm}(F)$. The four restriction category axioms are satisfied because they are satisfied in $\mathbb{X}$. $\square$

Notice that when $\mathbb{A}$ is the computable map category of a partial combinatory algebra $(A, \bullet)$ in partial functions on sets, and $\mathbb{X}$ is the category of partial functions on sets itself, with $I : \mathbb{A} \to \mathbb{X}$ the inclusion functor, the category of assemblies $\mathsf{asm}(I)$ is as follows

objects are restriction idempotents $\varphi \in \mathcal{O}(A \times X)$ in the category of partial functions on sets for any set X. Such an idempotent determines a relation between A and X, which in turn determines a function $X \to \mathcal{P}^*(A)$. Therefore, an equivalent characterization of the objects of $\mathsf{asm}(I)$ is as pairs (X, φ) where X is a set and $\varphi : X \to \mathcal{P}^*(A)$ is a function mapping each element of X to a nonempty subset of A.

arrows $(X, \varphi) \to (Y, \psi)$ are partial functions $f : X \to Y$ for which there exists a tracking map γ satisfying

[Tk.1]

$$\forall x \in X.\forall b \in A.(b \in \varphi(x) \wedge f(x)\downarrow \wedge \gamma(b)\downarrow) \Rightarrow \gamma(b) \in \psi(f(x))$$

[Tk.2]

$$\forall x \in X.\forall b \in A.(b \in \varphi(x) \wedge f(x)\downarrow) \Rightarrow \gamma(b)\downarrow$$

which is the case precisely when

$$\forall x \in X.\forall b \in A.(b \in \varphi(x) \wedge f(x)\downarrow) \Rightarrow (\gamma(b)\downarrow \wedge \gamma(b) \in \psi(f(x)))$$

holds. Since γ is an A-computable map, we can restate this in terms of a tracking *element* $a \in A$ such that

$$\forall x \in X.\forall b \in A.(b \in \varphi(x) \wedge f(x)\downarrow) \Rightarrow ((a \bullet b)\downarrow \wedge (a \bullet b) \in \psi(f(x)))$$

The only difference between $\mathsf{asm}(I)$ and the traditional category of assemblies is that maps in $\mathsf{asm}(I)$ are given by partial maps between sets, while traditionally these maps must be total. When we consider only the total maps, the tracking identities can be further simplified to

$$\forall x \in X.\forall b \in A.x \in \varphi(x) \Rightarrow ((a \bullet b)\downarrow \wedge (a \bullet b) \in \psi(f(x)))$$

since $f(x)\downarrow$ always. Thus, $\mathsf{total}(\mathsf{asm}(I))$ is the traditional category of assemblies of the partial combinatory algebra $(A, \bullet)$.

Having related it to the traditional one [36], we proceed to investigate the categorical structure of our construction. Partial product structure is straightforward to establish:

Proposition 5.3. *If $\mathbb{A}$ and $\mathbb{X}$ are cartesian restriction categories and $F : \mathbb{A} \to \mathbb{X}$ is* *a cartesian restriction functor, then* $\mathsf{asm}(F)$ *is a cartesian restriction category*

Proof. We must show that $\mathsf{asm}(F)$ has both a restriction terminal object, and restriction products. As in the proof that $\mathsf{asm}(F)$ is a restriction category, this involves showing that certain maps in the base category $\mathbb{X}$ have tracking maps in $\mathbb{A}$, and so give maps in $\mathsf{asm}(F)$. That these maps satisfy the required identities is then, as before, a consequence of the corresponding maps in $\mathbb{X}$ satisfying those identities.

The restriction terminal object in $\mathsf{asm}(F)$ is the assembly given by the identity map $1 \in \mathcal{O}(F(1) \times 1)$ in $\mathbb{X}$. The unique total map into the terminal object $\varphi \xrightarrow{!_\varphi} 1$ is given by the map $!_X : X \to 1$ in $\mathbb{X}$, which is tracked by $!_A : A \to 1$ in $\mathbb{A}$ since

[Tk.1]

$$\varphi(F(!)\times!)1 = \varphi(F(!)\times!)$$

[Tk.2]

$$\overline{\varphi(F(!)\times!)} = \overline{\varphi(\overline{F(!)} \times \overline{!})} = \overline{\varphi(1 \times \overline{!})} = \overline{\varphi(1\times!)}$$

and thus gives a map in $\mathsf{asm}(F)$. This map satisfies the requirements in the definition of restriction terminal object because $!_X$ satisfies those requirements in $\mathbb{X}$, and so $\mathsf{asm}(F)$ has a restriction terminal object.

The restriction product of two assemblies $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$ is defined by

$$(\varphi * \psi) \in \mathcal{O}(F(A \times B) \times X \times Y)$$
$$(\varphi * \psi) := \mathsf{ex}(\varphi \times \psi)\mathsf{ex}$$

where ex denotes the interchange map, the total map which swaps the position of the two middle components of a four-component product. We note that, since F preserves products, $F(A \times B) \times X \times Y = F(A) \times F(B) \times X \times Y$, allowing the use of the interchange map. For this to be an object of $\mathsf{asm}(F)$, $(\varphi * \psi)$ must be a restriction idempotent. Using that $\mathsf{exex} = 1$, we have

$$\overline{(\varphi * \psi)} = \overline{\mathsf{ex}(\varphi \times \psi)\mathsf{ex}} = \overline{\mathsf{ex}(\varphi \times \psi)} = \overline{\mathsf{ex}(\varphi \times \psi)}\mathsf{exex}$$

$$= \mathsf{ex}\overline{(\varphi \times \psi)}\mathsf{ex} = \mathsf{ex}(\overline{\varphi} \times \overline{\psi})\mathsf{ex} = \mathsf{ex}(\varphi \times \psi)\mathsf{ex} = (\varphi * \psi)$$

as required. The projection maps $(\varphi * \psi) \xrightarrow{\pi_0} \varphi$ and $(\varphi * \psi) \xrightarrow{\pi_1} \psi$ are the corresponding projections $\pi_0 : X \times Y \to X$, $\pi_1 : X \times Y \to Y$ in $\mathbb{X}$. The tracking maps are $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ respectively. For π_0:

[Tk.1]

$$(\varphi * \psi)(F(\pi_0) \times \pi_0)\varphi = \mathsf{ex}(\varphi \times \psi)\mathsf{ex}(\pi_0 \times \pi_0)\varphi = \mathsf{ex}(\varphi \times \psi)\pi_0\varphi$$

$$= \mathsf{ex}\langle \pi_0\varphi, \pi_1\psi\rangle\pi_0\varphi = \mathsf{ex}\overline{\pi_1\psi}\pi_0\varphi\varphi = \mathsf{ex}\overline{\pi_1\psi}\pi_0\varphi = \mathsf{ex}\langle \pi_0\varphi, \pi_1\psi\rangle\pi_0$$

$$= \mathsf{ex}(\varphi \times \psi)\pi_0 = \mathsf{ex}(\varphi \times \psi)\mathsf{ex}(F(\pi_0) \times \pi_0) = (\varphi * \psi)(F(\pi_0) \times \pi_0)$$

[Tk.2]

$$\overline{(\varphi * \psi)(F(\pi_0) \times \pi_0)} = \overline{(\varphi * \psi)(\overline{F(\pi_0)} \times \overline{\pi_0})} = \overline{(\varphi * \psi)(1 \times \pi_0)}$$

with the tracking identities for π_1 following from a similar argument. We proceed to show that the diagram

$$\varphi \xleftarrow{\pi_0} (\varphi * \psi) \xrightarrow{\pi_1} \psi$$

is a restriction product in $\mathsf{asm}(F)$. To that end, suppose that $\chi \in \mathcal{O}(F(C) \times Z)$ is an assembly, and that $\chi \xrightarrow{f} \varphi$ and $\chi \xrightarrow{g} \psi$ are arrows of $\mathsf{asm}(F)$ corresponding to arrows $f : Z \to X$ tracked by $\gamma : C \to A$ and $g : Z \to Y$ tracked by $\delta : C \to B$ respectively. We define the mediating map $\chi \xrightarrow{\langle f,g\rangle} (\varphi * \psi)$ to be the mediating map $\langle f, g\rangle : Z \to X \times Y$ in $\mathbb{X}$, which is tracked by $\langle \gamma, \delta\rangle : C \to A \times B$ as follows

[Tk.1]

$$\chi(F(\langle\gamma,\delta\rangle)\times\langle f,g\rangle)(\varphi*\psi) = \chi(F(\Delta)\times\Delta)(F(\gamma\times\delta)\times f\times g)(\varphi*\psi)$$

$$= \chi(F(\Delta)\times\Delta)(F(\gamma)\times F(\delta)\times f\times g)\mathsf{ex}(\varphi\times\psi)\mathsf{ex}$$

$$= \chi(\Delta\times\Delta)\mathsf{ex}(F(\gamma)\times f\times F(\delta)\times g)(\varphi\times\psi)\mathsf{ex}$$

$$= \chi\Delta((F(\gamma)\times f)\varphi\times(F(\delta)\times g)\psi)\mathsf{ex}$$

$$= \Delta(\chi(F(\gamma)\times f)\varphi\times\chi(F(\delta)\times g)\psi)\mathsf{ex}$$

$$= \Delta(\chi(F(\gamma)\times f)\times\chi(F(\delta)\times g))\mathsf{ex}$$

$$= \chi(\Delta\times\Delta)\mathsf{ex}(F(\gamma)\times f\times F(\delta)\times g)\mathsf{ex}$$

$$= \chi(F(\Delta)\times\Delta)(F(\gamma\times\delta)\times f\times g)\mathsf{exex}$$

$$= \chi(F(\langle\gamma,\delta\rangle)\times\langle f,g\rangle)$$

[Tk.2]

$$\overline{\chi(F(\langle\gamma,\delta\rangle)\times\langle f,g\rangle)} = \overline{\chi(F(\Delta)\times\Delta)(F(\gamma\times\delta)\times f\times g)}$$

$$= \overline{\chi(\Delta\times\Delta)(F(\gamma)\times F(\delta)\times f\times g)\mathsf{ex}} = \overline{\chi\Delta(F(\gamma)\times f\times F(\delta)\times g)}$$

$$= \overline{\Delta(\chi(F(\gamma)\times f)\times\chi(F(\delta)\times g))} = \overline{\Delta(\overline{\chi(F(\gamma)\times f)}\times\overline{\chi(F(\delta)\times g)})}$$

$$= \overline{\Delta(\overline{\chi(1\times f)}\times\overline{\chi(1\times g)})} = \overline{\Delta(\chi(1\times f)\times\chi(1\times g))}$$

$$= \overline{\chi(\Delta\times\Delta)\mathsf{ex}(1\times f\times 1\times g)} = \overline{\chi(F(\Delta)\times\Delta)(1\times 1\times f\times g)}$$

$$= \overline{\chi(1\times\langle f,g\rangle)}$$

and so $\langle f,g\rangle$ is well defined. That it is the unique map for which

$$\begin{tikzcd}
 & \chi \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
\varphi & (\varphi*\psi) \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & \psi
\end{tikzcd}$$

commutes with $\langle f,g\rangle\pi_0 = \overline{g}f$ and $\langle f,g\rangle\pi_1 = \overline{f}g$ follows from this being the case in

$\mathbb{X}$. In some sense the entire restriction product structure has been inherited from $\mathbb{X}$. We have now shown that restriction products exist in $\mathsf{asm}(F)$, and in turn that it is a cartesian restriction category. $\square$

Before moving on to more complex structure in the category of assemblies, we note that the functor $F : \mathbb{A} \to \mathbb{X}$ allows us to define another functor $H : \mathbb{A} \to \mathsf{asm}(F)$.

Lemma 5.4. *There is a restriction functor $H : \mathbb{A} \to \mathsf{asm}(F)$.*

Proof. On objects A of $\mathbb{A}$, define $H(A)$ to be the assembly $\overline{\Delta^{(-1)}} \in \mathcal{O}(F(A) \times F(A))$. On maps $f : A \to B$ in $\mathbb{A}$, define $H(f) : H(A) \to H(B)$ to be the map given by $F(f) : F(A) \to F(B)$ in $\mathbb{X}$. $F(f)$ is tracked by f in $\mathbb{A}$ as follows:

[Tk.1]

$$\begin{aligned}
&\overline{\Delta^{(-1)}}(F(f) \times F(f))\overline{\Delta^{(-1)}} = \Delta^{(-1)}\Delta(F(f) \times F(f))\Delta^{(-1)}\Delta \\
&= \Delta^{(-1)}F(f)\Delta\Delta^{(-1)}\Delta = \Delta^{(-1)}F(f)\Delta \\
&= \Delta^{(-1)}\Delta(F(f) \times F(f)) = \overline{\Delta^{(-1)}}(F(f) \times F(f))
\end{aligned}$$

[Tk.2]

$$\begin{aligned}
&\overline{\overline{\Delta^{(-1)}}(F(f) \times F(f))} = \overline{\Delta^{(-1)}\Delta\langle \pi_0 F(f), \pi_1 F(f)\rangle} \\
&= \overline{\Delta^{(-1)}\langle \Delta\pi_0 F(f), \Delta\pi_1 F(f)\rangle} = \overline{\Delta^{(-1)}\langle F(f), F(f)\rangle} \\
&= \overline{\Delta^{(-1)}\overline{F(f)}} = \overline{\Delta^{(-1)}\langle 1, F(f)\rangle} \\
&= \overline{\Delta^{(-1)}\langle \Delta\pi_0, \Delta\pi_1 F(f)\rangle} = \overline{\Delta^{(-1)}\Delta\langle \pi_0, \pi_1 F(f)\rangle} \\
&= \overline{\overline{\Delta^{(-1)}}(1 \times F(f))}
\end{aligned}$$

Thus, $H(f)$ is well-defined. That H is a restriction functor follows immediately from F being a restriction functor. $\square$

We remark that H inherits many other properties of F in the same way. For example, if F is a cartesian restriction functor, so is H, if F is faithful (or even an embedding), so is H, and if F preserves joins, so does H.

5.2 Adding Structure

Our eventual goal is to construct the realizability tripos associated with the category of assemblies. For this, we will require more categorical structure, which in turn requires stronger assumptions about the cartesian restriction categories involved. Recall that the traditional category of assemblies is regular, cartesian closed, and has finite colimits. Each of these properties has a restriction categorical counterpart. Regularity is related to having ranges [22] [23], discreteness ensures the existence of finite restriction colimits, and the counterpart of a cartesian closed category is a cartesian closed restriction category.

We show that, with no additional assumptions on the category of realizers, if the base category has ranges, so does the category of assemblies, and that if the base category is discrete, so is the category of assemblies. With the additional assumption that the category of realizers is a weakly cartesian closed restriction category (such as a Turing category) we show that if the base category is a cartesian closed restriction category, so is the category of assemblies. This is not all strictly necessary for the construction of the realizability tripos as every discrete cartesian closed restriction category necessarily has ranges, but it is interesting that the category of realizers is required only to be a cartesian restriction category for the category of assemblies to inherit the discrete and range structure of the base category. We complete the story by showing that if the base category has finite joins and the category of realizers is a discrete cartesian restriction category with finite interleaving, then the category of assemblies has finite joins.

Lemma 5.5. *If $\varphi \in \mathcal{O}(F(A) \times X)$ is an assembly and $e \in \mathcal{O}(X)$ is a restriction* *idempotent in $\mathbb{X}$, then e is tracked by the identity map 1_A, and gives a restriction idempotent $e \in \mathcal{O}(\varphi)$ in $\mathsf{asm}(F)$.*

Proof. We need only show that the two tracking identities are satisfied:

[Tk.1]

$$\varphi(F(1_A) \times e)\varphi = \varphi\overline{(F(1_A) \times e)}\varphi = \varphi\varphi\overline{(F(1_A) \times e)}$$
$$= \varphi(F(1_A) \times e)$$

[Tk.2]

$$\overline{\varphi(F(1_A) \times e)} = \overline{\varphi(1_{F(A)} \times e)}$$

as required. □

Corollary 5.6. *If $\mathbb{X}$ has universal quantification, so does $\mathsf{asm}(F)$. If $\mathbb{X}$ has existential quantification, so does $\mathsf{asm}(F)$. In particular, this means that if $\mathbb{X}$ has ranges, so does $\mathsf{asm}(F)$, as having existential quantification implies that each map is open. (See [22] [23]).*

Proposition 5.7. *If $\mathbb{A}$ is a cartesian restriction category, $\mathbb{X}$ is a* discrete *cartesian restriction category, and $F : \mathbb{A} \to \mathbb{X}$ is a cartesian restriction functor, then $\mathsf{asm}(F)$ is a discrete cartesian restriction category.*

Proof. It suffices to show that for two maps $f, g : \varphi \to \psi$ of $\mathsf{asm}(F)$ where $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times Y)$, with underlying maps $f, g : X \to Y$ in $\mathbb{X}$, tracked by $a, b : A \to B$ in $\mathbb{A}$ respectively, the meet $f \cap g : X \to Y$ is tracked, giving a map $\varphi \to \psi$ of $\mathsf{asm}(F)$. In that case, the meet identities are satisfied because they hold in $\mathbb{X}$, and $\mathsf{asm}(F)$ is therefore discrete. We actually have two tracking maps for $f \cap g$, as either of a, b suffices. We show that a tracks the meet

[Tk.1]

$$
\begin{aligned}
&\varphi(F(a) \times (f \cap g))\psi = \varphi(F(a) \times \overline{(f \cap g)}\, f)\psi \\
&= \varphi\overline{(1 \times (f \cap g))}(F(a) \times f)\psi = \overline{(1 \times (f \cap g))}\varphi(F(a) \times f)\psi \\
&= \overline{(1 \times (f \cap g))}\varphi(F(a) \times f) = \varphi\overline{(1 \times (f \cap g))}(F(a) \times f) \\
&= \varphi(F(a) \times \overline{(f \cap g)}\, f) = \varphi(F(a) \times (f \cap g))
\end{aligned}
$$

[Tk.2]

$$
\begin{aligned}
&\overline{\varphi(F(a) \times (f \cap g))} = \overline{\varphi\overline{(1 \times (f \cap g))}(F(a) \times f)} \\
&= \overline{\overline{(1 \times (f \cap g))\varphi(F(a) \times f)}} = \overline{\overline{(1 \times (f \cap g))}\varphi(1 \times f)} \\
&= \overline{\varphi(1 \times \overline{(f \cap g)}\, f)} = \overline{\varphi(1 \times (f \cap g))}
\end{aligned}
$$

as required. $\square$

Proposition 5.8. *If $\mathbb{A}$ is a weakly cartesian closed restriction category, $\mathbb{X}$ is a discrete cartesian closed restriction category, and $F : \mathbb{A} \to \mathbb{X}$ is a cartesian restriction functor, then $\mathsf{asm}(F)$ is a cartesian closed restriction category.*

Proof. We write $T_{A,B}$ for the weak exponential of A, B in $\mathbb{A}$, and write the corresponding evaulation map as $\tau_{A,B} : A \times T_{A,B} \to B$.

Let $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$ be objects of $\mathsf{asm}(F)$. We define the exponential assembly $\psi^{\varphi} \in \mathcal{O}(F(T_{A,B}) \times Y^X)$ as follows

$$
\begin{aligned}
\psi^{\varphi} :=\, &\overline{\lambda((\varphi \times 1)\mathsf{ex}(F(\tau_{A,B}) \times \mathsf{ev})) \cap \lambda((\varphi \times 1)\mathsf{ex}(F(\tau_{A,B}) \times \mathsf{ev})\psi)} \\
&\overline{\lambda((\varphi \times 1)\mathsf{ex}(F(\tau_{A,B}) \times \mathsf{ev})!) \cap \lambda((\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!)}
\end{aligned}
$$

where the typing is given by $\mathsf{ev} : X \times Y^X \to Y$. The idea being that a map of assemblies ψ^φ is a map Y^X of $\mathbb{X}$ together with a tracking map in $T_{A,B}$ of $\mathbb{A}$. Our exponential idempotent $\psi^\varphi \in \mathcal{O}(F(T_{A,B}) \times Y^X)$ is defined precisely when the $F(T_{A,B})$ component of the product tracks the Y^X component as a map $\varphi \to \psi$ in $\mathsf{asm}(F)$. The left and right conjuncts that form ψ^φ ensure that tracking identities [**Tk.1**] and [**Tk.2**] hold, respectively.

The evaluation map $\varphi * \psi^\varphi \xrightarrow{\mathsf{ev}} \psi$ is given by $\mathsf{ev} : X \times Y^X \to X$ in $\mathbb{X}$, with tracking map $\tau_{A,B} : A \times T_{A,B} \to B$. For the tracking identities, we have

[**Tk.1**]

$$
\begin{aligned}
&(\varphi * \psi^\varphi)(F(\tau) \times \mathsf{ev})\psi \\
&= \mathsf{ex}(1 \times \psi^\varphi)(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi \\
&= \mathsf{ex}(1 \times \psi^\varphi)(1 \times \lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi))\mathsf{ev} \\
&= \mathsf{ex}(1 \times \psi^\varphi\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi))\mathsf{ev} \\
&= \mathsf{ex}(1 \times \psi^\varphi\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})))\mathsf{ev} \\
&= \mathsf{ex}(1 \times \psi^\varphi)(1 \times \lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})))\mathsf{ev} \\
&= \mathsf{ex}(1 \times \psi^\varphi)(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev}) \\
&= (\varphi * \psi^\varphi)(F(\tau) \times \mathsf{ev})
\end{aligned}
$$

[**Tk.2**]

$$
\begin{aligned}
&\overline{(\varphi * \psi^\varphi)(F(\tau) \times \mathsf{ev})} \\
&= \overline{\mathsf{ex}(1 \times \psi^\varphi)(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!} \\
&= \overline{\mathsf{ex}(1 \times \psi^\varphi)(1 \times \lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!))\mathsf{ev}} \\
&= \overline{\mathsf{ex}(1 \times \psi^\varphi)(1 \times \lambda((\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!)\mathsf{ev}} \\
&= \overline{\mathsf{ex}(1 \times \psi^\varphi)(\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!} \\
&= \overline{(\varphi * \psi^\varphi)(1 \times \mathsf{ev})}
\end{aligned}
$$

as required.

We proceed to show that our construction is the exponential. Suppose $\chi \in \mathcal{O}(F(C) \times Z)$ is an assembly, and that $(\varphi * \chi) \xrightarrow{f} \psi$ is a map of $\mathsf{asm}(F)$ corresponding to $f : X \times Z \to Y$ in $\mathbb{X}$ with tracking map $\gamma : A \times C \to B$ in $\mathbb{A}$. We must define, for every $e \in \mathcal{O}(\chi)$, an exponential transpose $\chi \xrightarrow{\lambda_e(f)} \psi^\varphi$ in $\mathsf{asm}(F)$. This is the map given by $\lambda_e(f) : Z \to Y^X$ in $\mathbb{X}$, tracked by any weak transpose of γ, $h : C \to T_{A,B}$ in $\mathbb{A}$, as in

$$\begin{tikzcd} A \times T_{A,B} \arrow[r, "\tau"] & B \\ A \times C \arrow[u, "1\times h"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We show the tracking identities:

[Tk.1] We work with each of the two idempotents that make up ψ^φ separately. First, we have

$$\begin{aligned}
&\chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev}))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((1 \times \chi(F(h) \times \lambda_e(f)))(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev}))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((\varphi \times \chi)(1 \times 1 \times F(h) \times \lambda_e(f))\mathsf{ex}(F(\tau) \times \mathsf{ev}))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((\varphi \times \chi)\mathsf{ex}(1 \times F(h) \times 1 \times \lambda_e(f))(F(\tau) \times \mathsf{ev}))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{exex}(\varphi \times \chi)\mathsf{ex}((1 \times F(h))F(\tau) \times (1 \times \lambda_e(f))\mathsf{ev}))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{ex}(\varphi * \chi)(F(\gamma) \times f))\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{ex}(\varphi * \chi)(F(\gamma) \times f)\psi)\\
&= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((1 \times \chi(F(h) \times \lambda_e(f)))(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi)\\
&= \chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi)
\end{aligned}$$

which gives

$$\chi(F(h) \times \lambda_e(f))\overline{\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev}) \cap \lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})\psi)}$$

$$= \overline{\chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev}))}\chi(F(h) \times \lambda_e(f))$$

$$= \chi(F(h) \times \lambda_e(f))$$

Next, we have

$$\chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((1 \times \chi(F(h) \times \lambda_e(f)))(\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((\varphi \times \chi)\mathsf{ex}((1 \times F(h)) \times (1 \times \lambda_e(f))\mathsf{ev})!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{exex}(\varphi \times \chi)\mathsf{ex}((1 \times F(h)) \times f)!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{ex}(\varphi * \chi)(1 \times f)!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda(\mathsf{ex}(\varphi * \chi)(F(\gamma) \times f)!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((\varphi \times \chi)\mathsf{ex}(F(h) \times 1 \times \lambda_e(f) \times 1)(F(\tau) \times \mathsf{ev})!)$$

$$= \overline{\chi(F(h) \times \lambda_e(f))}\lambda((1 \times \chi(F(h) \times \lambda_e(f)))(\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!)$$

$$= \chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!)$$

which gives

$$\chi(F(h) \times \lambda_e(f))\overline{\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!) \cap \lambda((\varphi \times 1)\mathsf{ex}(1 \times \mathsf{ev})!)}$$

$$= \overline{\chi(F(h) \times \lambda_e(f))\lambda((\varphi \times 1)\mathsf{ex}(F(\tau) \times \mathsf{ev})!)}\chi(F(h) \times \lambda_e(f))$$

$$= \chi(F(h) \times \lambda_e(f))$$

Together, the above identities give immediately that

$$\chi(F(h) \times \lambda_e(f))\psi^{\varphi} = \chi(F(h) \times \lambda_e(f))$$

[Tk.2] Since h is total we have

$$\overline{\chi(F(h) \times \lambda_e(f))} = \overline{\chi(\overline{F(h)} \times \overline{\lambda_e(f)})} = \overline{\chi(1 \times \lambda_e(f))}$$

so our transpose $\chi \xrightarrow{\lambda_e(f)} \psi^\varphi$ is well-defined. Finally, we know $(1 \times \lambda_e(f))\mathsf{ev} = f$, $\overline{\lambda_e(f)} = e$, and that $\lambda_e(f)$ is the unique map with these properties since this is all the case in $\mathbb{X}$.

$\square$

We have now shown

Theorem 5.9. *If $\mathbb{A}$ is weakly cartesian closed restriction category, $\mathbb{X}$ is a discrete cartesian closed restriction category, and $F : \mathbb{A} \to \mathbb{X}$ is a cartesian restriction functor, then $\mathsf{asm}(F)$ is a discrete cartesian closed restriction category.*

In particular, we have shown that this holds when $\mathbb{A}$ is a Turing category, since every Turing category is weakly cartesian closed.

We also consider joins, and show how joins in the base category correspond to joins in the category of assemblies when the category of realizers admits *interleaving* of maps.

Proposition 5.10. *If $\mathbb{X}$ is a cartesian restriction category with finite joins, $\mathbb{A}$ is a discrete cartesian restriction category with finite interleaving and $F : \mathbb{A} \to \mathbb{X}$ is a cartesian restriction functor that preserves joins, then $\mathsf{asm}(F)$ has finite joins.*

Proof. We must show that $\mathsf{asm}(F)$ has restriction zeroes and binary joins.

The restriction zero $0_{\varphi,\psi} : \varphi \to \psi$ is given by $0_{X,Y}$ in $\mathbb{X}$, which is tracked by $r_A s_B : A \to B$ in $\mathbb{A}$. Using the fact that for any map $k : A \to B$ we have

$$(F(k) \times 0_{X,Y}) = \overline{(F(k) \times 0_{X,Y})}(F(k) \times 0_{X,Y})$$

$$= \overline{\pi_0 F(k)}\ \overline{\pi_1 0_{X,Y}}(F(k) \times 0_{X,Y})$$

$$= \overline{0}\ \overline{\pi_0 F(k)}(F(k) \times 0_{X,Y})$$

$$= 0\overline{\pi_0 F(k)}(F(k) \times 0_{X,Y})$$

$$= 0$$

we show the tracking identities as follows.

[Tk.1]

$$\varphi(F(r_A s_B) \times 0_{X,Y})\psi = \varphi 0 \psi = 0 = \varphi 0$$
$$= \varphi(F(r_A s_B) \times 0_{X,Y})$$

[Tk.2]

$$\overline{\varphi(F(r_A s_B) \times 0_{X,Y})} = \overline{\varphi 0} = \overline{\varphi(1 \times 0_{X,Y})}$$

Thus, our restriction zero is well-defined. That $0_{\varphi,\psi}$ is in fact the restriction zero in $\mathsf{asm}(F)$ follows immediately from the fact that $0_{X,Y}$ is the restriction zero in $\mathbb{X}$, as usual.

The join of two maps $f, g : \varphi \to \psi$ with $f \smile g$ is given by the join of the corresponding maps $f, g : X \to Y$ in the base. Clearly $f \smile g$ in $\mathbb{X}$, so $f \vee g : X \to Y$ is a map in $\mathbb{X}$. If f and g are tracked by $\gamma : A \to B$ and $\omega : A \to B$ respectively, then any interleaving $h : A \to B$ of γ and ω in $\mathbb{A}$ tracks $f \vee g$ as follows

[Tk.1]

$$\varphi(F(h) \times (f \vee g))\psi = \varphi(F((h \cap \gamma) \vee (h \cap \omega)) \times (f \vee g))\psi$$
$$= \varphi((F(h \cap \gamma) \vee F(h \cap \omega)) \times (f \vee g))\psi$$

$$
\begin{aligned}
&= \varphi((F(h \cap \gamma) \times f) \vee (F(h \cap \omega) \times g))\psi \\
&= \varphi((\overline{F(h \cap \gamma)}F(\gamma) \times f) \vee (\overline{F(h \cap \omega)}F(\omega) \times g))\psi \\
&= \varphi(\overline{F(h \cap \gamma)}F(\gamma) \times f)\psi \vee \varphi(\overline{F(h \cap \omega)}F(\omega) \times g)\psi \\
&= ((\overline{F(h \cap \gamma)} \times 1)\varphi(F(\gamma) \times f)\psi) \vee ((\overline{F(h \cap \omega)} \times 1)\varphi(F(\omega) \times g)\psi) \\
&= ((\overline{F(h \cap \gamma)} \times 1)\varphi(F(\gamma) \times f)) \vee ((\overline{F(h \cap \omega)} \times 1)\varphi(F(\omega) \times g)) \\
&= \varphi((F(h \cap \gamma) \times f) \vee (F(h \cap \omega) \times g)) = \varphi(F(h) \times (f \vee g))
\end{aligned}
$$

[Tk.2] Notice that

$$
\begin{aligned}
&\overline{\varphi(1 \times f)} = \overline{\varphi(F(\gamma) \times f)} = \overline{\varphi(F(\overline{\gamma}) \times \overline{f})} \\
&\leq \overline{\varphi(F(\overline{h}) \times \overline{f})} = \overline{\varphi(F(h) \times f)} \\
&\leq \overline{\varphi(F(h) \times (f \vee g))}
\end{aligned}
$$

and similarly $\overline{\varphi(1 \times g)} \leq \overline{\varphi(F(h) \times (f \vee g))}$. We now have

$$
\begin{aligned}
&\overline{\varphi(1 \times (f \vee g))} = \overline{\varphi((1 \vee 1) \times (f \vee g))} \\
&= \overline{\varphi((1 \times f) \vee (1 \times g))} = \overline{\varphi(1 \times f) \vee \varphi(1 \times g)} \\
&= \overline{\varphi(1 \times f)} \vee \overline{\varphi(1 \times g)} \\
&\leq \overline{\varphi(F(h) \times (f \vee g))}
\end{aligned}
$$

which gives $\overline{\varphi(F(h) \times (f \vee g))} = \overline{\varphi(1 \times (f \vee g))}$ as the reverse inequality clearly holds.

So $f \vee g : \varphi \to \psi$ is well-defined in $\mathsf{asm}(F)$. That it is the join follows from $f \vee g : X \to Y$ being the join in $\mathbb{X}$.

□

6 Latent Fibrations

Fibrations play an important role in categorical logic, where they are used to model systems of logic with universal or existential quantification [18]. We are interested in constructing such logical fibrations, in particular triposes, in our restriction categorical setting. However, the relevant techniques from categorical logic presuppose that the category on which we base our fibrations is one of total maps, and that the logical structure is presented accordingly. Logical structure in a restriction category is more nuanced due to the partiality of the maps, and the techniques must be adapted. In this chapter, we introduce the restriction categorical analogue of a fibration, called a *latent fibration*. We show some elementary properties of latent fibrations, and show that if a latent fibration reflects total maps, we can construct from it a fibration over the total map category of the original base category. Finally, we give two examples of latent fibrations, the properties of which will be investigated in some depth in the sequel.

6.1 Latent Fibrations

We begin with the definition:

Definition 6.1. Let $\partial : \mathbb{E} \to \mathbb{X}$ be a restriction functor, with $\mathbb{E}$ and $\mathbb{X}$ arbitrary restriction categories.

(i) An arrow $f : X' \to X$ in $\mathbb{E}$ is *prone in* ∂ in case whenever we have $g : Y \to X$ in $\mathbb{E}$ and $h : \partial(Y) \to \partial(X')$ in $\mathbb{X}$ such that $h\partial(f) \geq \partial(g)$,

$$\text{in } \mathbb{E} \quad \begin{tikzcd} Y \arrow[d, dotted, "\exists \widetilde{h}"'] \arrow[dr, "g"] \arrow[d, phantom, shift left=3ex, "\geq"] & \\ X' \arrow[r, "f"'] & X \end{tikzcd} \qquad \text{in } \mathbb{X} \quad \begin{tikzcd} \partial(Y) \arrow[d, "h"'] \arrow[dr, "\partial(g)"] \arrow[d, phantom, shift left=3ex, "\geq"] & \\ \partial(X') \arrow[r, "\partial(f)"'] & \partial(X) \end{tikzcd}$$

there is a *lifting* $\widetilde{h}: Y \to X'$ in $\mathbb{X}$ such that

(a) $\widetilde{h}$ is a *candidate lifting*. That is, $\widetilde{h}\, f \geq g$ and $\partial(\widetilde{h}) \leq h$.

(b) $\widetilde{h}$ is the *smallest* candidate lifting. That is, for any other candidate lifting $k : Y \to X'$ in $\mathbb{X}$, $\widetilde{h} \leq k$.

(ii) A restriction functor $\partial : \mathbb{E} \to \mathbb{X}$ is a *latent fibration* in case for each $f : A \to \partial(X)$ in $\mathbb{X}$ for some X in $\mathbb{E}$, there is a prone arrow over f with codomain X.

It is equivalent to require in the definition that $\widetilde{h}\, f = g$, instead of $\widetilde{h}\, f \geq g$, as follows.

Lemma 6.2. *If f is a prone arrow where $k\partial(f) \geq \partial(g)$ and $\widetilde{k}$ is the lifting of k*

$$\text{in } \mathbb{E} \quad \begin{tikzcd} \cdot \arrow[d, "\widetilde{k}"'] \arrow[dr, "g"] \arrow[dr, phantom, "\geq", near start, shift right=2ex] & \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd} \qquad\qquad \text{in } \mathbb{X} \quad \begin{tikzcd} \cdot \arrow[d, "k"'] \arrow[dr, "\partial(g)"] \arrow[dr, phantom, "\geq", near start, shift right=2ex] & \\ \cdot \arrow[r, "\partial(f)"'] & \cdot \end{tikzcd}$$

then $\overline{g}\, \widetilde{k} = \widetilde{k} = \widetilde{k}\, \overline{f}$ and $\widetilde{k}\, f = g$.

Proof. Since $\overline{g}\, \widetilde{k} \leq \widetilde{k}$ and $\widetilde{k}\, \overline{f} \leq \widetilde{k}$ trivially, it suffices to show that both $\overline{g}\, \widetilde{k}$ and $\widetilde{k}\, \overline{f}$ are candidate liftings of k. $\widetilde{k}$ is $\leq$ any other candidate lifting of k, and so this gives the identities immediately.

For $\overline{g}\, \widetilde{k}$, we have $g \leq \overline{g}\, \widetilde{k}\, f$ immediately, and using the fact that $\partial(\widetilde{k}) \leq k$,

$$\overline{\partial(\overline{g}\, \widetilde{k})}k = \overline{\partial(g)}\, \overline{\partial(\widetilde{k})}k = \overline{\partial(g)}\partial(\widetilde{k}) = \partial(\overline{g}\, \widetilde{k})$$

which gives $\partial(\overline{g}\, \widetilde{k}) \leq k$, meaning $\overline{g}\, \widetilde{k}$ is a candidate lifting of k, and so $\widetilde{k} = \overline{g}\, \widetilde{k}$.

For $\widetilde{k}\, \overline{f}$, we know $g \leq \widetilde{k}\, f = \widetilde{k}\, \overline{f} f$, and using that $\partial(\widetilde{k}) = k$,

$$\overline{\partial(\widetilde{k}\, \overline{f})}k = \overline{\partial(\widetilde{k})\partial(f)}k = \overline{k\partial(f)}k = k\overline{\partial(f)} = \partial(\widetilde{k})\partial(\overline{f}) = \partial(\widetilde{k}\, \overline{f})$$

which gives $\partial(\widetilde{k}\, \overline{f}) \leq k$, meaning $\widetilde{k}\, \overline{f}$ is a candidate lifting of k, and so $\widetilde{k} = \widetilde{k}\, \overline{f}$.

Finally, $\widetilde{k}\, f = \overline{g}\, \widetilde{k}\, f = g$. $\square$

Many results and constructions involving fibrations have a restriction categorical analogue for latent fibrations. For example, given a fibration, any two prone arrows over the same arrow induce an isomorphism between their domains. For a latent fibration, we show that the analogous situation induces a *partial* isomorphism instead.

Lemma 6.3. *If f, f' are prone arrows with codomain Y such that $\partial(f) = \partial(f')$*

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ X' \arrow[r, "f'"'] & Y \end{tikzcd}$$

then there is a unique partial isomorphism $\alpha : X \to X'$ such that $\partial(\alpha) \leq 1$, $\alpha f' = f$, and $\overline{\alpha} = \overline{f}$, and $\overline{\alpha^{-1}} = \overline{f'}$

Proof. Let β, β' be the liftings of the identity as in

$$\begin{tikzcd} X \arrow[d, dotted, "\beta"'] \arrow[dr, "f"] & \\ X' \arrow[r, "f'"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[d, dotted, "\beta'"'] \arrow[dr, "f'"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then by the previous lemma, $\overline{f'}\beta' = \beta' = \beta'\overline{f}$, $\overline{f}\beta = \beta = \beta\overline{f'}$, $\beta' f = f'$, and $\beta f' = f$.

Consider

$$\begin{tikzcd} X \arrow[d, dotted, "k"'] \arrow[dr, "f"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where k is the lifting of the identity. Since $\overline{f}$ is a candidate lifting of the identity, $k \leq \overline{f}$, which also means $k = \overline{k}$. We then have

$$k = \overline{k}\,\overline{f} = \overline{\overline{k}\,\overline{f}} = \overline{\overline{k} f} = \overline{kf} = \overline{f}$$

and therefore we have that $\overline{f}$ is the lifting of the identity in the above situation.

Now $\beta\beta' f = \beta f' = f \geq f$, and $\partial(\beta\beta') = \partial(\beta)\partial(\beta') \leq 1$, so $\beta\beta'$ is a candidate lifting of the identity as well, meaning that $\overline{f} \leq \beta\beta'$. Then we have

$$\beta\beta' = \overline{f}\beta\beta' = \overline{f}$$

and similarly obtain $\beta'\beta = \overline{f'}$. Finally,

$$\overline{\beta} = \overline{\overline{f}\beta} = \overline{f}\,\overline{\beta} = \overline{\beta}\,\overline{f} = \overline{\beta}\beta\beta' = \beta\beta' = \overline{f}$$

and similarly $\overline{\beta'} = \overline{f'}$, so $\beta\beta' = \overline{\beta}$ and $\beta'\beta = \overline{\beta'}$, which means that β is a partial isomorphism with $\beta^{-1} = \beta'$. To see that β is unique, suppose $h : X \to X'$ is a partial iso with $\partial(h) \leq 1$, $hf' = f$, and $\overline{h} = \overline{f}$. Then h is clearly a candidate lifting of the identity, and so $\beta \leq h$. This gives $\beta = \overline{\beta}h = \overline{f}h = \overline{h}h = h$ and we are done. The unique partial isomorphism is $\beta : X \to X'$, with $\beta^{-1} = \beta' : X' \to X$ as above.

□

Analogous to the fiber of an object in a fibration, we can talk about the fiber of an object in a latent fibration. Note that maps in a fiber of a latent fibration are not those above the identity map of an object, but those over any restriction idempotent on that object.

Definition 6.4. The *fiber* of B an object of $\mathbb{X}$, written $\partial^{-1}(B)$, is the category whose objects are objects X of $\mathbb{E}$ such that $\partial(X) = B$, and whose maps are maps f of $\mathbb{E}$ such that $\partial(f) \leq 1_B$. (That is, such that $\partial(f) \in \mathcal{O}(B)$).

Latent fibrations may be *cloven*, which means the same thing it does for ordinary fibrations:

Definition 6.5. A latent fibration *has a cleavage* (is *cloven*) in case there is a chosen prone arrow $f^*_X : f^*(X) \to X$ over each $f : A \to B$ for each X over B.

We construct the analogue for latent fibrations of reindexing functors between the fibers in a fibration. In general we will only be able to obtain restriction semifunctors, which are like restriction functors, but do not preserve the identity.

Definition 6.6. A *restriction semifunctor* $F : \mathbb{X} \to \mathbb{Y}$ consists of an assignment of objects A of $\mathbb{X}$ to objects $F(A)$ of $\mathbb{Y}$, and an assignment of maps $f : A \to B$ of $\mathbb{X}$ to maps $F(f) : F(A) \to F(B)$ of $\mathbb{Y}$ such that $F(fg) = F(f)F(g)$ and $F(\overline{f}) = \overline{F(f)}$.

Definition 6.7. Let $\partial : \mathbb{E} \to \mathbb{X}$ be a cloven latent fibration, and let $u : A \to B$ be a map in $\mathbb{X}$. We define the *reindexing semifunctor* $u^* : \partial^{-1}(B) \to \partial^{-1}(A)$ as follows: on arrows $f : X \to Y$ in $\partial^{-1}(B)$, $u^*(f)$ is the arrow $u^*(X) \to u^*(Y)$ in $\partial^{-1}(A)$ given by the lifting of the identity for

$$\text{in } \mathbb{E} \quad \begin{tikzcd} u^*(X) \arrow[dr, "u^*_X"] \arrow[dd, dotted, "u^*(f)"'] & & \\ & X \arrow[dr, "f"] & \\ u^*(Y) \arrow[rr, "u^*_Y"'] & & Y \end{tikzcd} \qquad \text{in } \mathbb{X} \quad \begin{tikzcd} A \arrow[dr, "u"] \arrow[dd, "1_A"'] & & \\ & B \arrow[dr, "\partial(f)"] \arrow[d, phantom, "\geq"] & \\ A \arrow[rr, "u"'] & {} & B \end{tikzcd}$$

where $u^*(X)$ is the domain of the prone map above u with codomain X.

we then have

Proposition 6.8. *If $\partial : \mathbb{E} \to \mathbb{X}$ is a latent fibration with a cleavage and $u : A \to B$ is a map in $\mathbb{X}$, then $u^* : \partial^{-1}(B) \to \partial^{-1}(A)$ as defined above is a restriction semifunctor.*

Proof. We must show that $u^*(\overline{f}) = \overline{u^*(f)}$ and $u^*(fg) = u^*(f)u^*(g)$.

First, suppose $f : X \to Y$ is a map in $\partial^{-1}(B)$. Then $u^*(\overline{f})$ is defined as the lifting of 1_A for

$$\begin{tikzcd} u^*(X) \arrow[dr, "u^*_X"] \arrow[dd, "u^*(\overline{f})"'] & & \\ & X \arrow[dr, "\overline{f}"] & \\ u^*(X) \arrow[rr, "u^*_X"'] & & X \end{tikzcd}$$

Now, $\overline{u^*(f)u_X^*} \geq u_X^*\overline{f}$ by

$$\overline{u_X^*\overline{f}}\,\overline{u^*(f)u_X^*} = \overline{u_X^*f}\,\overline{u^*(f)u_X^*} = \overline{u^*(f)u_Y^*}\,\overline{u^*(f)u_X^*}$$
$$= \overline{\overline{u^*(f)u_Y^*}u^*(f)u_X^*} = \overline{u^*(f)\overline{u_Y^*}u_X^*} = \overline{u^*(f)u_Y^*u_X^*}$$
$$= \overline{u_X^*f}u_X^* = u_X^*\overline{f}$$

and since ∂ is a restriction functor $\partial(\overline{u^*(f)}) \leq 1_A$, meaning that $\overline{u^*(f)}$ is a candidate lifting of 1_A, so we have $u^*(\overline{f}) \leq \overline{u^*(f)}$. In particular, note that this means $u^*(\overline{f})$ is a restriction idempotent. Then, we have

$$u^*(\overline{f}) = u^*(\overline{f})\overline{u_X^*} = \overline{u^*(\overline{f})u_X^*} = \overline{u_X^*\overline{f}} = \overline{u_X^*f}$$
$$= \overline{u^*(f)u_Y^*} = \overline{u^*(f)\overline{u_Y^*}} = \overline{u^*(f)}$$

and so u^* preserves the restriction combinator.

Next, suppose $f : X \to Y$ and $g : Y \to Z$ are maps of $\partial^{-1}(B)$. Then $u^*(fg)$ is defined as the lifting of 1_A for

$$\begin{tikzcd}
u^*(X) \arrow[dr, "u_X^*"] \arrow[dd, "u^*(fg)"'] & \\
& X \arrow[d, "fg"] \\
u^*(Z) \arrow[r, "u_Z^*"'] & Z
\end{tikzcd}$$

Now, $u^*(f)u^*(g)u_Z^* = u^*(f)u_Y^*g = u_X^*fg \geq u_X^*fg$, and clearly $\partial(u^*(f)u^*(g)) \leq 1_A$, so $u^*(f)u^*(g)$ is a candidate lifting of 1_A and we have $u^*(fg) \leq u^*(f)u^*(g)$. Then, we have

$$u^*(fg) = \overline{u^*(fg)}\,u^*(f)u^*(g) = \overline{u^*(fg)u_Z^*}\,u^*(f)u^*(g) = \overline{u_X^*fg}\,u^*(f)u^*(g)$$

$$
\begin{aligned}
&= \overline{u^*(f)u^*_Y g}\, u^*(f)u^*(g) = \overline{u^*(f)u^*(g)u^*_Z}\, u^*(f)u^*(g) \\
&= \overline{u^*(f)u^*(g)\overline{u^*_Z}}\, u^*(f)u^*(g) = \overline{u^*(f)u^*(g)}\, u^*(f)u^*(g) = u^*(f)u^*(g)
\end{aligned}
$$

and so u^* preserves composition, and is therefore a restriction semifunctor. □

Definition 6.9. A latent fibration $\partial : \mathbb{E} \to \mathbb{X}$ is said to *reflect total maps* if whenever f is total and $\partial g = f$, then g is also total.

These reindexing semifunctors become restriction functors precisely when our latent fibration reflects total maps.

Proposition 6.10. *If $\partial : \mathbb{E} \to \mathbb{X}$ is a latent fibration with a cleavage that reflects total maps $u : A \to B$ is a total map in $\mathbb{X}$, then $u^* : \partial^{-1}(B) \to \partial^{-1}(A)$ is a restriction functor.*

Proof. We have already shown that u^* is a restriction semifunctor. Since ∂ reflects total maps, we have that $u^*_X : u^*(X) \to X$ is total, as u is. This gives

$$
\begin{aligned}
u^*(1_X) &= u^*(\overline{1_X}) = \overline{u^*(1_X)} = \overline{u^*(1_X)\overline{u^*_X}} \\
&= \overline{u^*(1_X)u^*_X} = \overline{u^*_X} = 1_{u^*(X)}
\end{aligned}
$$

and so u^* is a restriction functor. □

Given a latent fibration over base category $\mathbb{X}$ which reflects total maps, there is a fibration (in the usual sense) over base category $\mathsf{total}(\mathbb{X})$. We use the fact that the category of restriction categories and restriction functors has finite limits [10], and in particular pullbacks, to construct this fibration.

Proposition 6.11. *The pullback of a latent fibration along any restriction functor is a latent fibration.*

Proof. Suppose $\partial : \mathbb{Z} \to \mathbb{Y}$ is a latent fibration, that $F : \mathbb{X} \to \mathbb{Y}$ is restriction functor, and consider the pullback

$$\begin{tikzcd} \mathbb{W} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathbb{Z} \arrow[d, "\partial"] \\ \mathbb{X} \arrow[r, "F"'] & \mathbb{Y} \end{tikzcd}$$

in which the category $\mathbb{W}$ is defined by

objects are pairs (X, Z) where X and Z are objects of $\mathbb{X}$ and $\mathbb{Z}$ respectively which satisfy $F(X) = \partial(Z)$.

maps of type $(X, Z) \to (X', Z')$ are pairs (f, g) where f and g are maps of $\mathbb{X}$ and $\mathbb{Z}$ respectively which satisfy $F(f) = \partial(g)$.

composition and **identities** are pairwise.

restriction is given by $\overline{(f, g)} = (\overline{f}, \overline{g})$, which is well defined since, if $F(f) = \partial(g)$, then $F(\overline{f}) = \overline{F(f)} = \overline{\partial(g)} = \partial(\overline{g})$.

and the pullback maps p_0, p_1 are the first and second projections.

We must show that p_0 is a latent fibration. To that end, suppose $f : X \to X'$ is a map of $\mathbb{X}$, with (X', Z) an object of $\mathbb{W}$. We define the prone map above f to be

$$(X, F(f)^*(Z)) \xrightarrow{(f, F(f)^*)} (X', Z)$$

where $F(f)^*$ is the prone map above $F(f)$ in ∂. $F(f) = \partial(F(f)^*)$ by definition, so $(f, F(f)^*)$ is a well-defined map of $\mathbb{W}$. We proceed to show that $(f, F(f)^*)$ is prone in p_0. Suppose we have

in $\mathbb{W}$
$$\begin{tikzcd} (Y, D) \arrow[dr, "{(g,k)}"] & \\ (X, F(f)^*(Z)) \arrow[r, "{(f,F(f)^*)}"'] & (X', Z) \end{tikzcd}$$
in $\mathbb{X}$
$$\begin{tikzcd} Y \arrow[d, "h"'] \arrow[dr, "g"] \arrow[d, phantom, shift left=12, "\geq"] & \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

and notice that in this case we must also have

$$\text{in } \mathbb{Z} \quad \begin{tikzcd} & D \arrow[dr, "k"] & \\ F(f)^*(Z) \arrow[rr, "F(f)^*"'] & & Z \end{tikzcd} \qquad\qquad \text{in } \mathbb{Y} \quad \begin{tikzcd} F(Y) \arrow[d, "F(h)"'] \arrow[dr, "F(g)"] \arrow[d, phantom, shift left=3ex, "\geq"] & \\ F(X) \arrow[r, "F(f)"'] & F(x') \end{tikzcd}$$

The properties of $\mathbb{W}$ ensure that $F(g) = \partial(k)$, and we already knew that $\partial(F(f)^*) = F(f)$. Since ∂ is a latent fibration, we thus have a lifting

$$\widetilde{F(h)}\colon D \to F(f)^*(Z)$$

of $F(h)$ in ∂. Recall that our current goal is to find a lifting of h in p_0. This lifting is $(h, \widetilde{F(h)}) : (Y, D) \to (X, F(f)^*(Z))$, which is a well-defined map of $\mathbb{W}$ since $F(h) = \partial(\widetilde{F(h)})$ by definition. We have $hf \geq g$ by assumption, and $\widetilde{F(h)}\ F(f)^* \geq k$ because ∂ is a latent fibration, so $(h, \widetilde{F(h)})(f, F(f)^*) = (hf, \widetilde{F(h)}\ F(f)^*) \geq (g, k)$, and our lifting is a candidate lifting. To see that it is the smallest candidate lifting, suppose (h, ω) is also a candidate lifting of h in p_0. Then $(h, \omega)(f, F(f)^*) \geq (g, k)$, and so in particular $\omega F(f)^* \geq k$. But we also know that since (h, ω) is a map of $\mathbb{W}$, $F(h) = \partial(\omega)$, and so ω is a candidate lifting of $F(h)$ in ∂, meaning that $\widetilde{F(h)} \leq \omega$. This in turn means that $(h, \widetilde{F(h)}) \leq (h, \omega)$, and so $(h, \widetilde{F(h)})$ is the smallest candidate lifting of h in p_0. We conclude that $(f, F(f)^*)$ is prone in p_0, and have therefore shown that p_0 is a latent fibration. $\square$

Next, we consider the pullback of a latent fibration $\partial : \mathbb{Z} \to \mathbb{X}$ which reflects total maps along the inclusion $\mathsf{total}(\mathbb{X}) \to \mathbb{X}$, a special case of the above situation.

Proposition 6.12. *If $\partial : \mathbb{Z} \to \mathbb{X}$ is a latent fibration which reflects total maps, then in the pullback*

$$\begin{tikzcd} \mathbb{W} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathbb{Z} \arrow[d, "\partial"] \\ \mathsf{total}(\mathbb{X}) \arrow[r] & \mathbb{X} \end{tikzcd}$$

the arrow p_0 is a fibration.

Proof. We have already shown that p_0 is a latent fibration. Notice that for every map (f, g) of $\mathbb{W}$, since f is a map of $\mathsf{total}(X)$, ∂ reflects total maps, and $\partial(g) = f$, we know that f and g are both total maps. Thus, $\overline{(f, g)} = (\overline{f}, \overline{g}) = (1, 1) = 1$, and so every map of $\mathbb{W}$ is total.

Observe that for total maps h and k, $h \leq k$ if and only if $h = k$. Since $p_0 : \mathbb{W} \to \mathsf{total}(\mathbb{X})$ is a latent fibration and the restriction is trivial in both $\mathbb{W}$ and $\mathsf{total}(\mathbb{X})$, the inequalities in the definition of a latent fibration are replaced with equalities, and p_0 is a fibration. $\square$

6.2 The Domain Latent Fibration

Let $\mathbb{X}$ be a restriction category, and define the restriction category $\mathcal{R}(\mathbb{X})$ by

objects: pairs (X, e) where X is an object of $\mathbb{X}$, and e is a restriction idempotent on X.

maps: (X, e) to (X', e') are maps $f : X \to X'$ of $\mathbb{X}$ such that $e \leq \overline{fe'}$, or equivalently $e = \overline{efe'}$.

composition: is composition in $\mathbb{X}$. This is well-defined since if

$$(X, e) \xrightarrow{f} (X', e') \xrightarrow{f'} (X'', e'')$$

are maps of $\mathcal{R}(\mathbb{X})$ given by $f : X \to X'$ and $f' : X' \to X''$ in $\mathbb{X}$, then we have

$$\begin{aligned}\overline{eff'e''} &= \overline{\overline{efe'}ff'e''} = \overline{e\overline{fe'}ff'e''} = \overline{efe'f'e''} \\ &= \overline{ef\overline{e'f'e''}} = \overline{efe'} = e\end{aligned}$$

meaning ff' gives a map $(X, e) \to (X'', e'')$ in $\mathcal{R}(\mathbb{X})$.

identities: as in $\mathbb{X}$. That is, $1_{(X,e)} = 1_X$. Trivially $e = \overline{e1_Xe}$, so this is well-defined.

restriction: also as in $\mathbb{X}$, with $\overline{f} : (X, e) \to (X, e)$ for $f : (X, e) \to (X', e')$ given by $\overline{f} : X \to X$. This is well-defined as

$$e = \overline{efe'} = \overline{e\overline{f}fe'} = \overline{ee\overline{f}fe'} = \overline{e\overline{f}efe'} = \overline{e\overline{f}\,\overline{efe'}} = \overline{e\overline{f}e}$$

Associativity of composition, the restriction combinator axioms, and the requirements on identity maps all hold in $\mathcal{R}(\mathbb{X})$ immediately since they hold in $\mathbb{X}$.

Now, we define a restriction functor $\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$ by mapping (X, e) to X, and mapping $f : (X, e) \to (X', e')$ to $f : X \to X'$. It is easy to see that this is in fact a restriction functor. In fact, we have:

Proposition 6.13. *$\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$ is a latent fibration.*

Proof. Suppose $f : X' \to X$ in $\mathbb{X}$ and let (X, e) be an object of $\mathcal{R}(\mathbb{X})$. Then $(X', \overline{fe})$ is also an object of $\mathcal{R}(\mathbb{X})$, and f gives a map $(X', \overline{fe}) \to (X, e)$ in $\mathcal{R}(\mathbb{X})$ since $\overline{fe} = \overline{\overline{fe}fe}$ trivially. It suffices to show that this map is prone. To that end, suppose that $g : (Y, e') \to (X, e)$ and $h : Y \to X'$ are maps in $\mathcal{R}(\mathbb{X})$ and $\mathbb{X}$ respectively such that

$$\begin{tikzcd}
Y \arrow[d, "h"'] \arrow[dr, "g"] \arrow[d, phantom, shift left=4ex, "\geq"] & \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

in $\mathbb{X}$. Then $\overline{g}h\overline{f}$ gives a map $(Y, e') \to (X', \overline{fe})$ in $\mathcal{R}(\mathbb{X})$ since

$$\overline{e'\overline{g}h\overline{f}\,\overline{fe}} = \overline{e'\overline{g}fe} = \overline{e'ge} = e'$$

and this map is a candidate lifting of h in $\mathbb{X}$ as clearly $\mathcal{O}(\overline{g}h\overline{f}) = \overline{g}h\overline{f} \leq h$ in $\mathbb{X}$ and $\overline{g}h\overline{f}f = \overline{g}hf = g \geq g$ in $\mathcal{R}(\mathbb{X})$ since this is the case in $\mathbb{X}$. To see that $\overline{g}h\overline{f}$ is the smallest candidate lifting, suppose $k : (Y, e') \to (X', \overline{fe})$ has $\partial(k) = k \leq h$ and

$kf \geq g$. Then $\overline{g}h\overline{f} \leq k$ since

$$\begin{aligned}
&\overline{\overline{g}h\overline{f}}k = \overline{\overline{g}hf}k = \overline{g}k = \overline{g}\,\overline{k}h = \overline{k}\,\overline{g}h \\
&\overline{k}\,\overline{\overline{g}kf}h = \overline{\overline{k}\,\overline{g}kf}h = \overline{\overline{g}kf}h = \overline{\overline{g}k\overline{f}}h \\
&= \overline{g}k\overline{f}
\end{aligned}$$

and so the lifting of h is $\overline{g}h\overline{f}$, meaning that f is a prone map in $\mathcal{R}(\mathbb{X})$ as required. $\square$

We refer to $\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$ as the *domain latent fibration*.

Notice that the fiber $\mathcal{O}^{-1}(X)$ of $\mathcal{O}$ are exactly the restriction idempotents on X, which we have been denoting $\mathcal{O}(X)$. We will abuse our notation and often write $\mathcal{O}(X)$ for the fiber over X in $\mathcal{O}$ in the sequel.

It is easy to see that $\mathcal{O} : \mathcal{R}(\mathbb{X}) \to \mathbb{X}$ reflects total maps. Thus, we can pull $\mathcal{O}$ back along the inclusion $\mathbb{X} \to \mathsf{total}(\mathbb{X})$

$$\begin{tikzcd}
\mathbb{W} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathcal{R}(\mathbb{X}) \arrow[d, "\mathcal{O}"] \\
\mathsf{total}(\mathbb{X}) \arrow[r] & \mathbb{X}
\end{tikzcd}$$

and in doing so obtain a fibration $p_0 : \mathbb{W} \to \mathbb{X}$.

Consider the restriction category $\mathbb{W}$. In this case, $\mathbb{W}$ is defined to be

objects: pairs $(X, (Y, e))$ where X, Y are objects of $\mathbb{X}$, $e \in \mathcal{O}(Y)$, and $X = \mathcal{O}(Y) = Y$. We may thus equivalently specify objects of $\mathbb{W}$ as pairs (X, e) where $e \in \mathcal{O}(X)$.

maps: $(X, e) \to (X', e')$ are then pairs (f, g) where $f : X \to X'$ is a map of $\mathsf{total}(\mathbb{X})$, and $g : X \to X'$ is a map of $\mathbb{X}$ with $e \leq \overline{ge'}$ and $f = \mathcal{O}(g) = g$. Note that it suffices to supply a total map $f : X \to X'$ of $\mathbb{X}$ with $e \leq \overline{fe'}$.

composition: is now composition of total maps in $\mathbb{X}$.

The **identity:** on (X, e) is given by 1_X in $\mathbb{X}$.

restriction: is trivial.

It is easy to see that in this situation $\mathbb{W}$ is isomorphic to $\mathsf{total}(\mathcal{R}(\mathbb{X}))$.

We define $\mathsf{total}(\mathcal{O}) : \mathsf{total}(\mathcal{R}(\mathbb{X})) \to \mathsf{total}(X)$ to be the fibration p_0 obtained this way, and refer to $\mathsf{total}(\mathcal{O})$ as *the domain fibration*, since the fiber of X in $\mathsf{total}(X)$ consists of the collection of restriction idempotents on X, which are sometimes called "domains of definition".

6.3 The Realizability Latent Fibration

Our second example of a latent fibration is constructed from a category of assemblies.

It turns out that if $\mathbb{X}$ is a cartesian restriction category and $F : \mathbb{A} \to \mathbb{X}$ is a restriction functor, the forgetful functor $\partial : \mathsf{asm}(F) \to \mathbb{X}$ is a latent fibration. Specifically, if $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times Y)$, and $f : \varphi \to \psi$ a map in $\mathsf{asm}(F)$, then $\partial(f)$ is $f : X \to Y$, the underlying map in $\mathbb{X}$. Clearly ∂ is a cartesian restriction functor.

Proposition 6.14. *$\partial : \mathsf{asm}(F) \to \mathbb{X}$ as defined above is a latent fibration.*

Proof. Suppose $f : X \to Y$ is a map in $\mathbb{X}$, and let $\psi \in \mathcal{O}(F(B) \times Y)$ be an assembly. Then $\overline{(1 \times f)\psi} \in \mathcal{O}(F(B) \times X)$ is also an assembly, and f gives a map $\overline{(1 \times f)\psi} \to \psi$ in $\mathsf{asm}(F)$. The tracking map is 1_B, and we verify the required identities:

[Tk.1] $\overline{(1 \times f)\psi}(1 \times f)\psi = (1 \times f)\psi\psi = (1 \times f)\psi = \overline{(1 \times f)\psi}(1 \times f)$

[Tk.2] is immediate.

Now, suppose that in $\mathsf{asm}(F)$ we have an assembly $\chi \in \mathcal{O}(F(C) \times Z)$ a map $g : \chi \to \psi$ given by $g : Z \to Y$ in $\mathbb{X}$ tracked by $\omega : C \to B$ in $\mathbb{A}$ such that for some $h : Z \to X$ in $\mathbb{X}$, $hf \geq g$

$$\text{in } \mathsf{asm}(F) \quad \begin{tikzcd} & \chi \arrow[dr, "g"] & \\ \overline{(1 \times f)\psi} \arrow[rr, "f"'] & & \psi \end{tikzcd} \qquad \text{in } \mathbb{X}: \quad \begin{tikzcd} Z \arrow[d, "h"'] \arrow[dr, "g"] \arrow[d, phantom, shift left=3ex, "\geq"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and define the lifting of h to be $\overline{g}h\overline{f}$, which is tracked by ω as a map $\chi \to \overline{(1 \times f)\psi}$ since

[Tk.1]

$$\begin{aligned}
&\chi(F(\omega) \times \overline{g}h\overline{f})\overline{(1 \times f)\psi} = \chi\overline{(F(\omega) \times \overline{g}h\overline{f}f)\psi}(F(\omega) \times \overline{g}h\overline{f}) \\
&= \overline{\chi(F(\omega) \times \overline{g}hf)\psi}(F(\omega) \times \overline{g}h\overline{f}) = \overline{\chi(F(\omega) \times g)\psi}(F(\omega) \times \overline{g}h\overline{f}) \\
&= \overline{\chi(F(\omega) \times g)}(F(\omega) \times \overline{g}h\overline{f}) = \overline{\chi(1 \times g)}(F(\omega) \times \overline{g}h\overline{f}) \\
&= \chi(1 \times \overline{g})(1 \times \overline{g}h\overline{f}) = \chi(1 \times \overline{g}h\overline{f})
\end{aligned}$$

[Tk.2]

$$\begin{aligned}
&\overline{\chi(F(\omega) \times \overline{g}h\overline{f})} = \chi(\overline{F(\omega)} \times \overline{\overline{g}h\overline{f}}) = \chi(\overline{F(\omega)} \times \overline{g}\,\overline{hf}) \\
&= \chi(\overline{F(\omega)} \times \overline{g})(1 \times \overline{hf}) = \overline{\chi(F(\omega) \times g)}(1 \times \overline{hf}) = \overline{\chi(1 \times g)}(1 \times \overline{hf}) \\
&= \chi(1 \times \overline{g})(1 \times \overline{hf}) = \chi(1 \times \overline{g}\,\overline{hf}) = \overline{\chi(1 \times \overline{g}h\overline{f})}
\end{aligned}$$

and so gives a map in $\mathsf{asm}(F)$. Clearly $\partial(\overline{g}h\overline{f}) = \overline{g}h\overline{f} \leq h$ and $\overline{g}h\overline{f}f = \overline{g}hf = g \geq g$, so $\overline{g}h\overline{f}$ is a candidate lifting for h. It remains to show that it is the smallest such candidiate lifting. To that end, suppose that k is also a candiate lifting for h. That is, $\partial(k) = k \leq h$ and $g \leq kf$. Then, we have $\overline{g}kf = g = \overline{g}hf$, and consequently $\overline{g}\,\overline{kf} = \overline{g}\,\overline{hf}$. Additionally, $k \leq h$ gives $\overline{g}k\overline{f} \leq \overline{g}h\overline{f}$. Now, we can derive

$$\begin{aligned}
&\overline{\overline{g}h\overline{f}}k = \overline{g}\,\overline{hf}k = \overline{g}\,\overline{kf}k = \overline{g}k\overline{f} = \overline{\overline{g}k\overline{f}}\,\overline{g}h\overline{f} \\
&= \overline{g}\,\overline{kf}\,\overline{g}h\overline{f} = \overline{g}\,\overline{hf}\,\overline{g}h\overline{f} = \overline{\overline{g}h\overline{f}}\,\overline{g}h\overline{f} = \overline{g}h\overline{f}
\end{aligned}$$

and so $\overline{g}h\overline{f} \leq k$, meaning that $\overline{g}h\overline{f}$ is the smallest candidate lifting for h, and is therefore *the* lifting for h in f. Thus, $f : \overline{(1 \times f)\psi} \to \psi$ is prone, and $\partial : \mathsf{asm}(F) \to \mathbb{X}$ is a latent fibrantion. $\square$

We refer to $\partial : \mathsf{asm}(F) \to \mathbb{X}$ as the *realizability latent fibration.*

It is easy to see that $\partial : \mathsf{asm}(F) \to \mathbb{X}$ reflects total maps. Thus, we can pull ∂ back along the inclusion $\mathsf{total}(\mathbb{X}) \to \mathbb{X}$

$$\begin{tikzcd} \mathbb{W} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathsf{asm}(F) \arrow[d, "\partial"] \\ \mathsf{total}(\mathbb{X}) \arrow[r] & \mathbb{X} \end{tikzcd}$$

to obtain a fibration p_0. We again consider what $\mathbb{W}$ is defined to be in this situation:

objects are pairs (X, φ) in which $\varphi \in \mathcal{O}(F(A) \times X)$ for some A of $\mathbb{A}$. Since φ alone determines the pair, an object of $\mathbb{W}$ is an object of $\mathsf{asm}(F)$.

maps of type $\varphi \to \psi$ where $\varphi \in \mathcal{O}(F(A) \times X)$, $\psi \in \mathcal{O}(F(B) \times X')$ for A, B objects of $\mathbb{A}$, X, X' objects of $\mathbb{X}$ are then total maps $f : X \to X'$ of $\mathbb{X}$ with a tracking element $A \to B$ in $\mathbb{A}$.

composition, **identities**, and **restriction** are all as in $\mathsf{asm}(F)$

Thus, $\mathbb{W}$ is isomorphic to $\mathsf{total}(\mathsf{asm}(F))$. We define $\mathsf{total}(\partial) : \mathsf{total}(\mathsf{asm}(F)) \to \mathsf{total}(\mathbb{X})$ to be the fibration p_0 obtained this way, and refer to $\mathsf{total}(\partial)$ as the *realizability fibration*, since it is how we construct the realizability tripos in the sequel.

7 Triposes

Introduced in [31], triposes play an important role in the construction of the classical realizability topos. A realizability tripos is constructed, and then the realizability topos is defined in terms of the internal language of the tripos. We construct our more general realizabilty topos in the same way, constructing a realizability tripos in the restriction categorical setting, with the associated realizability topos constructed from this tripos as it is in the classical case (see the final chapter). Further, we notice that from any tripos one can construct a *partial* topos in the sense of [13] whose total maps are the realizability topos. This construction involves a minor modification of the tripos-to-topos construction, and as such makes use of the internal language of a tripos. We use the internal language rather informally, and do not provide the details of the interpretation of proof of soundness here. For this we recommend [36].

7.1 Definition and Internal Logic

Definition 7.1. If $\mathbb{X}$ is a cartesian closed category, an $\mathbb{X}$-*tripos* is a fibration $p : \mathbb{E} \to \mathbb{X}$ satisfying:

(i) Each fiber of p is a Heyting pre-algebra (a preorder whose poset completion is a Heyting algebra), and for every map $f : X \to Y$ of $\mathbb{X}$, the reindexing $f^* : p^{-1}(Y) \to p^{-1}(X)$ is a morphism of Heyting pre-algebras.

(ii) p has universal and existential quantification. That is, for each map $f : X \to Y$ of $\mathbb{X}$, the reindexing functor $f^* : p^{-1}(Y) \to p^{-1}(X)$ has a right adjoint $\forall_f : p^{-1}(X) \to p^{-1}(Y)$ and a Frobenius left adjoint $\exists_f : p^{-1}(X) \to p^{-1}(Y)$, both of which satisfy the Beck-Chevalley condition. That is, if

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a pullback in $\mathbb{X}$, then $g^* \circ \forall_f \simeq \forall_k \circ h^*$, and $g^* \circ \exists_f \simeq \exists_k \circ h^*$.

(iii) p has a generic predicate. That is, an object σ in $p^{-1}(\Sigma)$ for some Σ in $\mathbb{X}$ such that for every object X in $\mathbb{X}$, every φ in $p^{-1}(X)$, there is a map $[\varphi] : X \to \Sigma$ of $\mathbb{X}$ such that φ is isomorphic to $[\varphi]^*(\sigma)$.

It is worth mentioning that it is possible to define a tripos over any *cartesian* category, not necessarily cartesian closed [31] [36]. In this more general setting we require a *membership predicate* ([36] Definition 2.1.2) instead of a generic predicate. In the cartesian closed case the definition is slightly easier to satisfy, and is general enough for our purposes.

We have already constructed one example of a tripos:

Proposition 7.2. *If* $\mathbb{X}$ *is a discrete cartesian closed restriction category, then the domain fibration* $\mathsf{total}(\mathcal{O}) : \mathsf{total}(\mathcal{R}(\mathbb{X})) \to \mathsf{total}(\mathbb{X})$ *is a tripos.*

Proof. Clearly $\mathsf{total}(\mathbb{X})$ is a cartesian closed category. Universal and existential quantification for $\mathsf{total}(\mathcal{O})$ are given by the (different notion of) universal and existential quantification in $\mathbb{X}$, which are both present since $\mathbb{X}$ is a discrete cartesian closed restriction category. Similarly, that each fiber of $\mathsf{total}(\mathcal{O})$ is a Heyting algebra follows from each $\mathcal{O}(X)$ for X in $\mathbb{X}$ being a Heyting algebra. We have also shown that for total maps f of $\mathbb{X}$ the pullback functor f^* is a Heyting algebra morphism, so we need only show the existence of a generic predicate. For this, we use the subobject classifier

$$\Omega \xrightarrow{\mathsf{ev}} 1$$

then $\overline{\mathsf{ev}} \in \mathcal{O}(\Omega)$ is the generic predicate. Suppose $e \in \mathcal{O}(X)$ for some object X of $\mathbb{X}$. Then we have $\chi_\varphi : X \to \Omega$ which satisfies

$$\overline{\chi_\varphi \overline{\mathsf{ev}}} = \overline{\chi_\varphi \mathsf{ev}} = \overline{\varphi !} = \varphi$$

as required. $\square$

The following well known lemma [27] [18] about existential quantification in a fibered Heyting algebra (such as a tripos) will be useful in the next section:

Lemma 7.3. *If* $\exists_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ *is a left adjoint to* $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ *for some map* $f : X \to Y$ *of a restriction category* $\mathbb{X}$ *such that both* $\mathcal{O}(X)$ *and* $\mathcal{O}(Y)$ *are Heyting algebras, then the Frobenius identity*

$$\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi))$$

holds if and only if

$$f^*(\varphi \Rightarrow \psi) = (f^*(\varphi) \Rightarrow f^*(\psi))$$

holds.

Proof. Suppose $\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi))$. First, we have

$$\begin{array}{c}
f^*(\varphi \Rightarrow \psi) \leq f^*(\varphi) \Rightarrow f^*(\psi) \\ \hline\hline
f^*(\varphi \Rightarrow \psi) \wedge f^*(\varphi) \leq f^*(\psi) \\ \hline\hline
\exists_f(f^*(\varphi \Rightarrow \psi) \wedge f^*(\varphi)) \leq \psi) \\ \hline\hline
\exists_f(f^*(\varphi \Rightarrow \psi)) \wedge \psi \leq \psi
\end{array}$$

and certainly $\exists_f(f^*(\varphi \Rightarrow \psi)) \wedge \psi \leq \psi$, so $f^*(\varphi \Rightarrow \psi) \leq f^*(\varphi) \Rightarrow f^*(\psi)$.

Next, we show the reverse inequality

$$\begin{array}{c}
f^*(\varphi) \Rightarrow f^*(\psi) \leq f^*(\varphi \Rightarrow \psi) \\ \hline\hline
\exists_f(f^*(\varphi) \Rightarrow f^*(\psi)) \leq \varphi \Rightarrow \psi \\ \hline\hline
\exists_f(f^*(\varphi) \Rightarrow f^*(\psi)) \wedge \varphi \leq \psi \\ \hline\hline
\exists_f((f^*(\varphi) \Rightarrow f^*(\psi)) \wedge f^*(\varphi) \leq \psi \\ \hline\hline
(f^*(\varphi) \Rightarrow f^*(\psi)) \wedge f^*(\varphi) \leq f^*(\psi) \\ \hline\hline
f^*(\varphi) \Rightarrow f^*(\psi) \leq f^*(\varphi) \Rightarrow f^*(\psi)
\end{array}$$

and certainly $f^*(\varphi) \Rightarrow f^*(\psi) \leq f^*(\varphi) \Rightarrow f^*(\psi)$, so $f^*(\varphi) \Rightarrow f^*(\psi) \leq f^*(\varphi \Rightarrow \psi)$, and therefore $f^*(\varphi \Rightarrow \psi) = f^*(\varphi) \Rightarrow f^*(\psi)$.

Now, suppose $f^*(\varphi \Rightarrow \psi) = f^*(\varphi) \Rightarrow f^*(\psi)$. We must show $\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi))$. Notice that

$$
\begin{array}{c}
\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi)) \\
\hline\hline
\exists_f(\varphi) \leq \psi \Rightarrow \exists_f(\varphi \wedge f^*(\psi)) \\
\hline\hline
\varphi \leq f^*(\psi \Rightarrow \exists_f(\varphi \wedge f^*(\psi))) \\
\hline\hline
\varphi \leq f^*(\psi) \Rightarrow f^*(\exists_f(\varphi \wedge f^*(\psi))) \\
\hline\hline
\varphi \wedge f^*(\psi) \leq f^*(\exists_f(\varphi \wedge f^*(\psi))) \\
\hline\hline
\exists_f(\varphi \wedge f^*(\psi)) \leq \exists_f(\varphi \wedge f^*(\psi))
\end{array}
$$

Clearly $\exists_f(\varphi \wedge f^*(\psi)) \leq \exists_f(\varphi \wedge f^*(\psi))$, and thus $\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi))$ as required.

$\square$

7.2 The Partial Topos of a Tripos

Every tripos has as its internal language a model of higher-order intuitionistic logic without equality. Given a tripos $p : \mathbb{E} \to \mathbb{X}$, formulas are interpreted as elements over some fiber $p(X)$ of the tripos. For a closed formula φ, say that $p \vDash \varphi$ in case the interpretation of φ into p is the top element of $p(1)$, the fiber of the terminal object. Working in this internal language, we define the category $\mathbb{X}[p]$ of *partial equivalance relations over* p as follows, writing $[\varphi]$ for the interpretation of φ as an element of some fiber of the tripos:

objects are pairs $(X, \sim)$ where $\sim \in p(X \times X)$ such that $\sim$ is symmetric and transitive. That is,

$$p \vDash \forall xy(x \sim y \to y \sim x)$$

$$p \vDash \forall xyz(x \sim y \wedge y \sim z \to x \sim z)$$

hold.

maps $(X, \sim) \to (Y, \sim)$ of $\mathbb{X}[p]$ are isomorphism classes of objects $F \in p(X \times Y)$ that are strict, relational, deterministic, and total. That is,

$$p \vDash \forall xy(F(x,y) \to x \sim x \wedge y \sim y)$$

$$p \vDash \forall xx'yy'(F(x,y) \wedge x \sim x' \wedge y \sim y' \to F(x',y')$$

$$p \vDash \forall xyy'(F(x,y) \wedge F(x,y') \to y \sim y')$$

$$p \vDash \forall x(x \sim x \to \exists y F(x,y))$$

hold.

composition is as in the category of relations. That is, the composite of $F : (X, \sim) \to (Y, \sim)$ and $G : (Y, \sim) \to (Z, \sim)$ is defined to be

$$FG := [\exists y(F(x,y) \wedge G(y,z))]$$

The **identity map** on $(X, \sim)$ is given by $\sim \in p(X \times X)$ itself.

So defined, the category $\mathbb{X}[p]$ is a topos [31].

There is a notion of *partial topos*, first introduced in [13], that can be phrased in terms of restriction categories as follows:

Definition 7.4. A *partial monic* in a restriction category is a map m such that if $fm = gm$, then $f\overline{m} = g\overline{m}$.

Definition 7.5. A *partial topos* is a discrete cartesian closed restriction category in which every partial monic has a partial inverse.

The total map category of such a partial topos is a topos, and the category of partial maps formed from the stable system of monics consisting of all monics of some topos is a partial topos.

Given the existence of these partial toposes and the fact that our category of assemblies is a restriction category whose total maps are the classical category of assemblies, it is natural to ask if there is a restriction category whose total maps are the realizability tripos we have constructed. We give a positive answer to this question by showing that when the requirement that maps in the category of partial equivalence relations be *total* relations is dropped, the result is a partial topos.

Definition 7.6. If $\mathbb{X}$ is a category with finite products and $p : \mathbb{E} \to \mathbb{X}$ is a tripos, define the restriction category $\mathbb{X}\{p\}$ as follows:

objects as in $\mathbb{X}[p]$.

maps $(X, \sim) \to (Y, \sim)$ are isomorphism classes of objects F in $p(X \times Y)$ which are strict, relational, and deterministic. That is,

$$p \vDash \forall xy(F(x,y) \to x \sim x \wedge y \sim y)$$
$$p \vDash \forall xx'yy'(F(x,y) \wedge x \sim x' \wedge y \sim y' \to F(x',y'))$$
$$p \vDash \forall xyy'(F(x,y) \wedge F(x,y') \to y \sim y')$$

hold. Note that we do not require our maps to be total, and that the maps of $\mathbb{X}[p]$ are precisely the maps of $\mathbb{X}\{p\}$ for which $p \vDash \forall x(x \sim x \to \exists y F(x,y))$ holds.

composition as in $\mathbb{X}[p]$.

identities as in $\mathbb{X}[p]$.

restriction If $F : (X, \sim) \to (Y, \sim)$ is a map of $\mathbb{X}\{p\}$, defined $\overline{F} : (X, \sim) \to (X, \sim)$ by

$$\overline{F} := [(x \sim x') \wedge \exists y F(x,y)]$$

in $p(X \times X)$.

Since maps of $\mathbb{X}\{p\}$ are defined as isomorphism classes of interpreted formulae, and the interpretation is sound, we may show two formulas give the same map when interpreted by showing that each implies the other. In fact, we can more generally show that $[\varphi] \leq [\psi]$ in the relevant fiber of p by showing that $\varphi \vDash \psi$ in the internal language. We make heavy use of this below.

Proposition 7.7. *$\mathbb{X}\{p\}$ as defined above is a restriction category.*

Proof. We must show that composition is well-defined and associative, that identities are well-defined and satisfy the definition of identity maps, and that restriction is well-defined and satisfies the restriction axioms.

For composition, we first show that for $F : (X, \sim) \to (Y, \sim)$, $G : (Y, \sim) \to (Z, \sim)$, the composite $FG := [\exists y(F(x, y) \wedge G(y, z))]$ is strict, relational, and deterministic. For strictness, observe that if $FG(x, z)$, then in particular $F(x, y)$ and $G(y, z)$, so since F and G are strict we know $x \sim x$ and $z \sim z$, meaning that FG is also strict. To see that FG is relational, suppose $FG(x, z)$, $x \sim x'$, and $z \sim z'$. Then for some y we know $F(x, y)$ and $G(y, z)$, and that $y \sim y$ since F is strict. Now we have $F(x', y)$ and $G(y, z')$ since F and G are relational, so we know $\exists y(F(x', y) \wedge G(y, z'))$, which means $FG(x', z')$, and so FG is relational. For determinism, suppose $FG(x, z)$ and $FG(x, z')$. Then for some y, y' we have $F(x, y)$,$G(y, z)$,$F(x, y')$,$G(y', z')$. Since F is deterministic we then know $y \sim y'$, which since G is relational gives $G(y, z')$, meaning that $z \sim z'$ since G is deterministic. Thus FG is well-defined.

To show that composition is associative, we suppose that $F : (X, \sim) \to (Y, \sim)$, $G : (Y, \sim) \to (Z, \sim)$, and $H : (Z, \sim) \to (W, \sim)$ are maps of $\mathbb{X}\{p\}$. It suffices to show that $(FG)H \dashv\vdash F(GH)$, and we have the following two-way inference:

$$\begin{array}{c}
(FG)G(x, w) \\
\hline\hline
\exists z(FG(x, z) \wedge H(z, w)) \\
\hline\hline
\exists z(\exists y(F(x, y) \wedge G(y, z)) \wedge H(z, w)) \\
\hline\hline
\exists y(F(x, y) \wedge \exists z(G(y, z) \wedge H(z, w))) \\
\hline\hline
F(GH)(x, w)
\end{array}$$

so composition is associative.

Next, we deal with identity maps, and begin by showing that if $(X, \sim)$ is an object of $\mathbb{X}\{p\}$ then $\sim \in p(X \times X)$ is strict, relational, and deterministic. If $x \sim x'$ then by symmetry $x' \sim x$ and transitivity gives $x \sim x$ and $x' \sim x'$, meaning $\sim$ is strict. To see that $\sim$ is relational, suppose $x_1 \sim x_2$, $x_1 \sim x_1'$, and $x_2 \sim x_2'$. Then $x_1' \sim x_1$ by symmetry, and transitivity gives $x_1' \sim x_2'$. For determinism, if $x_1 \sim x_2$ and $x_1 \sim x_2'$, then symmetry gives $x_2 \sim x_1$ and by transitivity we have $x_2 \sim x_2'$. Our identity maps are therefore well-defined.

We must also show that $\sim F = F = F \sim$ for $F : (X, \sim) \to (Y, \sim)$, and it suffices to show that $\sim F \dashv\vdash F \dashv\vdash F \sim$. For $\sim F \dashv\vdash F$, suppose $\sim F(x, y)$. Then $\exists x'$ with $x \sim x'$ and $F(x', y)$, and since F is strict and relational we have $F(x, y)$. Conversely if $F(x, y)$ then since F is strict $x \sim x$, and so $\exists x'(x \sim x' \wedge F(x', y))$, where x' is x, giveing $\sim F(x, y)$. Similarly we have $F \dashv\vdash F \sim$, and so your identity maps are indeed identity maps.

Finally, we turn our attention to restirction. Suppose $F : (X, \sim) \to (Y, \sim)$ is a map of $\mathbb{X}\{p\}$. We show that the domain of definition of F, defined by

$$\overline{F} := [(x \sim x') \wedge \exists y F(x, y)]$$

in $p(X \times X)$ defines a map by showing that it is strict, relational, and deterministic. For strictness, notice that if $\overline{F}(x, x')$ then in particular $x \sim x'$. Symmetry gives $x' \sim x$, and transitivity then gives $x \sim x$ and $x' \sim x'$ as required. To see that $\overline{F}$ is relational, suppose $x_1 \sim x_1'$,$x_2 \sim x_2'$, and $\overline{F}(x_1, x_2)$. Then by symmetry and transitivity we have $x_1' \sim x_1 \sim x_2 \sim x_2'$, and also $y \sim y$ since F is strict. As F is also relational we now know $x_1' \sim x_2' \wedge \exists y F(x_1', y)$, and so $\overline{F}$ is relational. For determinism, if $\overline{F}(x_1, x_2)$ and $\overline{F}(x_1, x_2')$ then in particluar $x_1 \sim x_2$ and $x_1 \sim x_2'$, so $x_2 \sim x_2'$ as required, and $\overline{F}$ is well-defined.

For the restriction axioms:

For $\overline{F}F = F$, it suffices to show $\overline{F}F \dashv\vdash F$. If $\overline{F}F(x,y)$ then $\exists x'(\overline{F}(x,x') \wedge F(x',y))$, and since F is strict and relational we know $F(x,y)$, so $\overline{F}F \vdash F$. Conversely, if $F(x,y)$ then $x \sim x$ since F is strict, so certainly we have $\exists x'((x \sim x') \wedge \exists y F(x,y) \wedge F(x,y))$ with $x' = x$, meaning $F \vdash \overline{F}F$, as required.

For $\overline{F}\,\overline{G} = \overline{G}\,\overline{F}$, it straightforward to verify that the following two-way inference holds:

$$
\begin{array}{c}
\overline{F}\,\overline{G}(x_1, x_2) \\ \hline\hline
\exists x(\overline{F}(x_1, x) \wedge \overline{G}(x, x_2)) \\ \hline\hline
\exists x(x_1 \sim x \wedge \exists y F(x_1, y) \wedge x \sim x_2 \wedge \exists z G(x, z)) \\ \hline\hline
\exists xyz(x_1 \sim x \wedge F(x_1, y) \wedge x \sim x_2 \wedge G(x, z)) \\ \hline\hline
\exists xyz(x_1 \sim x \wedge F(x, y) \wedge x \sim x_2 \wedge G(x_1, z)) \\ \hline\hline
\exists x(x_1 \sim x \wedge \exists z G(x_1, z) \wedge x \sim x_2 \wedge \exists y F(x, y)) \\ \hline\hline
\exists x(\overline{G}(x_1, x) \wedge \overline{F}(x, x_2)) \\ \hline\hline
\overline{G}\,\overline{F}(x_1, x_2)
\end{array}
$$

which means $\overline{F}\,\overline{G} \dashv\vdash \overline{G}\,\overline{F}$, and we are done.

For $\overline{\overline{F}G} = \overline{F}\,\overline{G}$,

$$
\begin{array}{c}
\overline{\overline{F}G}(x_1, x_2) \\ \hline\hline
x_1 \sim x_2 \wedge \exists z \overline{F}G(x_1, z) \\ \hline\hline
x_1 \sim x_2 \wedge \exists z \exists x(\overline{F}(x_1, x) \wedge G(x, z)) \\ \hline\hline
x_1 \sim x_2 \wedge \exists z \exists x(x_1 \sim x \wedge \exists_y F(x_1, y) \wedge G(x, z)) \\ \hline\hline
\exists xyz(x_1 \sim x_2 \wedge x_1 \sim x \wedge F(x_1, y) \wedge G(x, z)) \\ \hline\hline
\exists xyz(x_1 \sim x \wedge x \sim x_2 \wedge F(x_1, y) \wedge G(x, z)) \\ \hline\hline
\exists x(x_1 \sim x \wedge \exists_y F(x_1, y) \wedge x \sim x_2 \wedge G(x, z)) \\ \hline\hline
\exists x(\overline{F}(x_1, x) \wedge \overline{G}(x, x_2)) \\ \hline\hline
\overline{F}\,\overline{G}(x_1, x_2)
\end{array}
$$

For $F\overline{G} = \overline{FG}F$,

$$
\begin{array}{c}
F\overline{G}(x,y) \\
\hline\hline
\exists y'(F(x,y') \wedge \overline{G}(y',y)) \\
\hline\hline
\exists y'(F(x,y') \wedge y \sim y' \wedge \exists z G(y',z)) \\
\hline\hline
\exists y' z(y' \sim y \wedge F(x,y') \wedge G(y,z)) \\
\hline\hline
\exists x' y' z(x \sim x' \wedge F(x,y') \wedge G(y',z) \wedge F(x',y)) \\
\hline\hline
\exists x'(x \sim x' \wedge \exists z \exists y'(F(x,y') \wedge G(y',z)) \wedge F(x',y)) \\
\hline\hline
\exists x'(x \sim x' \wedge \exists z FG(x,z) \wedge F(x',y)) \\
\hline\hline
\exists x'(\overline{FG}(x,x') \wedge F(x',y)) \\
\hline\hline
\overline{FG}F(x,y)
\end{array}
$$

We have now shown that $\mathbb{X}\{p\}$ is a restriction category. $\square$

Observe that $\overline{F} = 1_{(X,\sim)}$ if and only if $p \vDash \forall x(x \sim x \rightarrow \exists y F(x,y))$ as follows: Suppose $p \vDash \forall x(x \sim x \rightarrow \exists y F(x,y))$. Then if $x \sim x'$, we know $x \sim x$, which means $\exists y F(x,y)$ by assumption and so $x \sim x' \wedge \exists y F(x,y)$. Conversely if $x \sim x' \wedge \exists y F(x,y)$ then certainly $x \sim x'$, and so $\overline{F} = \sim = 1_{(X,\sim)}$. On the other hand, if we assume that $\overline{F} = 1_{(X,\sim)}$, then if $x \sim x$, we have $\overline{F}(x,x)$, which is to say $x \sim x \wedge \exists y F(x,y)$, so $p \vDash \forall x(x \sim x \rightarrow \exists y F(x,y))$ holds.

This means that $\mathsf{total}(\mathbb{X}\{p\})$ is $\mathbb{X}[p]$.

Next, observe that $\mathbb{X}\{p\}$ is split. Notice that if $(X,\sim)$ is an object of $\mathbb{X}\{p\}$ and $F : (X,\sim) \rightarrow (Y,\sim)$, then the domain of defintion of F, $\overline{F} \in p(X \times X)$, is a symmetric and transitive relation, and so also defines an object $(X,\overline{F})$ of $\mathbb{X}\{p\}$. Then we a map $S : (X,\overline{F}) \rightarrow (X,\sim)$ defined by

$$S(x,x') := x \sim x'$$

e in $p(X \times X)$, and a map $R : (X,\sim) \rightarrow (X,\overline{F})$ defined by

$$R(x,x') := \overline{F}(x,x')$$

in $p(X \times X)$. $SR = \sim \overline{F} = \overline{F} = 1_{(X,\overline{F})}$ and $RS = \overline{F} \sim = \overline{F}$, meaning that we have shown $\overline{F}$ is split. Since F was an arbitrary map in $\mathbb{X}\{p\}$, we know that $\mathbb{X}\{p\}$ is a split restriction category.

Finally, we show that every monic in $\mathbb{X}\{p\}$ is a restriction monic as follows: A map $M : (X, \sim) \to (Y, \sim)$ is monic if and only if $p \vDash \forall x x' y (M(x, y) \wedge M(x', y) \to x \sim x')$. Thus, given a monic $M : (X, \sim) \to (Y, \sim)$, we define its retraction to be

$$R(y, x) := M(x, y)$$

We show that this is well-defined: R is obviously strict. To see that R is relational suppose $x \sim x'$,$y \sim y'$, and $R(y, x)$. Then $M(x, y)$ and since M is relational $M(x', y')$, meaning $R(y', x')$. To show R is deterministic suppose $R(y, x)$ and $R(y, x')$. Then $M(x, y)$ and $M(x', y)$, so since M is monic $x \sim x'$, as required. Now, we show that $1 \leq MR$ and $RM \leq 1$, establishing that MR is total and RM is a restriction idempotent, and thus that M is a restriction monic. For $1 \leq MR$, we use that M is total to obtain

$$\cfrac{\cfrac{\cfrac{\cfrac{x \sim x'}{x \sim x' \wedge \exists y M(x, y)}}{\exists y (M(x, y) \wedge M(x', y))}}{\exists y (M(x, y) \wedge R(y, x'))}}{MR(x, x')}$$

and for $RM \leq 1$, we have

$$\cfrac{\cfrac{\cfrac{RM(y, y')}{\exists x (R(y, x) \wedge M(x, y'))}}{\exists x (M(x, y) \wedge M(x, y'))}}{y \sim y'}$$

Thus the class of all monics in $\mathbb{X}\{p\}$ is the class $\mathcal{M}_{\mathbb{X}\{p\}}$ of restriction monics in $\mathbb{X}\{p\}$, which together with the fact that

$$\mathbb{X}\{p\} \simeq \mathsf{Par}(\mathsf{total}(\mathbb{X}\{p\}), \mathcal{M}_{\mathbb{X}\{p\}}) \simeq \mathsf{Par}(\mathbb{X}[p], \mathcal{M}_{\mathbb{X}\{p\}})$$

tells us that $\mathbb{X}\{p\}$ is a partial topos, since $\mathbb{X}[p]$ is a topos and $\mathcal{M}_{\mathbb{X}\{p\}}$ contains all the monic maps in $\mathbb{X}\{p\}$ (see [13] theorem 6.1).

8 The Realizability Fibration

Let $\mathbb{A}$ be a Turing category, $\mathbb{X}$ a discrete cartesian closed restriction category, and $F : \mathbb{A} \to \mathbb{X}$ a cartesian restriction functor. In this section we are concerned with the properties of the fibration

$$\mathsf{total}(\partial(\mathbb{X})) : \mathsf{total}(\mathsf{asm}(F)) \to \mathsf{total}(\mathbb{X})$$

Specifically, we show that it is a tripos. Most of our reasoning will be about the realizability latent fibration $\partial : \mathsf{asm}(F) \to \mathbb{X}$.

8.1 Existential Quantification

Existential quantification for $\mathsf{total}(\partial)$ is defined directly in terms of existential quantification for $\mathsf{total}(\mathcal{O})$. Universal quantification is a bit more complicated.

Proposition 8.1. $\mathsf{total}(\partial)$ *has existential quantification. That is, for every map* $f : X \to Y$ *of* $\mathsf{total}(\mathbb{X})$*, the reindexing functor* $f^* : \mathsf{total}(\partial)^{-1}(Y) \to \mathsf{total}(\partial)^{-1}(X)$ *has a left adjoint which satisfies the Beck-Chevalley and Frobenius conditions.*

Proof. We use the existential quantification in $\mathsf{total}(\mathcal{O})$ to define the existential quantification in $\mathsf{total}(\partial)$. Suppose X is an object of $\mathbb{X}$, $\varphi \in \mathcal{O}(F(A) \times X)$ is an object of $\mathsf{total}(\partial)^{-1}(X)$, and $f : X \to Y$ is a map of $\mathsf{total}(\mathbb{X})$. Define $\exists_f(\varphi) \in \mathcal{O}(F(A) \times Y)$ by

$$\exists_f(\varphi) := \exists_{(1\times f)}(\varphi)$$

We must show that the mapping defined by $\exists_f : \mathsf{total}(\partial)^{-1}(X) \to \mathsf{total}(\partial)^{-1}(Y)$ is a functor, that it is left adjoint to $f^* : \mathsf{total}(\partial)^{-1}(Y) \to \mathsf{total}(\partial)^{-1}(X)$, and that it satifies the Beck-Chevalley and Frobenius conditions.

First, we show that $\exists_f$ is left adjoint to f^*, the reindexing functor over f in $\mathsf{total}(\partial)$. We must prove that if $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$ are objects in the fiber over X and Y respectively, then

$$\frac{\exists_{1\times f}(\varphi) = \exists_f(\varphi) \vdash \psi}{\varphi \vdash f^*(\psi) = \overline{(1 \times f)\psi}}$$

It suffices to show that $\gamma : A \to B$ in $\mathbb{A}$ is a realizer for $\exists_{1\times f}(\varphi) \vdash \psi$ if and only if it is a realizer for $\varphi \vdash \overline{(1 \times f)\psi}$. Notice that

$$\frac{\dfrac{\exists_{(1\times f)}(\varphi) = \overline{\exists_{(1\times f)}(\varphi)(F(\gamma) \times 1)}}{\exists_{(1\times f)}(\varphi) \leq \overline{(F(\gamma) \times 1)}}}{\dfrac{\varphi \leq \overline{(1 \times f)\overline{(F(\gamma) \times 1)}} = \overline{(F(\gamma) \times 1)}}{\overline{\varphi(F(\gamma) \times 1)} = \varphi}}$$

and so [**Tk.2**] in each case implies [**Tk.2**] in the other, and we may restrict our attention to [**Tk.1**] . Suppose that $\exists_{1\times f}(\varphi) \vdash \psi$ is realized by γ. Then we have

$$\exists_{(1\times f)}(\varphi) = \overline{\exists_{(1\times f)}(\varphi)(F(\gamma) \times 1)} = \overline{\exists_{(1\times f)}(\varphi)(F(\gamma) \times 1)\psi}$$
$$= \exists_{(1\times f)}(\varphi)\overline{(F(\gamma) \times 1)\psi}$$

meaning $\exists_{(1\times f)}(\varphi) \leq \overline{(F(\gamma) \times 1)\psi}$. But then

$$\frac{\exists_{(1\times f)}(\varphi) \leq (F(\gamma) \times 1)\psi}{\varphi \leq \overline{(1 \times f)\overline{(F(\gamma) \times 1)\psi}} = \overline{(F(\gamma) \times f)\psi}}$$

which gives [**Tk.1**] :

$$\varphi(F(\gamma) \times 1)\overline{(1 \times f)\psi} = \varphi\overline{(F(\gamma) \times 1)(1 \times f)\psi}(F(\gamma) \times 1)$$
$$= \varphi\overline{(F(\gamma) \times f)\psi}(F(\gamma) \times 1) = \varphi(F(\gamma) \times 1)$$

and so γ realizes $\varphi \vdash \overline{(1 \times f)\psi}$. For the converse, suppose γ realizes $\varphi \vdash \overline{(1 \times f)\psi}$. Then we have

$$\begin{aligned}
\varphi &= \overline{\varphi(F(\gamma) \times 1)} = \overline{\varphi(F(\gamma) \times 1)\overline{(1 \times f)\psi}} \\
&= \overline{\varphi(1 \times f)\overline{(F(\gamma) \times 1)\psi}} = \varphi\,\overline{(1 \times f)\overline{(F(\gamma) \times 1)\psi}}
\end{aligned}$$

meaning $a \leq \overline{(1 \times f)\overline{F(\gamma) \times 1)b}}$. But then

$$\frac{a \leq \overline{(1 \times f)\overline{(F(\gamma) \times 1)b}}}{\exists_{(1 \times f)}(a) \leq \overline{(F(\gamma) \times 1)b}}$$

which gives [**Tk.1**] :

$$\begin{aligned}
&\exists_{(1 \times f)}(\varphi)(F(\gamma) \times 1)\psi = \exists_{(1 \times f)}(\varphi)\overline{(F(\gamma) \times 1)\psi}(F(\gamma) \times 1) \\
&= \exists_{(1 \times f)}(\varphi)(F(\gamma) \times 1)
\end{aligned}$$

and so γ realizes $\exists_{1 \times f}(\varphi) \vdash \psi$. Thus, $\exists_f$ is left adjoint to reindexing over f in $\mathsf{total}(\partial)$.

This also gives that $\exists_f$ is a functor of preorders. If $\varphi \vdash \psi$, then $\varphi \vdash \psi \vdash f^*(\exists_f(\psi))$, and we have $\exists_f(\varphi) \vdash \exists_f(\psi)$, as required.

Next, we show that if $f : X \to Y$ in $\mathsf{total}(\mathbb{X})$ then $\exists_f$ in $\mathsf{total}(\partial)$ satisfies the Beck-Chevalley condition.

Suppose that the square

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback in $\mathsf{total}(\mathbb{X})$, and that $\varphi : \mathcal{O}(F(A) \times X)$ is an object of $\mathsf{total}(\partial)^{-1}(X)$. Then the square

$$\begin{tikzcd}
F(A) \times W \arrow[r, "1 \times h"] \arrow[d, "1 \times k"'] & F(A) \times X \arrow[d, "1 \times f"] \\
F(A) \times Z \arrow[r, "1 \times g"'] & F(A) \times Y
\end{tikzcd}$$

is also a pullback in $\mathsf{total}(\mathbb{X})$, and since the Beck-Chevalley condition holds for existential quantification in $\mathsf{total}(\mathcal{O})$, we know

$$\overline{(1 \times g)\exists_{1\times f}(\varphi)} = \exists_{1\times k}(\overline{(1 \times h)\varphi})$$

which says exactly that

$$g^*(\exists_f(\varphi)) = \exists_k(h^*(\varphi))$$

in $\mathsf{total}(\partial)$, as required.

For the Frobenius condition, we must show that for $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$,

$$\exists_f(\varphi) \wedge \psi \vdash \exists_f(\varphi \wedge f^*(\psi))$$

Specifically, we must construct a realizer $\gamma : A \times B \to A \times B$ of $\mathbb{A}$ for this entailment.

Notice that if

$$\exists_f(\varphi) \wedge \psi \leq \exists_f(\varphi \wedge f^*(\psi))$$

then we know that $1_{A\times B} : A \times B \to A \times B$ is our realizer as follows:

[Tk.1]

$$\begin{aligned}
&(\exists_f(\varphi) \wedge \psi)(F(1) \times 1)(\exists_f(\varphi \wedge f^*(\psi))) \\
&= (\exists_f(\varphi) \wedge \psi)(\exists_f(\varphi \wedge f^*(\psi))) \\
&= \exists_f(\varphi) \wedge \psi
\end{aligned}$$

[Tk.2] immediately

$$\overline{(\exists_f(\varphi) \wedge \psi)(F(1) \times 1)} = \exists_f(\varphi) \wedge \psi$$

Using that existential quantification over $\mathsf{total}(\mathcal{O})$ satisfies the Beck-Chevalley and Frobenius conditions, we have

$$\begin{aligned}
\exists_f(\varphi) \wedge \psi &= \overline{(F(\pi_0) \times 1)\exists_{1\times f}(\varphi)}\ \overline{(F(\pi_1) \times 1)\psi} \\
&= \exists_{1\times f}(\overline{(F(\pi_0) \times 1)\varphi}) \wedge \overline{(F(\pi_1) \times 1)\psi} \\
&\leq \exists_{1\times f}(\overline{(F(\pi_0) \times 1)\varphi} \wedge \overline{(F(\pi_1) \times 1)(1 \times f)\psi}) \\
&= \exists_f(\varphi \wedge f^*(\psi))
\end{aligned}$$

as required.

Thus, $\mathsf{total}(\partial)$ has existential quantification.

$\square$

8.2 Fibered Heyting algebra Structure

Consider $\mathsf{total}(\partial)^{-1}(X)$ for some object X of $\mathbb{X}$. Its objects are assemblies over X, and if $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times X)$ are two such assemblies, a map $f : \varphi \to \psi$ in $\mathsf{total}(\partial)^{-1}(X)$ is a map $f : \varphi \to \psi$ in $\mathsf{asm}(F)$ with $\partial(f) = 1_X$. This means there is at most one map between any two objects φ and ψ of $\mathsf{total}(\partial)^{-1}(X)$. That is, each fiber of $\mathsf{total}(\partial)^{-1}(X)$ is a preorder. We write $\varphi \vdash \psi$ to indicate that a map $\varphi \to \psi$ exists, and note that this is the case if and only if there is a tracking map $\gamma : A \to B$ for $1_X : X \to X$. So, to show $\varphi \vdash \psi$, it is necessary and sufficient to exhibit a map $\gamma : A \to B$ of $\mathbb{A}$ such that

[Tk.1] $\varphi(F(\gamma) \times 1_X)\psi = \varphi(F(\gamma) \times 1_X)$

[Tk.2] $\overline{\varphi(F(\gamma) \times 1_X)} = \varphi$

hold.

In fact, each fiber of $\mathsf{total}(\partial)$ is a Heyting algebra. We show this now, beginning with the bounded meet semilattice structure.

Proposition 8.2. *Each fiber of* $\mathsf{total}(\partial)$ *has binary meets and a top element. Further, reindexing over* $\mathsf{total}(\partial)$ *preserves the meet and top element.*

Proof. Let X be an object of $\mathbb{X}$. We define the top element of $\mathsf{total}(\partial)^{-1}(X)$ to be the identity map on $F(1) \times X$.

$$\top := 1 \in \mathcal{O}(F(1) \times X)$$

If $\varphi \in \mathcal{O}(F(A) \times X)$ is an object of $\mathsf{total}(\partial)^{-1}(X)$, we have $\varphi \vdash \top$ since $!_A : A \to 1$ tracks 1_X:

[Tk.1]

$$\varphi(F(!) \times 1)1 = \varphi(F(!) \times 1)$$

[Tk.2]

$$\overline{\varphi(F(!) \times 1)} = \varphi$$

as required. To see that reindexing over $\mathsf{total}(\partial)$ preserves the top element, note that since f is total,

$$f^*(\top) = \overline{(1 \times f)\top} = 1 = \top$$

Now, suppose $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times X)$ are objects of $\mathsf{total}(\partial)^{-1}(X)$. We define their meet, $(\varphi \wedge \psi) \in \mathcal{O}(F(A \times B) \times X)$ to be

$$(\varphi \wedge \psi) := \overline{(F(\pi_0) \times 1)\varphi}\ \overline{(F(\pi_1) \times 1)\psi}$$

We show $\varphi \wedge \psi \vdash \varphi$ using π_0 as the tracking map for 1_X:

[Tk.1]

$$(\varphi \wedge \psi)(F(\pi_0) \times 1)\varphi = \overline{(F(\pi_1) \times 1)\psi}\ \overline{(F(\pi_0) \times 1)\varphi}(F(\pi_0) \times 1)\varphi$$

$$= \overline{(F(\pi_1) \times 1)\psi}(F(\pi_0) \times 1)\varphi\varphi = \overline{(F(\pi_1) \times 1)\psi}(F(\pi_0) \times 1)\varphi$$

$$= \overline{(F(\pi_1) \times 1)\psi}\ \overline{(F(\pi_0) \times 1)\varphi}(F(\pi_0) \times 1)$$

[Tk.2]

$$\overline{(\varphi \wedge \psi)(F(\pi_0) \times 1)} = (\varphi \wedge \psi)$$

Similarly, π_1 tracks 1_X to give $(\varphi \wedge \psi) \vdash \psi$.

Next, suppose that $\chi \in \mathcal{O}(F(C) \times X)$ is an object of $\mathsf{total}(\partial)^{-1}(X)$ with $\chi \vdash \varphi$ and $\chi \vdash \psi$. Let $\gamma : C \to A$ and $\omega : C \to B$ be the tracking maps for 1_X that then exist. We obtain $\chi \vdash (\varphi \wedge \psi)$ by showing that $\langle \gamma, \omega \rangle : C \to A \times B$ is a tracking map for 1_X:

[Tk.1]

$$\chi(F(\langle \gamma, \omega \rangle) \times 1)(\varphi \wedge \psi) =$$

$$= \chi(F(\langle \gamma, \omega \rangle) \times 1)\overline{(F(\pi_0) \times 1)\varphi}\ \overline{(F(\pi_1) \times 1)\psi}$$

$$= \overline{\chi(F(\overline{\omega}\gamma) \times 1)\varphi}\ \overline{\chi(F(\overline{\gamma}\omega) \times 1)\psi}\, \chi(F(\langle \gamma, \omega \rangle) \times 1)$$

$$= \overline{\overline{\chi(F(\omega) \times 1)}(F(\gamma) \times 1)\varphi}\ \overline{\overline{\chi(F(\gamma) \times 1)}(F(\omega) \times 1)}\, \chi(F(\langle \gamma, \omega \rangle) \times 1)$$

$$= \overline{\chi(F(\gamma) \times 1)\varphi}\ \overline{\chi(F(\omega) \times 1)\psi}\, \chi(F(\langle \gamma, \omega \rangle) \times 1)$$

$$= \overline{\chi(F(\gamma) \times 1)}\ \overline{\chi(F(\omega) \times 1)}\, \chi(F(\langle \gamma, \omega) \times 1)$$

$$= \chi(F(\langle \gamma, \omega \rangle) \times 1)$$

[Tk.2]

$$\overline{\chi(F(\langle \gamma, \omega \rangle) \times 1)} = \overline{\chi(F(\overline{\langle \gamma, \omega \rangle}) \times 1)}$$

$$= \overline{\chi(F(\overline{\gamma}) \times 1)(F(\overline{\omega}) \times 1)} = \overline{\chi\, \overline{(F(\gamma) \times 1)}\; \overline{(F(\omega) \times 1)}}$$

$$= \overline{\overline{\chi(F(\gamma) \times 1)}\; \overline{\chi(F(\omega) \times 1)}\, \chi}$$

$$= \chi$$

Finally, we show that reindexing over $\mathsf{total}(\partial)$ preserves the meet:

$$f^*(\varphi \wedge \psi) = \overline{(1 \times f)\, \overline{(F(\pi_0) \times 1)\varphi}\; \overline{(F(\pi_1) \times 1)\psi}}$$

$$= \overline{\overline{(1 \times f)(F(\pi_0) \times 1)\varphi}\; \overline{(1 \times f)(F(\pi_1) \times 1)\psi}\, (1 \times f)}$$

$$= \overline{(F(\pi_0) \times 1)\overline{(1 \times f)\varphi}}\; \overline{(F(\pi_1) \times 1)\overline{(1 \times f)\psi}}$$

$$= (f^*(\varphi) \wedge f^*(\psi))$$

□

We make the following useful observation:

Lemma 8.3. $(\varphi \wedge \psi) = \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times \psi)}$

Proof.

$$\overline{(1 \times \Delta)\mathsf{ex}(\varphi \times \psi)} = \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times 1)(1 \times \psi)}$$

$$= \overline{\overline{(1 \times \Delta)\mathsf{ex}(\varphi \times 1)}\; \overline{(1 \times \Delta)\mathsf{ex}(1 \times \psi)}(1 \times \Delta)\mathsf{ex}}$$

$$= \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times 1)}\; \overline{(1 \times \Delta)\mathsf{ex}(1 \times \psi)}$$

$$= \overline{(1 \times \Delta)\mathsf{ex}\pi_0\varphi}\; \overline{(1 \times \Delta)\mathsf{ex}\pi_1\psi}$$

$$= \overline{(F(\pi_0) \times 1)\varphi}\; \overline{(F(\pi_1) \times 1)\psi} = (\varphi \wedge \psi)$$

□

Next, we deal with the Heyting implication

Proposition 8.4. *Each fiber of* $\mathsf{total}(\partial)$ *has a Heyting implication. Further, reindexing over* $\mathsf{total}(\partial)$ *preserves the Heyting implication.*

Proof. Let $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times X)$ be objects of $\mathsf{total}(\partial)^{-1}(X)$. Define $(\varphi \Rightarrow \psi) \in \mathcal{O}(F(T_{A,B}) \times X)$ to be

$$(\varphi \Rightarrow \psi) := \overline{\lambda((\varphi \wedge 1_{F(T_{A,B}) \times X})(F(\tau_{A,B}) \times 1)) \cap \lambda((\varphi \wedge 1_{F(T_{A,B}) \times X})(F(\tau_{A,B}) \times 1)\psi}$$
$$\overline{\lambda((\varphi \wedge 1_{F(T_{A,B}) \times X})(F(\tau_{A,B}) \times 1)!) \cap \lambda((\varphi \wedge 1_{F(T_{A,B}) \times X})!)}$$

We must show

$$\frac{(\chi \wedge \varphi) \vdash \psi}{\chi \vdash (\varphi \Rightarrow \psi)}$$

Suppose $(\chi \wedge \varphi) \vdash \psi$. Then also $(\varphi \wedge \chi) \vdash \psi$, and this is realized by some $\gamma : A \times C \to B$ in $\mathbb{A}$. Let $h : C \to T_{A,B}$ be a weak exponential transpose of γ, as in

$$\begin{tikzcd} A \times T_{A,B} \arrow[r, "\tau"] & B \\ A \times C \arrow[u, "1 \times h"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We show that h realizes $\chi \vdash (\varphi \Rightarrow \psi)$:

[Tk.1] Observe that

$$\begin{aligned}
&\chi(F(h) \times 1)\lambda((\varphi \wedge 1)(F(\tau) \times 1)) \\
&= \chi(F(h) \times 1)\lambda((1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)) \\
&= \chi\lambda((1 \times (F(h) \times 1))(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)) \\
&= \chi\lambda((1 \times \Delta)\mathsf{ex}(\varphi \times \chi)\mathsf{ex}(1 \times \Delta^{(-1)})(F((1 \times h)\tau) \times 1)) \\
&= \chi\lambda((\varphi \wedge \chi)(F(\gamma) \times 1)) = \chi\lambda((\varphi \wedge \chi)(F(\gamma) \times 1)\psi) \\
&= \chi\lambda((1 \times \Delta)\mathsf{ex}(\varphi \times \chi)\mathsf{ex}(1 \times \Delta^{(-1)})(F((1 \times h)\tau) \times 1)\psi)
\end{aligned}$$

$$= \chi(F(h) \times 1)\lambda((\varphi \wedge 1)(F(\tau) \times 1)\psi)$$

and similarly

$$\begin{aligned}
&\chi(F(h) \times 1)\lambda((\varphi \wedge 1)(F(\tau) \times 1)!) \\
&= \chi\lambda((1 \times (F(h) \times 1))(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)) \\
&= \chi\lambda((\varphi \wedge \chi)(F(\gamma) \times 1)!) = \chi\lambda((\varphi \wedge \chi)!) \\
&= \chi\lambda((1 \times \Delta)\mathsf{ex}(\varphi \times \chi)\mathsf{ex}(1 \times \Delta^{(-1)})!) \\
&= \chi\lambda((1 \times \Delta)\mathsf{ex}(\varphi \times \chi)\mathsf{ex}(1 \times \Delta^{(-1)})((1 \times F(h)) \times 1)!) \\
&= \chi\lambda((1 \times \chi(F(h) \times 1))(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})!) \\
&= \chi(F(h) \times 1)\lambda((\varphi \wedge 1)!)
\end{aligned}$$

which gives

$$\begin{aligned}
&\chi(F(h) \times 1)(\varphi \Rightarrow \psi) \\
&= \overline{\chi(F(h) \times 1)\lambda((\varphi \wedge 1)F(\tau) \times 1)}\ \overline{\chi(F(h) \times 1)\lambda((\varphi \wedge 1)!)}\,\chi(F(h) \times 1) \\
&= \chi(F(h) \times 1)
\end{aligned}$$

[Tk.2] Since h is total we have

$$\overline{\chi(F(h) \times 1)} = \chi$$

For the converse, suppose $\chi \vdash (\varphi \Rightarrow \psi)$. Then some map $\omega : C \to T_{A,B}$ of $\mathbb{A}$ realizes this. We show that $(1 \times \omega)\tau : A \times C \to B$ realizes $(\varphi \wedge \chi) \vdash \psi$ as follows:

[Tk.1]

$$(\varphi \wedge \chi)(F((1 \times \omega)\tau) \times 1)\psi = \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times \chi)}(F((1 \times \omega)\tau) \times 1)\psi$$

$$= \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times \chi)}(1 \times \Delta)\mathsf{ex}(1 \times (F(\omega) \times 1))\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)\psi$$

$$= (1 \times \Delta)\mathsf{ex}(\varphi \times \chi(F(\omega) \times 1))\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)\psi$$

$$= (1 \times \chi(F(\omega) \times 1))(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)\psi$$

$$= (1 \times \chi(F(\omega) \times 1)(\varphi \Rightarrow \psi))((\varphi \wedge 1)(F(\tau) \times 1)\psi)$$

$$= (1 \times \chi(F(\omega) \times 1)(\varphi \Rightarrow \psi)(1 \times \lambda((\varphi \wedge 1)(F(\tau) \times 1)\psi))\mathsf{ev}$$

$$= (1 \times \chi(F(\omega) \times 1)(\varphi \Rightarrow \psi)(1 \times \lambda((\varphi \wedge 1)(F(\tau) \times 1))\mathsf{ev}$$

$$= (1 \times \chi(F(\omega) \times 1)(\varphi \wedge 1)(F(\tau) \times 1)$$

$$= (1 \times \chi(F(\omega) \times 1)(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)$$

$$= (\varphi \wedge \chi)(F((1 \times \omega)\tau) \times 1)$$

[Tk.2]

$$\overline{(\varphi \wedge \chi)(F((1 \times \omega)\tau) \times 1)}$$

$$= \overline{(1 \times \chi(F(\omega) \times 1)(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\mathsf{ex}(1 \times \Delta^{(-1)})(F(\tau) \times 1)!}$$

$$= \overline{(1 \times \chi(F(\omega) \times 1)(\varphi \Rightarrow \psi))(1 \times \lambda((\varphi \wedge 1)(F(\tau) \times 1)!))\mathsf{ev}}$$

$$= \overline{(1 \times \chi(F(\omega) \times 1)(\varphi \Rightarrow \psi))(1 \times \lambda((\varphi \wedge 1)!))\mathsf{ev}}$$

$$= \overline{(1 \times \chi(F(\omega) \times 1))(1 \times \Delta)\mathsf{ex}(\varphi \times 1)!}$$

$$= \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\overline{(1 \times \chi(F(\omega) \times 1))}}$$

$$= \overline{(1 \times \Delta)\mathsf{ex}(\varphi \times 1)\chi} = (\varphi \wedge \chi)$$

So our Heyting implication is in fact a Heyting implication.

Reindexing over $\mathsf{total}(\partial)$ preserves our Heyting implication since existential quantification for $\mathsf{total}(\partial)$ satisfies the Frobenius condition.

$\square$

We move on the the join semilattice structure

Proposition 8.5. *Each fiber of* $\mathsf{total}(\partial)$ *has binary joins and a bottom element. Further, reindexing over* $\mathsf{total}(\partial)$ *preserves the join and bottom element.*

Proof. Let X be an object of $\mathbb{X}$. Since $\mathbb{X}$ is a discrete cartesian closed restriction category, we know that $\mathcal{O}(F(T) \times X)$ has a bottom element. We define the bottom element of $\mathsf{total}(\partial)^{-1}(X)$ to be

$$\perp := \perp \in \mathcal{O}(F(T) \times X)$$

If $\varphi \in \mathcal{O}(F(A) \times X)$ is an object of $\mathsf{total}(\partial)^{-1}(X)$, then since any $e \in \mathcal{O}(F(T) \times X)$ has $\perp \leq e$, any map $\gamma : T \to A$ is a tracking map for 1_X as a map $\perp \to \varphi$ in $\mathsf{asm}(F)$:

[Tk.1]

$$\perp(F(\gamma) \times 1)\varphi = \perp \overline{(F(\gamma) \times 1)\varphi}\,(F(\gamma) \times 1) = \perp(F(\gamma) \times 1)$$

[Tk.2]

$$\overline{\perp(F(\gamma) \times 1)} = \overline{\perp\,\overline{(F(\gamma) \times 1)}} = \perp$$

and since each object A of $\mathbb{A}$ is a retract of T, the retraction $r_A : T \to A$ is such a map and $\perp \vdash \varphi$ for all φ in $\mathsf{total}(\partial)^{-1}(X)$. To see that reindexing over $\mathsf{total}(\partial)$ preserves the bottom element, we recall that reindexing over $\mathsf{total}(\mathcal{O})$ does, and so

$$f^*(\perp) = \overline{(1 \times f)\perp} = \perp$$

For the join of $\varphi \in \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times X)$, we use the join in the fibers $\mathsf{total}(\mathcal{O})$ to define

$$(\varphi \vee \psi) := (\exists_{(F(\kappa)\times 1)}(\varphi) \vee \exists_{(F(\kappa')\times 1)}(\psi)) \in \mathcal{O}(F(A\#B) \times X)$$

where

$$A \xrightarrow{\kappa} A\#B \xleftarrow{\kappa'} B$$

is the weak coproduct of A and B in $\mathbb{A}$. We show that this is actually the join. $\varphi \vdash (\varphi \vee \psi)$ since $\kappa : A \to A\#B$ is a tracking map for 1_X:

[Tk.1] Notice that we have

$$\begin{array}{c} \exists_{F(\kappa)\times 1}(\varphi) \leq (\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi)) \\ \hline\hline \varphi \leq \overline{(F(\kappa)\times 1)(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi))} \\ \hline\hline \varphi = \varphi\,\overline{(F(\kappa)\times 1)(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi))} \end{array}$$

which gives

$$\begin{aligned} &\varphi(F(\kappa)\times 1)(\varphi \vee \psi) = \varphi(F(\kappa)\times 1)(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi)) \\ &= \varphi\,\overline{(F(\kappa)\times 1)(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi))}(F(\kappa)\times 1) \\ &= \varphi(F(\kappa)\times 1) \end{aligned}$$

as required.

[Tk.2] κ is total so immediately

$$\overline{\varphi(F(\kappa)\times 1)} = \varphi$$

Similarly, $\kappa' : B \to A\#B$ is a tracking map for 1_X to give $\psi \vdash (\varphi \vee \psi)$. Next, we suppose that for some $\chi \in \mathcal{O}(F(C)\times X)$ we have $\varphi \vdash \chi$ and $\psi \vdash \chi$, realized by $\gamma : A \to C$ and $\omega : B \to C$ in $\mathbb{A}$ respectively. We take ν to be their weak coproduct map, as in

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[dr, "\gamma"'] & A\#B \arrow[d, "\nu"] & B \arrow[l, "\kappa'"'] \arrow[dl, "\omega"] \\ & C & \end{tikzcd}$$

and now $(\varphi \vee \psi) \vdash \chi$ since ν is a tracking map for 1_X as follows:

[**Tk.1**] Notice that we have

$$\begin{aligned}
&\varphi\,\overline{(F(\kappa)\times 1)\overline{(F(\nu)\times 1)\chi}} = \overline{\varphi(F(\kappa)\times 1)(F(\nu)\times 1)\chi} \\
&= \overline{\varphi(F(\gamma)\times 1)\chi} = \overline{\varphi(F(\gamma)\times 1)} = \varphi
\end{aligned}$$

and that this gives

$$\begin{array}{c}
\varphi\,\overline{(F(\kappa)\times 1)\overline{(F(\nu)\times 1)\chi}} = \varphi \\
\hline\hline
\varphi \leq \overline{(F(\kappa)\times 1)\overline{(F(\nu)\times 1)\chi}} \\
\hline\hline
\exists_{F(\kappa)\times 1}(\varphi) \leq \overline{(F(\nu)\times 1)\chi} \\
\hline\hline
\overline{(F(\nu)\times 1)\chi}\,\exists_{F(\kappa)\times 1}(\varphi) = \exists_{F(\kappa)\times 1}(\varphi)
\end{array}$$

Similarly, we obtain

$$\overline{(F(\nu)\times 1)\chi}\,\exists_{F(\kappa')\times 1}(\psi) = \exists_{F(\kappa')\times 1}(\psi)$$

and then we have

$$\begin{aligned}
&(\varphi\vee\psi)(F(\nu)\times 1)\chi = (\varphi\vee\psi)\overline{(F(\nu)\times 1)\chi}(F(\nu)\times 1) \\
&= \overline{(F(\nu)\times 1)\chi}(\varphi\vee\psi)(F(\nu)\times 1) \\
&= \overline{(F(\nu)\times 1)\chi}(\exists_{F(\kappa)\times 1}(\varphi)\vee\exists_{F(\kappa')\times 1}(\psi))(F(\nu)\times 1) \\
&= (\overline{(F(\nu)\times 1)\chi}\exists_{F(\kappa)\times 1}(\varphi)\vee\overline{(F(\nu)\times 1)}\exists_{F(\kappa')\times 1}(\psi))(F(\nu)\times 1) \\
&= (\exists_{F(\kappa)\times 1}(\varphi)\vee\exists_{F(\kappa')\times 1}(\psi))(F(\nu)\times 1) = (\varphi\vee\psi)(F(\nu)\times 1)
\end{aligned}$$

[**Tk.2**] Using again that

$$\overline{(F(\nu)\times 1)\chi}\,\exists_{F(\kappa)\times 1}(\varphi) = \exists_{F(\kappa)\times 1}(\varphi)$$

and

$$\overline{(F(\nu)\times 1)\chi}\,\exists_{F(\kappa')\times 1}(\psi) = \exists_{F(\kappa')\times 1}(\psi)$$

we have

$$\overline{(\varphi \vee \psi)(F(\nu) \times 1)} = \overline{(F(\nu) \times 1)}(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi))$$
$$= \overline{(F(\nu) \times 1)}\exists_{F(\kappa)\times 1}(\varphi) \vee \overline{(F(\nu) \times 1)}\exists_{F(\kappa')\times 1}(\psi)$$
$$= \exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi) = (\varphi \vee \psi)$$

and so our join is in fact a join. We proceed to show that reindexing over $\mathsf{total}(\partial)$ preserves the join. Since

$$\begin{tikzcd} {} \arrow[r, "1\times f"] \arrow[d, "F(\kappa)\times 1"'] & {} \arrow[d, "F(\kappa)\times 1"] \\ {} \arrow[r, "1\times f"'] & {} \end{tikzcd}$$

is a pullback square in $\mathsf{total}(\mathbb{X})$ and $\exists$ satisfies the Beck-Chevalley condition in $\mathsf{total}(\mathcal{O})$, we have

$$\overline{(1 \times f)\exists_{F(\kappa)\times 1}(\varphi)} = \exists_{F(\kappa)\times 1}(\overline{(1 \times f)\varphi})$$

and similarly

$$\overline{(1 \times f)\exists_{F(\kappa')\times 1}(\psi)} = \exists_{F(\kappa')\times 1}(\overline{(1 \times f)\psi}$$

which gives

$$f^*(\varphi \vee \psi) = \overline{(1 \times f)(\exists_{F(\kappa)\times 1}(\varphi) \vee \exists_{F(\kappa')\times 1}(\psi))}$$
$$= \overline{(1 \times f)\exists_{F(\kappa)\times 1}(\varphi)} \vee \overline{(1 \times f)\exists_{F(\kappa')\times 1}(\psi)}$$
$$= \exists_{F(\kappa)\times 1}(\overline{(1 \times f)\varphi}) \vee \exists_{F(\kappa')\times 1}(\overline{(1 \times f)\psi})$$
$$= (f^*(\varphi) \vee f^*(\psi))$$

$\square$

We have now shown

Theorem 8.6. *Each fiber of* $\mathsf{total}(\partial)$ *is a Heyting algebra. Further, reindexing over* $\mathsf{total}(\partial)$ *is a Heyting algebra morphism.*

8.3 Universal Quantification

We require a technical lemma before we can define the universal quantification.

Lemma 8.7. *In a Turing category, for every object A, there is an object T_A and a map $\tau_A : T_A \to A$ such that for any map $\gamma : B \to A$ there is a total map $\delta : B \to T_A$ such that $\delta\tau_A = \gamma$, as in*

$$\begin{tikzcd}
T_A \arrow[r, "\tau_A"] & A \\
B \arrow[u, dotted, "\delta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Proof. Fix an object A. Using the weakly cartesian closed notation, we define $T_A := 1 \times T_{1,A}$ and $\tau_A := \tau_{1,A}$. Suppose $\gamma : B \to A$. Then we have

$$\begin{tikzcd}
1 \times T_A \arrow[r, "\tau_{1,A}"] & A \\
1 \times B \arrow[u, dotted, "1\times h"] \arrow[ur, "\pi_1\gamma"'] & \\
B \arrow[u, "\langle !_B, 1_B\rangle"] \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

and the $\delta := \langle !_B, 1_B\rangle(1 \times h)$. ◻

With this in hand, we move on to the universal quantification.

Proposition 8.8. $\mathsf{total}(\partial)$ *has universal quantification. That is, for every map $f : X \to Y$ of* $\mathsf{total}(\mathbb{X})$, *the reindexing functor $f^* : \mathsf{total}(\partial)^{-1}(X) \to \mathsf{total}(\partial)^{-1}(Y)$ has a right adjoint which satisfies the Beck-Chevalley condition.*

Proof. Define, for $\varphi \in \mathcal{O}(F(A) \times X)$ an assembly over X,

$$\forall_f(\varphi) := \forall_{1\times f}(\overline{(F(\tau_A) \times 1)\varphi})$$

where the quantifier on the right hand side is the universal quantification for $\mathsf{total}(\mathcal{O})$.

First, we show that $\forall_f$ is right adjoint to reindexing over f^* in $\mathsf{total}(\partial)$. It suffices to show that

$$\frac{f^*(\varphi) \vdash \psi}{\varphi \vdash \forall_f(\psi)}$$

Suppose that for $\varphi : \mathcal{O}(F(A) \times X)$ and $\psi \in \mathcal{O}(F(B) \times Y)$, the map $\gamma : A \to B$ realizes $f^*(\varphi) = \overline{(1 \times f)\varphi} \vdash \psi$. That is,

[Tk.1] $\overline{(1 \times f)\varphi}(F(\gamma) \times 1)\psi = \overline{(1 \times f)\varphi}(F(\gamma) \times 1)$

[Tk.2] $\overline{\overline{(1 \times f)\varphi}(F(\gamma) \times 1)} = \overline{(1 \times f)\varphi}$

Now, let $\delta : A \to T_B$ be defined by

$$\begin{tikzcd} T_B \arrow[r, "\tau_B"] & B \\ A \arrow[u, dotted, "\delta"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We show that δ realizes $\varphi \vdash \forall_{1 \times f}(\overline{(F(\tau_B) \times 1)\psi}) = \forall_f(\psi)$:

[Tk.1] Notice that if

$$\varphi \leq \overline{(F(\delta) \times 1)\forall_{1 \times f}(\overline{(F(\tau_B) \times 1)\psi})}$$

then we have the required identity as follows

$$\begin{aligned} &\varphi(F(\delta) \times 1)\forall_{1 \times f}(\overline{(F(\tau_B) \times 1)\psi}) \\ &= \varphi\, \overline{(F(\delta) \times 1)\forall_{1 \times f}(\overline{(F(\tau_B) \times 1)\psi})}(F(\delta) \times 1) \\ &= \varphi(F(\delta) \times 1) \end{aligned}$$

Now, since δ is total,

$$
\begin{tikzcd}
{} \arrow[r, "F(\delta)\times 1"] \arrow[d, "1\times f"'] & {} \arrow[d, "1\times f"] \\
{} \arrow[r, "F(\delta)\times 1"'] & {}
\end{tikzcd}
$$

is a pullback square in $\mathsf{total}(\mathbb{X})$, and so since universal quantification in $\mathsf{total}(\mathcal{O})$ satisfies the Beck-Chevalley condition, we have

$$
\begin{aligned}
&\overline{F(\delta)\forall_{1\times f}(\overline{(F(\tau_B)\times 1)\psi})}\\
&= \forall_{1\times f}(\overline{(F(\delta\tau_B)\times 1)\psi})\\
&= \forall_{1\times f}(\overline{(F(\gamma)\times 1)\psi})
\end{aligned}
$$

and so

$$
\begin{array}{c}
\varphi \leq \overline{(F(\delta)\times 1)\forall_{1\times f}(\overline{(F(\tau_B)\times 1)\psi})}\\
\hline\hline
\varphi \leq \forall_{1\times f}(\overline{(F(\gamma)\times 1)\psi})\\
\hline\hline
\overline{(1\times f)\varphi} \leq \overline{F(\gamma)\times 1)\psi}
\end{array}
$$

but using our assumption that γ realizes $\overline{(1\times f)\varphi} \vdash \psi$ we have

$$
\begin{aligned}
&\overline{(1\times f)\varphi}\;\overline{(F(\gamma)\times 1)\psi} = \overline{\overline{(1\times f)\varphi}\,(F(\gamma)\times 1)\psi}\\
&= \overline{\overline{(1\times f)\varphi}\,(F(\gamma)\times 1)} = \overline{(1\times f)\varphi}
\end{aligned}
$$

meaning $\overline{(1\times f)\varphi} \leq \overline{(F(\gamma)\times 1)\psi}$, as required.

[Tk.2] Since δ is total, we immediately have

$$\overline{\varphi(F(\delta)\times 1)} = \varphi$$

For the converse, suppose $\delta : A \to T_B$ realizes $\varphi \vdash \forall_f(\psi)$. We show that $\delta\tau_B : A \to B$ realizes $\overline{(1 \times f)\varphi} = f^*(\varphi) \vdash \psi$:

Notice that $\overline{(1 \times f)\varphi} \leq \overline{(F(\delta\tau_B) \times 1)\psi}$ since

$$
\begin{aligned}
&\overline{(1 \times f)\varphi} = \overline{(1 \times f)\varphi(F(\delta) \times 1)} \\
&= \overline{(1 \times f)\varphi(F(\delta) \times 1)\forall_f(\psi)} \\
&= \overline{(1 \times f)\varphi(F(\delta) \times 1)\forall_{1\times f}(\overline{(F(\tau_B) \times 1)\psi})} \\
&= \overline{\overline{(1 \times f)\varphi}(F(\delta) \times 1)\overline{(1 \times f)\forall_{1\times f}(\overline{(F(\tau_B) \times 1)\psi})}} \\
&\leq \overline{\overline{(1 \times f)\varphi}(F(\delta) \times 1)\overline{(F(\tau_B) \times 1)\psi}} \\
&= \overline{\overline{(1 \times f)\varphi}(F(\delta\tau_B) \times 1)\psi} \\
&= \overline{(1 \times f)\varphi}\ \overline{(F(\delta\tau_B) \times 1)\psi}
\end{aligned}
$$

Using this, we have

[Tk.1]

$$
\begin{aligned}
&\overline{(1 \times f)\varphi}(F(\delta\tau_B) \times 1)\psi = \overline{(1 \times f)\varphi}\ \overline{(F(\delta\tau_B) \times 1)\psi}\,(F(\delta\tau_B) \times 1) \\
&= \overline{(1 \times f)\varphi}(F(\delta\tau_B) \times 1)
\end{aligned}
$$

[Tk.2] since $\overline{(1 \times f)\varphi} \leq \overline{(F(\delta\tau_B) \times 1)\psi} \leq \overline{(F(\delta\tau_B) \times 1)}$,

$$
\overline{\overline{(1 \times f)\varphi}\,(F(\delta\tau_B) \times 1)} = \overline{(1 \times f)\varphi}\ \overline{(F(\delta\tau_B) \times 1)} = \overline{(1 \times f)\varphi}
$$

and we have established the required adjunction.

This also gives that $\forall_f$ is a preorder morphism, as if $\varphi \vdash \psi$, then $f^*(\forall_f * \varphi) \vdash \varphi \vdash \psi$, and so $\forall_f(\varphi) \vdash \forall_f(\psi)$.

Finally, we show that universal quantification over $\mathsf{total}(\partial)$ satisfies the Beck-Chevalley condition. To that end, suppose

$$
\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "k"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}
$$

is a pullback square in $\mathsf{total}(\mathbb{X})$. Then

$$
\begin{tikzcd}
F(T_B) \times W \arrow[r, "1\times h"] \arrow[d, "1\times k"'] & F(T_B) \times X \arrow[d, "1\times f"] \\
F(T_B) \times Z \arrow[r, "1\times g"'] & F(T_B) \times Y
\end{tikzcd}
$$

is also a pullback square in $\mathsf{total}(\mathbb{X})$. Now, the Beck-Chevalley condition is satisfied by universal quantification in $\mathsf{total}(\mathcal{O})$, so we have

$$\overline{(1 \times g)\forall_{1\times f}(\overline{(F(\tau_B) \times 1)\psi})} = \forall_{1\times k}(\overline{(1 \times h)(F(\tau_B) \times 1)\psi})$$

and then because $\overline{(1 \times h)(F(\tau_B) \times 1)\psi} = \overline{(F(\tau_B) \times 1)\overline{(1 \times h)\psi}}$ we have shown that in $\mathsf{total}(\partial)$

$$g^*(\forall_f(\psi)) = \overline{(1 \times g)\forall_f(\psi)} = \forall_k(\overline{(1 \times h)\psi}) = \forall_k(h^*(\psi))$$

and so certainly we have

$$g^*(\forall_f(\psi)) \dashv\vdash \forall_k(h^*(\psi))$$

as required. Thus $\mathsf{total}(\partial)$ has universal quantificaiton.

□

8.4 The Generic Predicate

Proposition 8.9. $\mathsf{total}(\partial)$ *has a generic predicate.*

Proof. The genereic predicate is the assembly $\overline{\mathsf{ev}} \in \mathcal{O}(F(T) \times 1^{F(T)})$ where T is the Turing object of $\mathbb{A}$. Let $\varphi \in \mathcal{O}(F(A) \times X)$ be an object of $\mathsf{total}(\partial)^{-1}(X)$ for some object X of $\mathbb{X}$. We use the fact that T is a universal object to define $\overline{(F(r_A) \times 1)\varphi} \in$

$\mathcal{O}(F(T) \times X)$, and then define $h : X \to 1^{F(T)}$ in $\mathsf{total}(\mathbb{X})$ by $h := \lambda(\overline{(\overline{F(r_A) \times 1)\varphi}\,!})$, as in

$$\begin{tikzcd} F(T) \times 1^{F(T)} \arrow[r, "\mathsf{ev}"] & 1 \\ F(T) \times A \arrow[u, dotted, "1\times h"] \arrow[ur, "\overline{(F(r_A)\times 1)\varphi}\,!"'] & \end{tikzcd}$$

We must show that $\varphi \dashv\vdash h^*(\mathsf{ev}) = \overline{(1 \times h)\overline{\mathsf{ev}}} = \overline{(1 \times h)\mathsf{ev}} = \overline{(F(r_A) \times 1)\varphi}$.

For $\varphi \vdash \overline{(F(r_A) \times 1)\varphi}$, the realizer is $s_A : A \to T$ as follows:

[Tk.1]

$$\varphi(F(s_A) \times 1)\overline{(F(r_A) \times 1)\varphi} = \varphi\overline{(F(s_A r_A) \times 1)\varphi}(F(s_A) \times 1)$$

$$= \varphi\varphi(F(s_A) \times 1) = \varphi(F(s_A) \times 1)$$

[Tk.2] $\overline{\varphi(F(s_A) \times 1)} = \varphi$ immediately as s_A is total.

For $\overline{(F(r_A) \times 1)\varphi} \vdash \varphi$, the realizer is $r_A : T \to A$ as follows:

[Tk.1] $\overline{(F(r_A) \times 1)\varphi}(F(r_A) \times 1)\varphi = (F(r_A) \times 1)\varphi\varphi = \overline{(F(r_A) \times 1)\varphi}(F(r_A) \times 1)$

[Tk.2] $\overline{\overline{(F(r_A) \times 1)\varphi}(F(r_A) \times 1)} = \overline{(F(r_A) \times 1)\varphi}$

and so $\overline{\mathsf{ev}} : \mathcal{O}(F(T) \times 1^{F(T)})$ is a generic predicate for $\mathsf{total}(\partial)$. $\square$

Note that we have now shown

Theorem 8.10. $\mathsf{total}(\partial)$ *is a tripos.*

We call $\mathsf{total}(\partial)$ the *realizability tripos* of $F : \mathbb{A} \to \mathbb{X}$ where $\mathbb{A}$ is a Turing category, $\mathbb{X}$ is a discrete cartesian closed restriction category, and F is a restriction functor. Similarly, we call $\mathbb{X}[\mathsf{total}(\partial)]$ the *realizability topos* of F.

We conclude by observing that when the realizability fibration is a tripos, the constant objects functor [31] can be modified to give a restriction functor from the base category into the partial topos of the realizability tripos. Define $\Delta_\partial : \mathbb{X} \to \mathsf{total}(\mathbb{X})\{\mathsf{total}(\partial)\}$ by, for X an object of $\mathbb{X}$ and $f : X \to Y$ in $\mathbb{X}$:

$$
\begin{aligned}
\Delta_\partial(X) &:= \exists_{\Delta_X}(1_{F(1)\times X}) \in \partial^{-1}(X \times X) \\
\Delta_\partial(f) &:= \exists_{\langle 1,f\rangle}(1_{F(1)} \times \overline{f}) \in \partial^{-1}(X \times Y)
\end{aligned}
$$

It is straightforward to verify that this gives a restriction functor. A more detailed proof for the total case can be found in [36], and the details are largely the same in the partial case. The main difference is that instead of having $\Delta_\partial(f)$ be a total relation, we require that $\Delta_\partial(\overline{f}) = \overline{\Delta_\partial(f)}$. That is, we require

$$
[\exists_{\langle 1,\overline{f}\rangle}(1 \times \overline{f})(x,y)] \simeq [\exists_{\Delta_X}(1)(x,x') \wedge \exists_y(\exists_{\langle 1,f\rangle}(1 \times \overline{f}))(x,y)]
$$

which is immediate since $\Delta_\partial(f) : \Delta_\partial(X) \to \Delta_\partial(Y)$ defines a strict relation in the internal language of the tripos (see [36]).

Recall that in a discrete cartesian closed restriction category we can define a range combinator by $\widehat{f} = \exists_f(\overline{f})$. We observe that on objects

$$
\Delta_\partial(X) = \exists_{\Delta_X}(1_{F(1)\times X}) = \exists_{1\times\Delta_X}(\overline{1 \times \Delta_X}) = (\widehat{1 \times \Delta_X}) = (1 \times \widehat{\Delta_X}) = (1 \times \overline{\Delta^{(-1)}})
$$

and similarly, on maps

$$
\Delta_\partial(f) = \exists_{\langle 1,f\rangle}(1_F(1) \times \overline{f}) = \exists_{1\times\langle 1,f\rangle}(\overline{1 \times \langle 1,f\rangle}) = \widehat{1 \times \langle 1,f\rangle} = 1 \times \widehat{\langle 1,f\rangle}
$$

offering another way to understand the constant objects functor in which we use the range combinator to construct a restriction idempotent $\widehat{\langle 1,f\rangle}$ corresponding to the *graph* of a function f.

Now, we construct a functor from the category of realizers into the partial topos of the realizability tripos by composing $H : \mathbb{A} \to \mathsf{asm}(F)$ (**lemma 5.4**) and the realizability latent fibration $\partial : \mathsf{asm}(F) \to \mathbb{X}$ with our modified constant objects functor:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathbb{X} \arrow[r, "\Delta_\partial"] & \mathsf{total}(\mathbb{X})\{\mathsf{total}(\partial)\} \\
\mathsf{asm}(F) \arrow[ur, "\partial"'] & &
\end{tikzcd}$$

or equivalently by composing $F : \mathbb{A} \to \mathbb{X}$ with Δ_∂. This has the potential to be an interesting functor, since it captures the sense in which the category of realizers is present in the realizability (partial) topos.

9 Conclusions and Further Research

We have constructed categories of assemblies in a way that generalizes existing constructions of such categories, and investigated their structure. We have introduced latent fibrations, and have shown that each category of assemblies defines a related latent fibration. We have shown that for a cartesian restriction functor with a Turing category and discrete cartesian closed restriction category as its domain and codomain respectively, this latent fibration defines a fibration, and this fibration is a tripos. We have also noticed that given a tripos, we can construct a partial topos, and that the total maps of this partial topos are exactly the topos associated with the tripos.

There are many directions for further research. Perhaps the most glaring omission in this thesis is the absence of a definition of "latent tripos". As mentioned in the chapter on restriction categories, there ought to be such a thing, and our domain latent fibration and realizability latent fibration ought to be examples of it. More generally, finding latent fibrational analogues of fibrational structure in categorical logic and connecting these to the work already done on the logical structure of restriction categories would be a good thing to do.

Having constructed our more general categories of assemblies, it would be nice to have some more general examples. There are likely to be interesting categories of assemblies in which the domain and/or codomain of the associated functor have less structure than they do in the classical examples. What are they, and how do they relate to the classical examples? From here, one might investigate the realizability latent fibration of interesting categories of assemblies. What structure does it have? This ties in to the first avenue of further research on logical structure in latent fibrations. Lars Birkedal's work in [4] deals with a part of this question for fibrations, and would be a good place to start.

Another potentially interesting area of study is how relationships between the functors associated with different categories of assemblies induce relationships between the categories of assemblies themselves. For example, an applicative morphism between two partial combinatory algebras in the category of sets and partial functions induces a regular functor between the corresponding categories of assemblies [29]. What other things like this happen? This extends, of course, to induced relationships between the associated realizability fibrations as well.

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