UNIVERSITY OF CALGARY

Categories of Kirchhoff Relations

by

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A THESIS

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Abstract

We define the category of Kirchhoff relations to consist of those Lagrangian relations that conserve total momentum – a condition that can also be interpreted as Kirchhoff's current law. We study and characterize different subcategories of Kirchhoff relations and present universal sets of generators of the different subcategories of Kirchhoff relations. These generators can be interpreted as junctions of ideal wires, resistances, voltage sources and current sources.

Preface

This thesis includes work that has been submitted to Quantum Physics and Logic 2022, it has been also been submitted for a talk at the Applied Category Theory 2022 Conference. Its available as a pre-print entitled "Categories of Kirchhoff Relations" [1].

Acknowledgements

Firstly, I would like to thank my supervisor Prof. Robin Cockett, from whom I learnt a lot about category theory. I greatly appreciate the endless hours he spent with me in front of the blackboard, writing and explaining proofs and calculations. I am grateful to him for his time and patience.

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I struggle to capture in words my gratefulness and appreciation of the support and unconditional love of my family. To my parents and older brother Aarat therefore, I can only quote Wittgenstein:

"Whereof One Cannot Speak, Thereof One Must Be Silent"

Dedicated to Prof. P.S Satsangi Sahab,

who inspired me to do Mathematics,

and whose work, personal advice and suggestions

directly influenced the topics of this thesis.

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List of Symbols, Abbreviations, and Nomenclature

Symbol	Definition
H	The parity-check matrix for a relation.
G	The generator matrix for a relation.
LinRel_F	Category of linear relations over a field F.
$\mathbb{A}, \mathbb{B} \ldots \mathbb{Z}$	Arbitrary categories.
AffRel_F	Category of affine relations over a field F.
LagRel_F	Category of Lagrangian relations over a field F.
$\mathsf{AffLagRel}_F$	Category of affine Lagrangian relations over a field F.
KirRel_F	Category of Kirchhoff relations over a field F.
$\mathsf{AffKirRel}_F$	Category of affine Kirchhoff relations over a field F.
$\mathsf{LosLagRel}_F$	Category of Lossless Lagrangian relations over a field F.
$\mathsf{LosKirRel}_F$	Category of Lossless Kirchhoff relations over a field F.
ResRel_F	Category of resistor relations over a field F.
KCL	Kirchhoff's Current Law.
KVL	Kirchhoff's Voltage Law.

Epigraph

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretched out his arms for others.

- Carl Friedrich Gauss

Chapter 1

Introduction

1.1 Overview

The program of categorical quantum mechanics was first initiated in its modern form by Abramsky and Coecke in [4]. Categorical methods have been used in the areas of quantum computation [5–7], quantum information [8, 9] and quantum foundations [10, 11]. In particular these methods have been applied with considerable success to give new insights into certain aspects of quantum error correction [12–14] and quantum circuit simplification [15–17]. The most popular categorical framework used to model quantum computation is called the ZX-Calculus, this calculus has been shown to be complete for stabilizer qubit quantum mechanics [18] and more recently it was shown that a version of this calculus is complete for qudit stabilizer quantum mechanics [19].

In a different direction categories of Lagrangian relations have been used to capture the behaviour of resistive electrical networks [20, 21]. In this framework circuits are regarded as morphisms with specific input and output terminals, the symplectic structure of the category of Lagrangian relations is used to give a natural grading of currents and voltages, this in turn is used to model resistor networks [20, 21].

Graphical Linear Algebra (GLA) proposed by Zanasi in [22] gives a complete graphical calculus for reasoning about linear relations, In this calculus the objects are natural numbers (n represents F^n) and the maps are linear relations i.e linear subspaces (thus, $R : n \to m$ means $R \subseteq F^n \times F^m$). GLA was used in [23] to capture the behaviour of linear electrical networks with no sources. This graphical language was then extended to deal with affine relations and the corresponding graphical language was called Graphical Affine Algebra (GAA) [2]. The calculus uses the more general affine relations to model electrical circuits with current and voltage sources. This provides a rigorous framework for reasoning about electrical circuits and has been used to re-derive several well known electrical network theorems completely graphically using the rules of GAA [24].

More recently, in the work of Comfort and Kissinger [3], a functor between Lagrangian relations and stabilizer quantum mechanics for qudits of odd prime dimensions was described. This functor was shown to be an equivalence. In addition a universal set of generators were proposed for Lagrangian relations in [3] and these generators were then interpreted in the context of stabilizer quantum mechanics. A graphical description of these generators was also given using the graphical calculus of GAA.

The relationship between symplectic vector spaces and qudit quantum computation in odd prime dimensions was developed in [25–29]. These papers discuss the relevant results on the mathematical structure of the Clifford group for qudits of odd prime dimension. They also discuss the stabilizer formalism for qudit quantum error correction and the phase space formalism of qudit quantum operations in odd prime dimensions.

In the electrical network theory, work on using the concept of "through" and "across" variables to model electrical networks was discussed in [30–34] and some connections to quantum computation were also established. The through and across variables can be thought of as the X and Z grading in the context of Lagrangian relations. Some graph theoretic methods were also discussed in the context of circuit simplification strategies using state space methods where electrical network analysis was carried out using matrices acting on vectors of currents and voltages [35, 36].

Inspired by these results, in this thesis a subcategory of Lagrangian relations is defined with the goal of making the connections between electrical networks and quantum circuits more transparent. To make the connection to qudit quantum mechanics, a characterization of Lagrangian relations and important subcategories thereof is made in terms of parity-check and generator matrices. Parity-check and generator matrices are used in both classical and quantum error correction.

The subcategory, consisting of Kirchhoff relations, is defined as the category consisting of those Lagrangian relations that conserve total momentum or, equivalently, satisfy Kirchhoff's current law. Maps in this subcategory can be generated by electrical components (generalized for the field F): namely resistors, current dividers, current and voltage sources. The "source" electrical components deliver the affine nature of the maps while current dividers add an interesting quasi-stochastic aspect. The fact that all the generators in the category of Kirchhoff relations have natural interpretations as electrical elements is particularly useful in establishing connections between electrical networks and stabilizer quantum mechanics: this was not true for the set of universal generators given in the work of Comfort and Kissinger. These set of generators of Kirchhoff relations have also been interpreted as quantum maps.

Deterministic relations and, in particular, deterministic Kirchhoff relations, are also characterized and using the structure of the parity-check matrices the category of deterministic Kirchhoff relations is isolated as the category which corresponds to electrical circuits that are generated by resistors circuits.

Using the concept of "input power" two important subcategories: Lossless Lagrangian relations and Lossless Kirchhoff relation are defined and their properties characterized.

A standard form of the parity-check matrix for each of these subcategories which correspond to different types of electrical networks is studied. In addition to this a set of generators for all these categories is provided and proofs for universality for each of these subcategories is given explicitly. Interpretation of these generators in terms of quantum maps is also discussed in the last chapter of the thesis.

1.2 Outline of the Thesis

Chapter 2 contains a review of basic category theory, GAA and Lagrangian relations, Chapters 3 and 4 contain original results based on analyzing parity-check matrices. In the conclusion, the relationship of Kirchhoff relations to qudit quantum mechanics is outlined and some potential future directions of work are discussed.

An outline of the thesis is as follows:

Chapter 2 Preliminaries: This is an introductory chapter and is meant to provide the background material required for the later chapters in thesis. In this chapter we review the basic theory of monoidal categories, Graphical Affine Algebra (GAA) and the theory of Lagrangian relations. The results and definitions in this chapter are available in the literature.

Chapter 3 Parity Check Matrices and Lagrangian Relations: In this chapter a novel characterization of Lagrangian relations in terms of parity-check matrices is given, a new subcategory of Lagrangian relations called Kirchhoff relations is introduced and its properties are characterized using parity check matrices. Deterministic relations are then considered and deterministic Kirchhoff and Lagrangian relations are characterized. Using the concept of "power input" the categories of lossless Lagrangian and Kirchhoff relations are defined and characterized.

Chapter 4 Universality for Kirchhoff Relations: This chapter provides generating relations corresponding to electrical components, for the different subcategories of Kirchhoff relations.

Chapter 5 Discussions and Conclusion: In the last chapter of the thesis the connection to qudit quantum computation is discussed and the generators of the category of Kirchhoff relations are interpreted as quantum maps.

1.3 Contributions

The contributions of the thesis are as follows:

- A characterization of the parity-check matrix of a Lagrangian relation was made in Theorem 3.4.3.
- Kirchhoff relations are defined in Definition 3.5.1 and the properties of translational invariance and Kirchhoff Current Law (KCL) are discussed in section Lemmas 3.5.5 and 3.5.6.
- Based on the concept of "power input" Lossless Lagrangian and Kirchhoff relations are defined using parity-check matrices. It is shown that the category of lossless Lagrangian relation is isomorphic to a particular subcategory (determined by the L functor) of Lagrangian relations in Lemma 3.8.7.
- Isolating different subcategories of Kirchhoff relations depending on the structure of parity check matrices and showing how these subcategories correspond to different types of electrical networks.
- In particular Theorem 3.7.2 shows how graph states are related to resistor networks called "meshes".
- In section 3.6 different deterministic relations are characterized, in particular it is explicitly shown that the composition of deterministic relations is well-defined.
- Universality proofs for different subcategories of Kirchhoff Relations are given in Theorems 4.5.3, 4.6.1 and 4.7.1. These results are proven in Chapter 4.
- A translation from electrical components to qudit quantum computation explicitly given in chapter 5.

Chapter 2

Graphical Affine Algebra and Lagrangian Relations

2.1 Introduction

In the first half of this chapter we review the basics of monoidal categories, theory of linear and affine relations. In the second half we review the basic theory of Lagrangian relations. All the results in this chapter are well known and discussed in detail in [2, 3, 21, 22, 24]. In what follows let F be a field. We will mainly focus our attention on finite fields, in which case $|F|$ denotes the total number of elements in the field and p denotes the characteristic.

2.2 Preliminaries

We begin with reviewing the basic notion of categories and functors. For more details one can consult [7, 37–39].

Definition 2.2.1. (See definition 1.1.1 in [39]) A **category** $\mathbb{C}$ consists of the following data:

- A collection of **objects** and a collection of **morphisms**, every morphism f has an associated **domain** and **codomain**, thus a morphism is $f : A \to B$, where A is the domain of f and B is the codomain of f and, A and B are objects of $\mathbb{C}$.

such that:

- For each pair of morphisms $f : A \to B$ and $g : B \to C$ which are composable in the sense that the codomain of f is the domain of g, there is a **composite** morphism: $g \circ f : A \to C$ in $\mathbb{C}$ (we use applicative notation).

- Each object A has an identity morphism $1_A : A \to A$.

The morphisms are subject to the following two laws:

Identity law: $\forall\ f : A \to B \in \mathbb{C},\ \ 1_B \circ f = f = f \circ 1_A$

Associative law: For morphisms: $f : A \to B$, $g : B \to C$ and $h : C \to D$ the associative law holds, that is: $h \circ (g \circ f) = (h \circ g) \circ f$

The class of maps between objects A and B are denoted as $\mathbb{C}(A, B)$.

We now give a couple of examples of categories:

Example 2.2.2. The category of sets and functions, Set, consists of:

Objects: Sets.

Maps : Functions.

Composition: Function composition.

Identity: Identity function $1_A : A \to A;\ a \mapsto a$.

Example 2.2.3. The category of sets and relations, Rel, consists of:

Objects: Sets

Maps : Relations $\mathcal{R} : A \to B$ which are subsets of $A \times B$

Composition : Relational composition $\mathcal{R} : A \to B$ and $\mathcal{R}' : B \to C$:

$$\mathcal{R}' \circ \mathcal{R} = \{(a, c) \mid \exists\ b \in B :\ (a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{R}'\}$$

Identity: The identity relation on a set A is:

$$1_A : A \to A := \{(a, a) \in A \times A \mid a \in \mathcal{R}\}$$

For every category one can define the “dual category”, this is sometimes called the opposite category:

Definition 2.2.4. (See definition 2.1.7 in [37]) The **opposite or dual** category $\mathbb{C}^{op}$ of $\mathbb{C}$ is defined as category with the same objects as $\mathbb{C}$ but with the arrows revered that is for a morphism $f : A \to B \in \mathbb{C}$ there is a morphism $f : B \to A \in \mathbb{C}^{op}$.

Definition 2.2.5. There are some special maps called **isomorphisms** which satisfy for a map f:

$$f^{-1} \circ f = 1 = f \circ f^{-1}$$

In Set these maps are called bijections.

A functor, is a map between categories which respects the categorical structure:

Definition 2.2.6. (See definition 0.16 in [7] and definition 1.3.1 in [39]) Consider two categories $\mathbb{C}$ and $\mathbb{D}$. A **functor** $F : \mathbb{C} \to \mathbb{D}$ associates with every object $A \in \mathbb{C}$, an object $F(A) \in \mathbb{D}$ and for every morphism $f : A \to B \in \mathbb{C}$, a morphism $F(f) : F(A) \to F(C) \in \mathbb{D}$. such that the following is satisfied:

Preservation of identity: For all objects $A \in \mathbb{C}$, the identities are mapped to identities that is: $F(1_A) = 1_{F(A)}$.

Preservation of composition: For all composable maps $f : A \to B$ and $g : B \to C$ $F(g \circ f) = F(g) \circ F(f)$.

An equivalence of functors is defined as follows:

Definition 2.2.7. (See definition 0.17 in [7])

(i) A functor is **full** if the morphism $F_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}'(F(A), F(B))$ is surjective for every $A, B \in \mathbb{C}$.

(ii) A functor is **faithful** if the morphism $F_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}'(F(A), F(B))$ is injective for every $A, B \in \mathbb{C}$

(iii) A functor is **essentially surjective** if for every object $B \in \mathbb{C}'$ there is an object $A \in \mathbb{C}$ such that there is an isomorphism $f : B \to F(A)$.

A functor $F : \mathbb{C} \to \mathbb{C}'$ is an **equivalence** if it satisfies the above three properties.

Maps between functors are called natural transformations and are defined as follows:

Definition 2.2.8. (See definition 0.19 in [7] and definition 2.1.1 in [37]) Given two functors $F, G : \mathbb{A} \to \mathbb{B}$, a **natural transformation** $\alpha : F \implies G$ is a class of maps $\alpha_A : F(A) \to G(A)$ for each object $A \in \mathbb{A}$ such that for all morphisms $f : A \to B \in \mathbb{A}$, $F\alpha_B \circ F(f) = G(f) \circ \alpha_A$ that is the diagram below commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

If each component α_A is an isomorphism then the natural transformation is called a **natural isomorphism**.

2.3 Monoidal Categories

The unitary group on n qubits, $U(2^n)$, is often considered to be the mathematical structure associated with quantum computing. But, in many useful models of quantum computing, the total number of qubits is not fixed – at any stage one can add ancilla qubits or discard them. One can also perform irreversible operations, like measurement, or post-selection on a measurement outcome. In the context of fault-tolerant quantum computing, there are usually various limitations imposed on the gate sets, the states one can prepare, or on other classical operations. As emphasized in [6] strict symmetric monoidal categories, first introduced by MacLane in [40]. are ideally suited to capture the behaviour of quantum processes.

We first review the theory of monoidal categories and then specialize to the case of a strict symmetric monoidal category. For a complete discussion the reader can refer to [6, 7, 37].

Definition 2.3.1. (See definition 2.2.1 [37] and definition 1.1 in [7]) A **monoidal category** $\mathbb{C}$ is a category equipped with the following data:

Tensor Product: A functor $- \otimes - : X \otimes X \to X$.

Tensor Unit: A unit object $I \in \mathbb{C}$.

Associator: A natural isomorphism α with components $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$

Left Unitor: A natural isomorphism $\lambda_X : I \otimes X \to X$

Right Unitor: A natural isomorphism $\rho_X : X \otimes I \to X$

This data satisfies the following triangle and pentagon equations for all objects $X, Y, Z, W \in \mathbb{C}$ as follows:

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1"'] & & X \otimes (1 \otimes Y) \arrow[dl, "1 \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "\alpha_{X,Y,Z} \otimes 1_W"] \arrow[d, "\alpha_{X \otimes Y,Z,W}"'] & (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "\alpha_{X,Y \otimes Z,W}"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "1_X \otimes \alpha_{Y,Z,W}"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "\alpha_{X,Y,Z \otimes W}"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

Definition 2.3.2. (See definition 2.2.1 in [37]) A monoidal category is **symmetric** if in addition to being monoidal it is equipped with a natural isomorphism σ referred to as the "symmetry" such that $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ and the following diagrams commute:

$$
\begin{tikzcd}
X \otimes Y \arrow[d, "\sigma_{X,Y}"'] \arrow[dr, "1_{X \otimes Y}"] & \\
Y \otimes X \arrow[r, "\sigma_{Y,X}"'] & X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes I \arrow[rr, "\sigma_{X,I}"] \arrow[dr, "\rho_X"'] & & I \otimes X \arrow[dl, "\lambda_X"] \\
& X &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\sigma_{X,Y} \otimes 1_Z"] \arrow[d, "\alpha_{X,Y,Z}"'] & (Y \otimes X) \otimes Z \arrow[d, "\alpha_{Y,X,Z}"] \\
X \otimes (Y \otimes Z) \arrow[d, "\sigma_{X,Y \otimes Z}"'] & Y \otimes (X \otimes Z) \arrow[d, "1_Y \otimes \sigma_{X,Z}"] \\
(Z \otimes Y) \otimes X \arrow[r, "\alpha_{Z,Y,X}"'] & Y \otimes (Z \otimes X)
\end{tikzcd}
$$

The last diagram governs the interaction of the symmetry map with the associator.

Some examples of monoidal categories are:

Example 2.3.3. (See examples in [7]) The category of sets, Set, with a monoidal structure which is given by the Cartesian product:

Tensor Product: Cartesian product that is given $f : X \to Y$ and $g : V \to W$ such that $f \times g : X \times V \to Y \times W; (x, v) \mapsto (f(x), g(v))$.

Unit: The unit object is given by the singleton set $\{*\}$.

Left Unitor: $\lambda_X : I \otimes X \to A$ is given by functions $(*, x) \to x$.

Right Unitor: $\rho_X : X \otimes I \to X$ is given by functions $(x, *) \to x$.

Associator: $\alpha_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z);\ ((x, y), z) \mapsto (x, (y, z))$

Example 2.3.4. (See examples in [7]) We now describe a monoidal structure on the category Rel induced by the Cartesian product:

Tensor Product: Cartesian product given as $\mathcal{R}_1 : X \to Y$ and $\mathcal{R}_2 : Z \to W$ given by $(x, z)(\mathcal{R}_1 \times \mathcal{R}_2)(y, w)$ iff $x\mathcal{R}_1 y$ and $z\mathcal{R}_2 w$.

Unit: The unit is given by the singleton set $\{*\}$.

Left Unitor: $\lambda_X : I \otimes X \to X$ are given by relations $(*, x) \sim x$.

Right Unitor: $\rho_X : X \otimes I \to X$ are given by relations defined by $(x, *) \sim x$.

Associator: $\alpha_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$ is given by relations $((x, y), z) \sim (x, (y, z))$.

Now we will define a special symmetric monoidal category called a strict symmetric monoidal category:

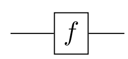
Definition 2.3.5. A **strict symmetric monoidal category** is a monoidal category where the $\rho_X, \lambda_X, \alpha_{X,Y,Z}$ natural isomorphims are all required to be identity morphisms.

Note that every monoidal category is equivalent to a strict monoidal category by Maclane's coherence theorem [38].

2.4 Graphical Calculi, Props and Monoidal Functors

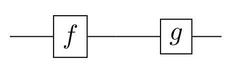
Maps in a monoidal category have an elegant graphical representation in terms of circuits, that is:

- A map $f : X \to Y$ can be represented as follows:

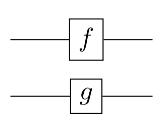


- The identity morphism has the following graphical representation (a single wire):

- The composition of morphism $f : X \to Y$ and $g : Y \to Z$ is graphically depicted as follows:



- The tensor product or parallel composition of maps $f : X \to Y$ and $g : Z \to W$ is given by $f \otimes g : X \otimes Z \to Y \otimes W$ which can be represented graphically as follows:



Graphical calculi associated with monoidal categories provides a mathematically rigorous framework to reason about processes using diagrams [41–43]. A graphical calculus allows for an intuitive diagrammatic reasoning for maps in a monoidal category. Since this diagrammatic reasoning is rigorous but intuitively clear, it helps in simplifying many proofs and in particular, has been used to provide a graphical syntax to reason about quantum operations.

Examples of graphical calculi include the ZX [44, 45, 5], ZW [46] and ZH calculus [47], which are frequently used in quantum computing as well as the graphical languages of GAA and GLA which have found applications in modelling linear systems, signal flow graphs and electrical circuits [22, 24].

We now define the concept of a "Product and Permutation" category (prop) as follows:

Definition 2.4.1. (See definition 2.2.4 in [37]) A strict symmetric monoidal category which is generated by a single object X, where all objects are given by n fold tensor products $X^{\otimes n}$ is called a "product and permutation category" or a **prop**.

We will refer to a product and permutation category as a prop throughout the thesis. ZX calculus, ZH calculus, Graphical Linear and Affine Algebra are all examples of props.

2.5 Monoidal Functors

We will now define monoidal functors, these functors are used to understand different graphical calculi and their relationship with each other:

Definition 2.5.1. A monoidal functor between two categories $\mathbb{A}$ and $\mathbb{B}$ consists of:

- A functor $F : \mathbb{A} \to \mathbb{B}$
- A morphism $m_I : I \to F(I)$
- A natural transformation: $m_\otimes : F(X) \otimes F(Y) \to F(X \otimes Y)$

such that the following diagrams commute:

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\alpha"] \arrow[d, "m_\otimes \otimes 1"'] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "1 \otimes m_\otimes"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "m_\otimes"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "m_\otimes"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha)"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes F(X) \arrow[r, "m_I \otimes 1"] \arrow[d, "\lambda_X"'] & F(I) \otimes F(X) \arrow[d, "m_\otimes"] \\
F(X) & F(I \otimes X) \arrow[l, "F(\lambda_X)"]
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \otimes I \arrow[r, "1 \otimes m_I"] \arrow[d, "\rho_X"'] & F(X) \otimes F(I) \arrow[d, "m_\otimes"] \\
F(X) \arrow[r, "F(\rho_X)"'] & F(I \otimes X)
\end{tikzcd}$$

A symmetric monoidal functor in addition has the following diagram which commutes:

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\sigma"] \arrow[d, "m_\otimes"'] & F(Y) \otimes F(X) \arrow[d, "m_\otimes"] \\
F(X \otimes Y) \arrow[r, "\sigma"'] & F(Y \otimes X)
\end{tikzcd}$$

A strict monoidal functor is one in which m_I and $m_\otimes$ are identities. We will now discuss some properties of a graphical calculus with respect to the following interpretation map $[\![-]\!] : \mathbb{A} \to \mathbb{B}$ which interprets the category $\mathbb{A}$ in $\mathbb{B}$.

Definition 2.5.2. (See [37]) An interpretation of a category $\mathbb{A}$ in $\mathbb{B}$ given by the functor $[\![-]\!] : \mathbb{A} \to \mathbb{B}$ is said to be **sound** if the interpretation morphism is a strict symmetric monoidal functor, is said to be **universal** if the functor is full and surjective on objects and, **complete** if the interpretation is surjective on objects and faithful.

2.6 The Categories of Linear and Affine Relations

In this section we review the theory of linear and affine relations, some references which the author found particularly useful are [2, 24]. We begin with defining the prop of relations over a field F LinRel_F:

Definition 2.6.1. (See definition 2 in [2]) Given a field F let LinRel_F be the prop of relations over a field F where:

Objects: $n \in \mathbb{N}$, which maybe regarded as F^n

Arrows: $\mathcal{R} : m \to n$ where $\mathcal{R} \subseteq F^m \times F^n$. The morphisms are linear subspaces. To see this note that if $(x, y) \in \mathcal{R}$ and $(x', y') \in \mathcal{R}$ then $(x + x', y + y') \in \mathcal{R}$ since $\mathcal{R}$ is a linear relation, similarly if $(x, y) \in \mathcal{R}$ and c is a scalar then $(cx, cy) \in \mathcal{R}$.

Composition : Given two relations $\mathcal{R}_1 : m \to n$ and $\mathcal{R}_2 : n \to p$, the composite relation $\mathcal{R}_2 \circ \mathcal{R}_1 : m \to p$ is given as follows:

$$\mathcal{R}_2 \circ \mathcal{R}_1 := \{(x, z) : x \in F^m, z \in F^p \mid \exists\ y \ \ (x, y) \in \mathcal{R}_1 \text{ and } (y, z) \in \mathcal{R}_2\}$$

To see that composition is well-defined suppose $(x, y), (x', y') \in \mathcal{R}_2 \circ \mathcal{R}_1$ then there are y and y' with $(x, y),\ (x', y') \in \mathcal{R}_1$ and $(y, z),\ (y', z') \in \mathcal{R}_2$. As $\mathcal{R}_1$ and $\mathcal{R}_2$ are linear $\alpha(x, y) + \beta(x', y') = (\alpha x + \beta x', \alpha y + \beta y') \in \mathcal{R}_1$ and $\alpha(y, z) + \beta(y', z') = (\alpha y + \beta y', \alpha z + \beta z') \in \mathcal{R}_2$ whence $\alpha(x, z) + \beta(x', y') = (\alpha x + \beta x', \alpha y + \beta y') \in \mathcal{R}_2 \circ \mathcal{R}_1$. Thus $R_1 \circ R_2$ is a linear subspace.

Monoidal Product: Given $\mathcal{R} : m \to n$ and $\mathcal{R}' : m' \to n'$, their monoidal product $\mathcal{R} \oplus S : m + m' \to n + n'$ is the relation:

$$\{(((x_1, x_2), (y_1, y_2)) : x_1 \in F^m\ x_2 \in F^{m'}\ y_1 \in F^n\ y_2 \in F^{n'} \mid (x_1, y_1) \in \mathcal{R},\ (x_2, y_2) \in \mathcal{R}'\}$$

that is the direct sum of the relational subspaces defined by the relations.

Symmetry : $\sigma_{m,n} : n + m \to m + n$ is the subspace $\{(x, y), (y, x) \mid x \in F^n \ and\ y \in F^m\}$.

We will now define the prop of affine relations over a field F, AffRel_F, by analogy to affine subspaces:

Definition 2.6.2 (See definition 3 and 4 in [2])**.** An affine subspace of F^d is a (possibly empty) subset $V \subseteq F^d$ for which there exists a vector $c \in F^d$ and a linear subspace V' of F^d such that $V := c + V' = \{c + x \mid x \in V'\}$. An affine relation $m \to n$ over a field F is defined to be a affine subspace of $F^m \times F^n$.

Now to show $\mathsf{AffRel}_\mathsf{F}$ forms a prop, we need to show that the composition is well defined that is: the composition of two affine relations is an affine relation.

Recall that one can associate to every affine relation a linear relation using homogenisation. Essentially this is done by embedding an affine relation into a space with one extra dimension.

Definition 2.6.3 (See definition 5 in [2])**.** Let $S : m \to n$ be an affine relation given by $(a, b) + S'$, its homogenisation is the linear relation $\hat{S} : m + 1 \to n$ defined as follows:

$$\hat{S} = \{((a, r), b) \mid (a, b) + r(a, b),\ (a, b) \in S\}$$

Now using the homogenisation construction on affine relations $R : m \to n$ and $S : n \to p$ we associate linear relations $\hat{R} : m + 1 \to n$ and $\hat{S} : n + 1 \to p$, but to compose these linear relations we interpret the above construction in the co-Kleisli category [39] of the $(-)+1$ comonad which is equipped with the following natural transformations:

$$\delta : (n) + 1 \to (n) + 2; \quad (a, r) \mapsto (a, r, r)$$

$$\epsilon : (n) + 1 \to n; \quad (a, r) \mapsto a$$

For $\hat{R} : m + 1 \to n$ and $\hat{S} : n + 1 \to p$ the co-Kleisli composition gives the relation $\hat{S} \circ \hat{R} : m + 1 \to p$ and hence composition of two affine relations is well defined.

Lemma 2.6.4 (See proposition 6 in [2])**.** Composition of two affine relations is an affine relation

Proof. This is a consequence of the co-Kleisli construction discussed above. □

2.7 Graphical Affine and Linear Algebras

In this section, we present a review of a graphical calculus for linear relations over a field F, known as Graphical Linear Algebra (GLA) and a closely related graphical calculus for affine relations, known as Graphical Affine Algebra (GAA). We closely follow the discussions in [2, 48, 49] which are summarized in [3], and readers are encouraged to consult these references for more complete discussion. Our notation is intended to resemble that of [2], however, like [49] we consider our diagrams flowing from down to up, rather than left to right.

We will present a set of generators and equations for GAA, which form a universal, sound and complete graphical calculus for AffRel [2]. Some of the generators and rewrite rules may be redundant – and in this thesis, we make no claims about the minimality of either for any of the graphical calculi we consider. By omitting some generators and equations, we also obtain various subcategories of GAA corresponding to graphical calculi for linear relations GLA, as well as graphical calculi for linear functions and affine functions.

The generators of the GAA are presented in Table 2.1.

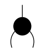
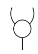
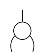
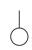
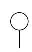
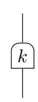
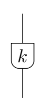
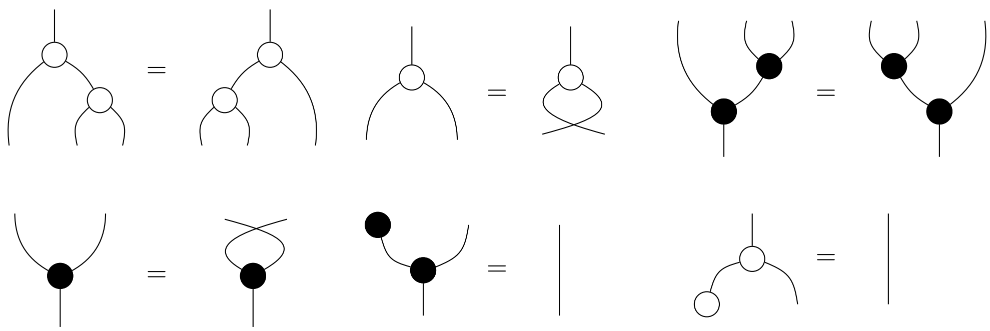
Generator	Symbol	Interpretation as a Linear Relation
1. “copy”		$\{x, (x, x) \mid x \in F\}$
2. “co-copy”		$\{((x, x), x) \mid x \in F\}$
3. “anything”		$\{(*, x) \mid x \in F\}$
4. “delete”		$\{(x, *) \mid x \in F\}$
5. “co-add”		$\{(x, (x - y, y)) \mid x, y \in F\}$
6. “add”		$\{((x, y), x + y \mid x, y \in F)\}$
7. “zero”		$\{(*, 0)\}$
8. “post-select: zero”		$\{(0, *)\}$
9. “multiply by k”		$\{(x, kx) \mid x \in F\}$
10. “multiply by k^{-1}”		$\{(kx, x) \mid x \in F\}$
11. “affine unit”		$\{(*, 1)\}$

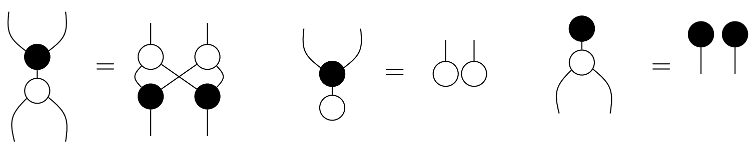
Table 2.1: **Generators of the Graphical Affine Algebra**. These generate all affine relations between F^n and F^m. We regard the external wires at the bottom of the diagram as “input”, and external wires at the top of the diagram as “output”. The variables x and y are assumed to take on values in F, so, e.g., “co–add” encodes the relation $\forall x, y \in F, x \sim (x - y, y)$.

Graphical Affine Algebra (GAA) is a graphical algebra that captures the behaviour of affine relations using only string diagrams. GAA is the prop generated by generators $1-9$ in Table 2.1 modulo certain equations. These rules, also known as the axioms of the affine interacting Hopf algebras [2,22] consist of the following rules, note here we do not list all the rules as there is a dual symmetry given by vertically reflecting the diagrams.

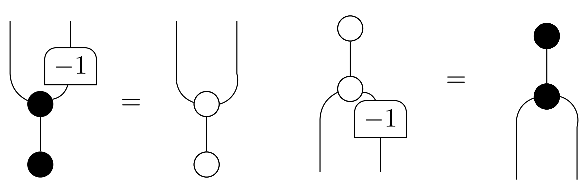
- The commutative and associativity rules for the monoid "add" (white) and "copy" (black).



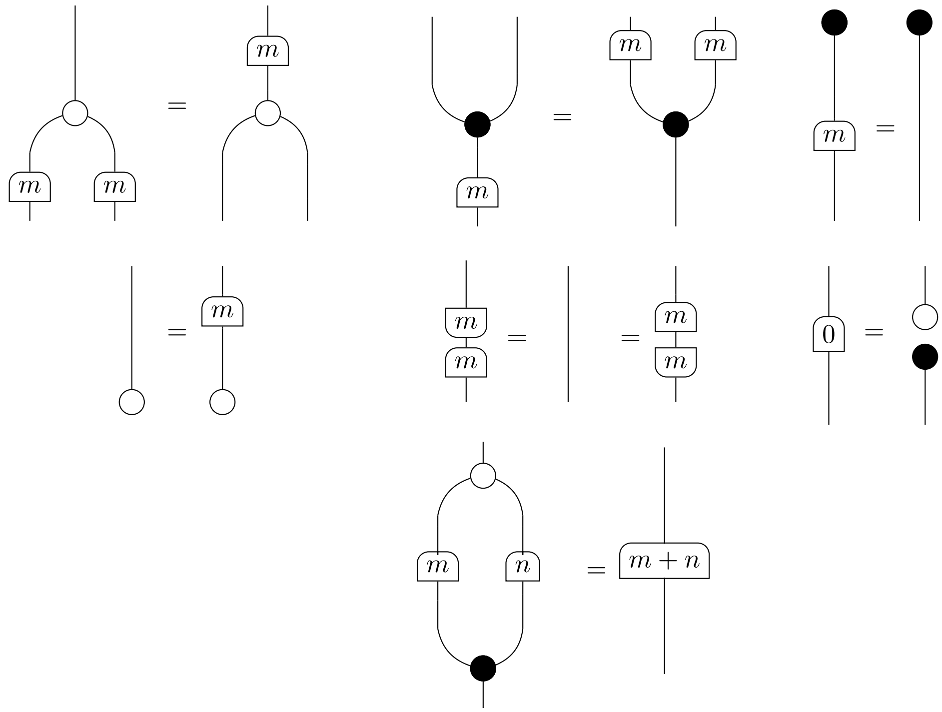
- Rules for interactions between the monoid "add"(white) and comonoid "copy" (black).



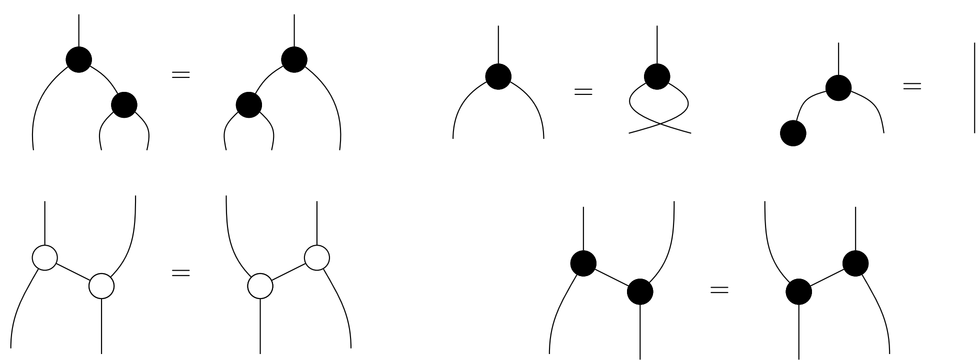
The generator "multiply by -1" is also sometimes known as the *antipode*. The antipode interacts with the addition and copy monoid and co-monoid via the following rules:



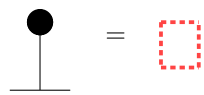
- The rules governing behaviour of scalars with respect to the monoid and comonoid are given as follows:



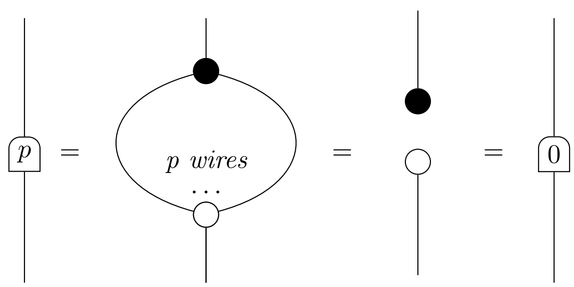
- The commutative, associativity and Frobenius rules for the relation "co-add" and relation "co-copy".



- The rules that govern the affine unit are given below:



The generator number 9 in Table 2.1 can be interpreted as a multiplier which multiplies the input by a scalar k. If $k \in \mathbb{N}$, this generator can be interpreted as "syntactic sugar", i.e., a notational device, for a diagram consisting of k wires summed. However, in general, for instance, if one is working within a finite field such as $GF(p^2)$ or $\mathbb{R}$, k need not be a natural number, so it is necessary to regard the generator for scalar multiplication being independent. In a field of characteristic p, the following equation holds:



(2.1)

One can interpret the generators numbered 1 and 6 in Table 2.1 as the functions "copy" and "add" respectively, and the generators 2 and 5 in Table 2.1 as the relations "co-copy" and "co-add".

Using the associative rules, each of these can be used to define spiders, that take multiple inputs and outputs, which we refer to as the "copy spider" and the "addition spider".

It was shown in [2] that GAA forms a universal, sound and complete graphical calculus for AffRel_R, where R is a rig (which is a ring without negatives). In this thesis we will focus on the case of a finite field F. When the affine unit (generator 10 in Table 2.1) and its associated identities are omitted, the theory becomes the Graphical Linear Algebra (GLA). It was shown in [23] that GLA forms a universal, sound and complete graphical calculus for LinRel_R. See also [2, 3, 23, 24] for more discussion.

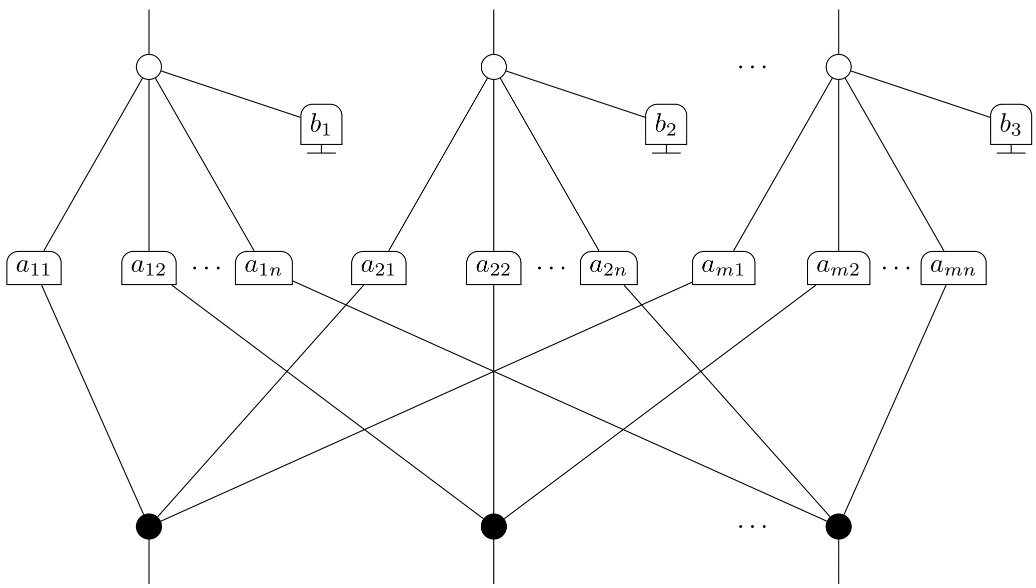
Let us sketch the proof of universality. First, let us consider the case of linear and affine functions. Here, we simply need to demonstrate the fact that any affine function,

$$y = Ax + B \tag{2.2}$$

can be written using the generators copy, addition and scalar multiplication. Let us list the components of A and B explicitly,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{1n} \\ \vdots & \vdots & \dots \vdots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix}^T$$

The following diagram encodes equation (2.2):



To show that GAA is universal for affine relations, note that any affine relation can be expressed, non-uniquely, as a linear relation, composed with a constant affine shifts applied to the output as shown in Figure 2.1.

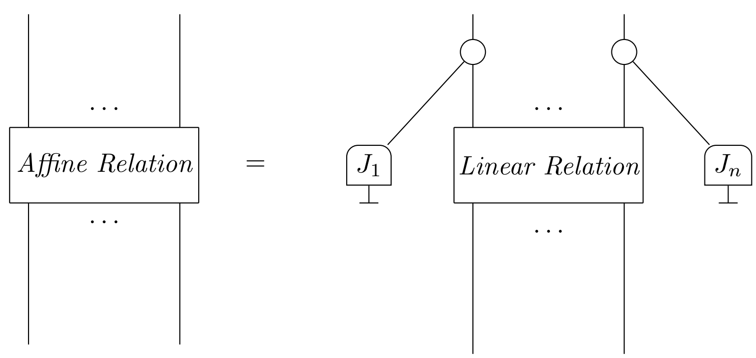


Figure 2.1: Any affine relation can be expressed (non-uniquely) in terms of a linear transformation, with additional shifts on the outputs.

2.8 Graphical Affine Algebra and Electrical Circuits

As discussed in [2] one can use the generators of GAA to capture the behaviour of electrical elements, this is summarized in the Figure 2.2, where we give the representation of resistors, voltage sources, current sources and current dividers in terms of the generators of the graphical affine algebra.

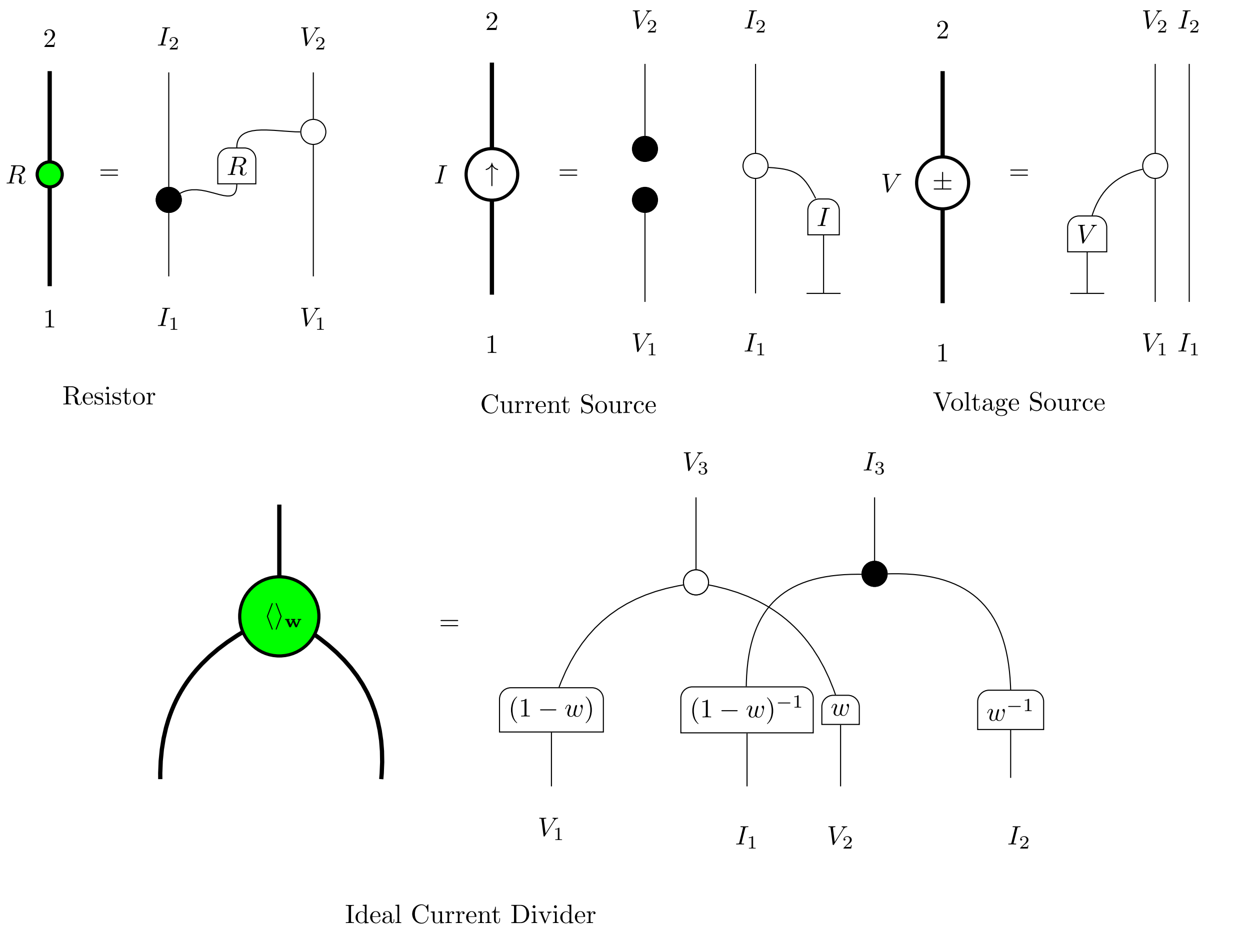


Figure 2.2: GAA representation of electrical circuit elements [2]

2.9 Orthogonal Complementation

Inspection of the equations of GLA reveals that there is a symmetry [50]: the rules remain unchanged under the combined operation of

- interchange of "copy" and "addition" spiders
- interchange of scalar multiplication by k with scalar multiplication by k^{-1}. Note the scalar 0 maps to itself.

More precisely, there exists a functor from LinRel to itself, known as *orthogonal complementation*, defined by its action on generators, shown in Figure 2.3, which is an involution on the category linear relations, maps copy to addition spiders and vice versa. However, the equations under this functor remain unchanged. We therefore regard orthogonal complementation as a conjugation or an covariant automorphism on the category of linear relations.

Orthogonal complementation functor can be defined at the level of linear relations. We first need to define two inner products. Let $e(\ ,\ )$ denote the Euclidean inner product on the vector space F^n:

$$e(v,w) = v^T w = \sum_{i=1}^{n} v_i w_i, \quad v,w \in F^n$$

Let $\bar{e}(\ ,\ )$ denote the "anti-Euclidean" inner product on the vector space F^n::

$$\bar{e}(v,w) = -v^T w = -\sum_{i=1}^{n} v_i w_i, \quad v,w \in F^n.$$

Definition 2.9.1. Consider a linear relation $\mathcal{R}$, which is a subspace of $F^a \oplus F^b$. We define the orthogonal complement of $\mathcal{R}$, denoted as $\mathcal{R}^\perp$, to be the set of all vectors in $F^a \oplus F^b$ that are orthogonal to all vectors in $\mathcal{R}$, under the inner product $\bar{e} \oplus e$.

By direct computation, one can check that the action of $(\ )^\perp$ on the generators of GLA obtained via definition 2.9.1 gives rise to the functor depicted in Figure 2.3.

The **orthogonal complement** of a state V is another state $V^\perp$. Observe that there is an inclusion functor $\iota : \mathsf{Mat}_F \to \mathsf{LinRel}_F;\ M \mapsto \{(x,y) \mid y = Mx\}$

Lemma 2.9.2. If $M : m \to n \in \mathsf{Mat}_F$ then the orthogonal complement of $\iota(M)$ is:

$$\iota(M)^\perp = \{(x,y) \mid x = M^T y\}$$

Proof. First note that $\dim \iota(M)^\perp = n$, so we need construct a subspace V of $F^m \oplus F^n$ of dimension n. The subspace encoded by $\iota(M)$ is: (x, Mx) for all $x \in F^m$. Define V as $(M^T y, y)$ for all $y \in F^n$. V has dimension n. The inner product of any vector in V with any vector in $\iota(M)$, computed as:

$$x^T M^T y - x^T M^T y = 0.$$

Therefore $\iota(M)^\perp = V$. □

When we include the affine generator, this symmetry of the orthogonal functor is no longer present – the equations for interactions with the affine unit with copy spiders and addition spiders are not symmetric under interchange of black and white.

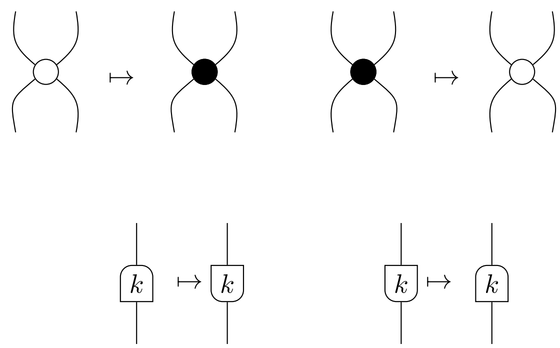


Figure 2.3: Functor of orthogonal complementation $()^{\perp}$ [3].

2.10 Lagrangian Relations

In this section we will review the theory of Lagrangian relations, we will closely be following [3,21]. We start by briefly reviewing the theory of symplectic vector spaces, for more details the reader should consult [51].

Definition 2.10.1 (See definition 2.1 in [3] and definition 6.32 in [21])**.** A **symplectic form** is a bilinear map $\langle \_, \_ \rangle : V \otimes V \to F$ which satisfies:

- $\langle x, x \rangle = 0$ for every $x \in V$;
- Any x such that $\langle x, y \rangle = 0$ for every $y \in V$ implies $x = 0$.

A vector space V equipped with a symplectic form $\omega(\_, \_)$ is called a **symplectic vector space.**

One can always express a bilinear form in matrix form as $\omega(x, y) := x^T S y$. For a symplectic form one always has $S^{-1} = -S = S^T$. Furthermore, a symplectic vector space always has a Darboux basis: this is a basis $q_1, ..., q_n, p_1, ..., p_n$ for which $\omega(q_i, p_j) = -\omega(p_j, q_i) = \delta_{ij}$ and $\omega(q_i, q_j) = \omega(p_i, p_j) = 0$ (where $\delta_{i,j} = 1$ when $i = j$ and zero otherwise). In particular, this means that a symplectic vector space always has even dimension. In the special case when the canonical basis of the vector space is a Darboux basis, the symplectic form can be expressed using the $2n \times 2n$ block matrix J:

$$\langle x, y \rangle := \begin{pmatrix} q^T & p^T \end{pmatrix} J \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

When a symplectic vector space has its canonical basis a Darboux basis, its vectors are naturally graded. In classical physics one regards $q^T = (q_1 \dots, q_n)$ as position and $p^T = (p_1 \dots p_n)$ as momenta coordinates.

Definition 2.10.2. (See Definition 2.1, 2.3 in [3] and Definition 6.32 in [21])

(i) A **symplectomorphism** is a linear map which preserves the symplectic form and is also an isomorphism.

(ii) The **symplectic dual** of a linear subspace $U \subseteq V$ of a symplectic vector space V is the linear subspace $U^\omega := \{u' \in V \mid \forall u \in U, \omega(u', u) = 0\} \subseteq V$.

(iii) A **Lagrangian subspace** U of a symplectic vector space V is linear subspace which is its own symplectic dual, so that $U = U^\omega$.

Definition 2.10.3. (See definition 2.5 in [3]) The prop of Lagrangian relations over a field F LagRel_F is defined as the follows:

Objects $n \in \mathbb{N}$: correspond to the graded vector spaces $F^n \oplus F^n$ equipped with the cannonical symplectic form given by J, described above.

Maps $\mathcal{L} : n \to m$: are relations $\mathcal{R} \subseteq F^{n+m} \oplus F^{n+m}$ where the state $\mathcal{L} = \{((q, p), (q', -p'))|(p, q), (p', q')) \in \mathcal{R}\}$ is Lagrangian.

Tensor: Given by addition on objects and the (graded) direct sum on maps.

An alternate way to define Lagrangian relations is to use the double construction which is discussed below:

For a symmetric monoidal category $\mathbb{X}$, the category $\mathsf{Double}(\mathbb{X})$ is defined as follows:

Objects: $X \in \mathbb{X}$

Maps: For maps $f : X \to Y$ in $\mathbb{X}$ we have maps: $f : X \otimes X \to Y \otimes Y \in \mathbb{X}$.

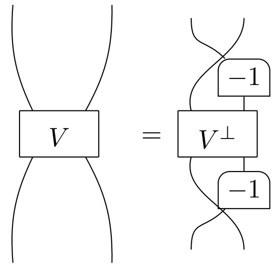
Composition: As in $\mathbb{X}$

Tensor: Given maps $f : X \to X'$ and maps $g : Y \to Y'$ in $\mathsf{Double}(\mathbb{X})$ we have maps $f : X \otimes X \to X' \otimes X'$ and $g : Y \otimes Y \to Y' \otimes Y'$ in $\mathbb{X}$ the map $f \otimes g$ is given by the following commuting square:

$$\begin{tikzcd}
X \otimes X \otimes Y \otimes Y \arrow[r, "f\otimes g"] & X' \otimes X' \otimes Y' \otimes Y' \arrow[d, "ex"] \\
X \otimes Y \otimes X \otimes Y \arrow[u, "ex"] \arrow[r, "f\underline{\otimes} g"] & X' \otimes Y' \otimes X' \otimes Y'
\end{tikzcd}$$

This then gives a map $f \underline{\otimes} g : X \otimes Y \to X' \otimes Y'$ in $\mathsf{Double}(\mathbb{X})$ where $\underline{\otimes}$ is the tensor product in $\mathsf{Double}(\mathbb{X})$.

Given $\mathsf{Double}(\mathbb{X})$, we can define the Lagrangian relations as a subcategory where $\mathbb{X} = \mathsf{LinRel}_F$ of which maps satisfy:

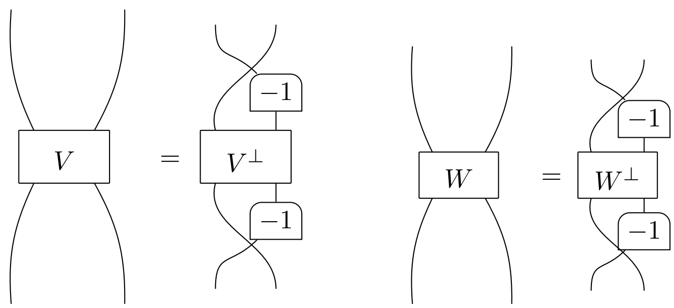


(2.3)

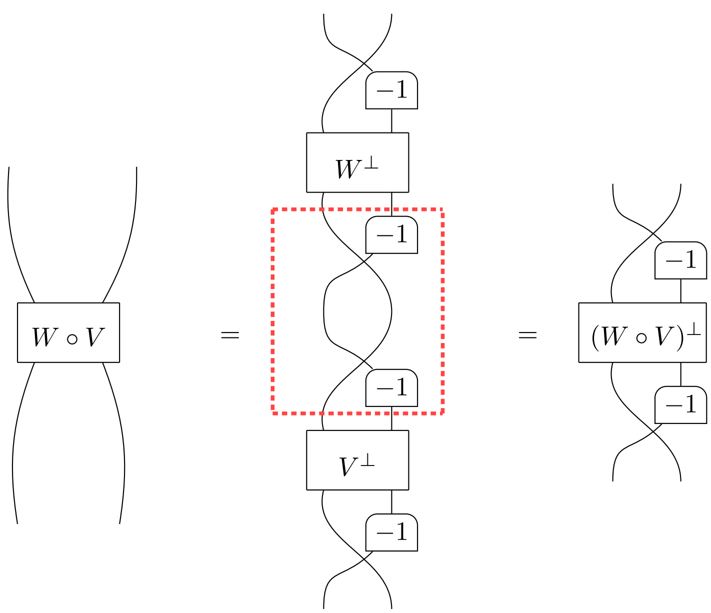
One can use the above formulation to prove the following:

Lemma 2.10.4. The composition of two Lagrangian relations is Lagrangian.

Proof. Consider two composable Lagrangian relations V and W both of which satisfy the following equation:



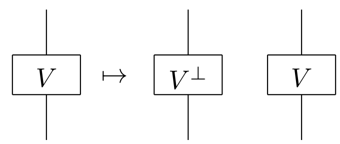
The composite relation $W \circ V$ is given as follows, note that the -1's in the dotted red box cancel out to give the following equality:



The last equality implies the composite relation $W \circ V$ is Lagrangian. □

As shown in [3] there exists a functor L from linear relations to Lagrangian relations. This functor takes a linear relation V and "doubles" it, that is it pair the relation V with its orthogonal complement $V^{\perp}$.

Lemma 2.10.5. (See lemma 2.7 in [3]) There is a symmetric strong monoidal functor $L : \mathsf{LinRel}_{\mathsf{F}} \to \mathsf{LagRel}_{\mathsf{F}}$ given graphically by the following diagram:

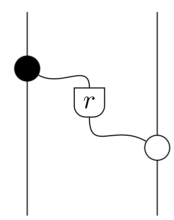


Proof. See [3] □

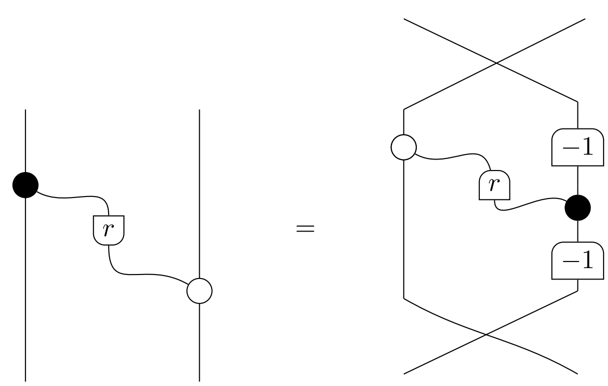
Lagrangian relations are closely connected to electrical circuits and this connection was explored in detail in [3]. In particular here we explicitly show that resistors and current dividers are examples of Lagrangian relations:

Lemma 2.10.6. Resistance (as shown in Figure 2.2) is a Lagrangian Relation.

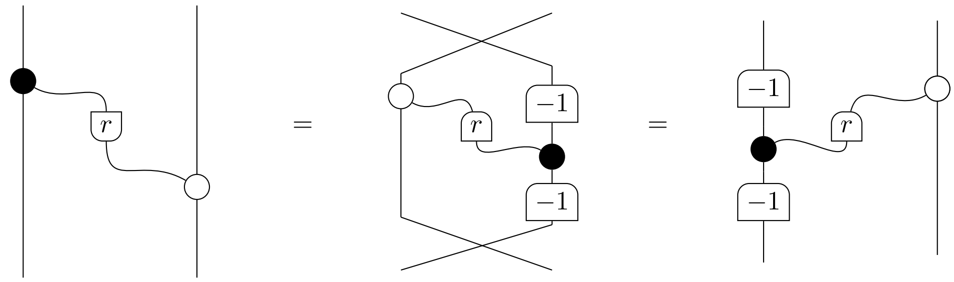
Proof. The representation of a resistance using the white and black dot generators of graphical affine algebra is the following:



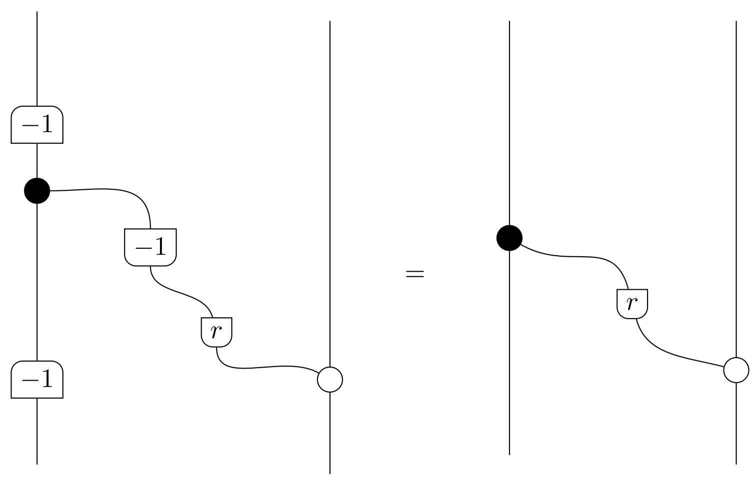
To show that this relation is Lagrangian we need to show that it satisfies Equation 2.3. For this we need to show that:



To show this we begin on the LHS and straighten wires to get the following diagram:



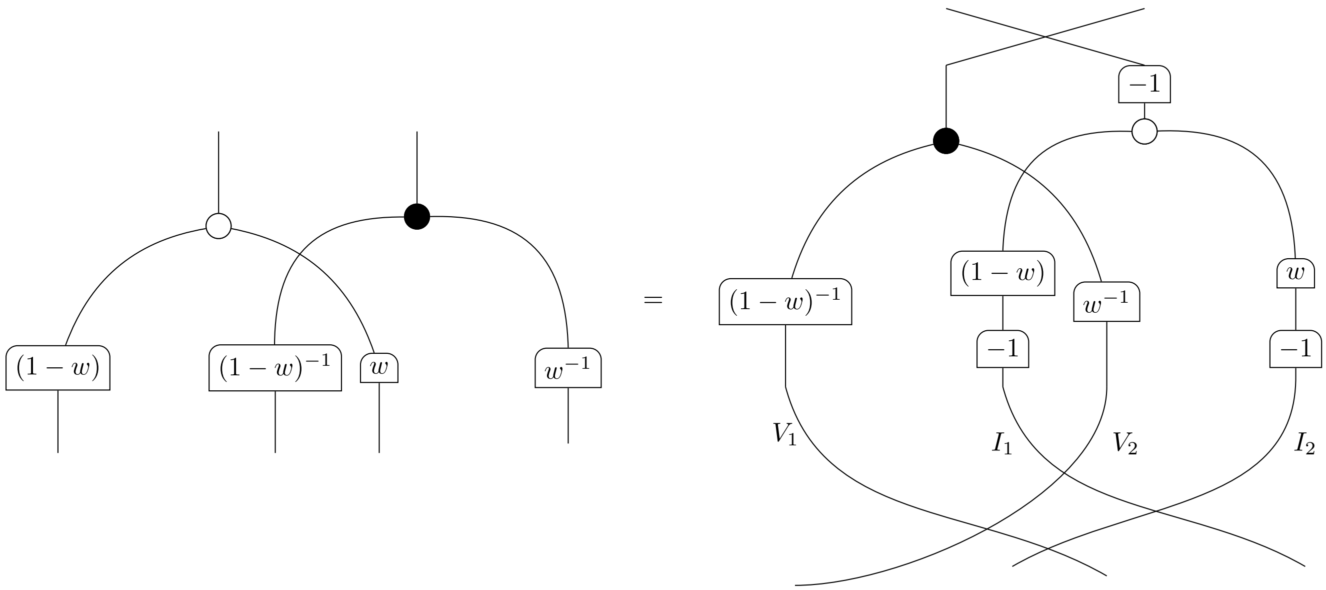
Finally when we simplifying the above diagram by pushing the black dot up by using a cap, this induces a -1 and we get the following equality:



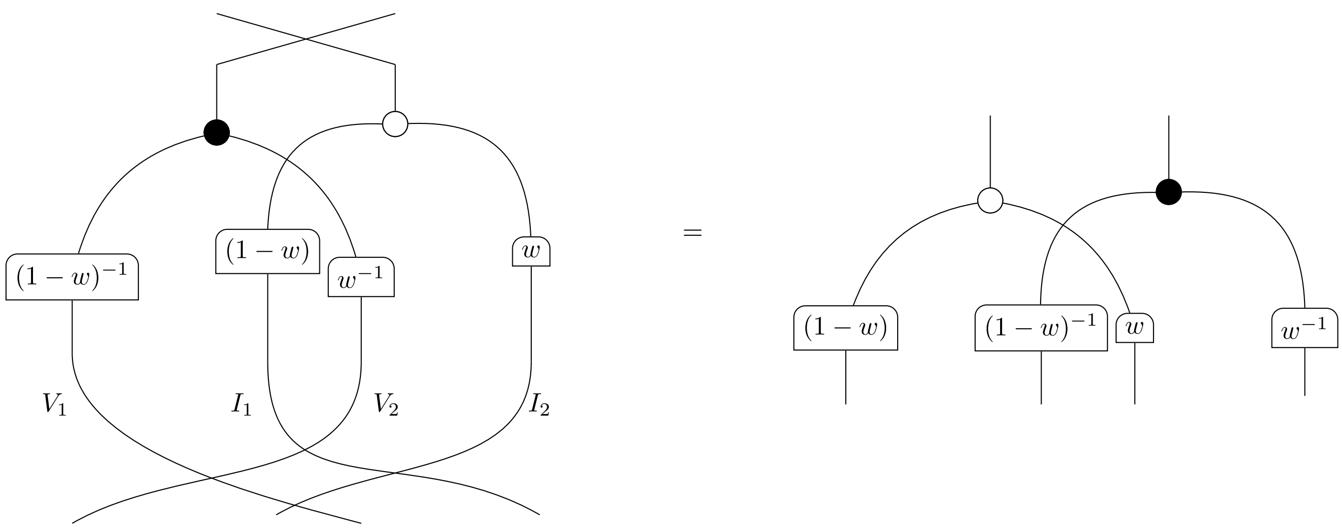
This shows that the resistance is a Lagrangian relation. □

Lemma 2.10.7. The ideal current divider (as shown in Figure 2.2) is a Lagrangian relation.

Proof. To show that the current divider is a Lagrangian relation it should satisfy equation 2.3, in the case of the current divider this will take the following form:



Starting from the RHS we push the -1 through and straightening the wires to get the following diagram:



which shows that the current divider is a Lagrangian relation as it satisfies 2.3. □

To conclude this section we mention the relation between affine Lagrangian relations and stabilizer qudit quantum mechanics which was explored and made precise in great detail in [3].

Definition 2.10.8. (See definition 4.4 in [3]) The prop of $\mathsf{AffLagRel}_F$ of affine Lagrangian relations over a field F is defined as follows:

Objects: Symplectic vector spaces

Maps: Generated by the image of $\mathsf{LagRel}_F \xrightarrow{E} \mathsf{LinRel}_F \to \mathsf{AffRel}_F$.

Tensor: Direct Sum

Note that $E : \mathsf{LagRel}_F \to \mathsf{LinRel}_F$ is a strong symmetric faithful functor described in [3].

Recall that the category of stabilizers is defined as follows:

Definition 2.10.9. (See definition 4.14 in [3]) The category of $\mathsf{Stab_p}$ is a subcategory of $\mathsf{Mat_C}$ which is generated upto scalars by p-dimensional qudit Clifford group, $|0\rangle$ and $\langle 0|$.

The precise relationship between $\mathsf{AffLagRel}_F$ and Stab_p was proven in [3] and is stated as follows:

Lemma 2.10.10. (See lemma 4.15 in [3]) There is an isomorphism $G : \mathsf{AffLagRel_F} \to \mathsf{Stab_p}$ where p is an odd prime.

We discuss this relationship in more detail in Chapter 5.

Chapter 3

Parity-Check Matrices of Linear, Affine and Lagrangian Relations

3.1 Introduction

In this chapter linear, affine and Lagrangian relations are viewed through the lens of parity-check and generator matrices, we classify various subcategories of Lagrangian relations based on the parity-check matrix structure. In particular a momentum conserving subcategory of Lagrangian relations is studied and its properties are characterized based on the structure of the parity-check matrices. Deterministic relations are characterized and their properties studied. Furthermore the concept of power input is defined and based on this concept the categories of lossless Kirchhoff and Lagrangian relations are studied. This chapter is based on the paper [1].

3.2 Linear and Affine Relations and Parity-Check Matrices

Let F be a field. Our attention will be focused on finite fields, in which case $|F|$ denotes the total number of elements in the field and p denotes the characteristic. Recall that a **linear relation** $R : m \to n$ between the vector spaces F^m and F^n is a linear subspace of $F^m \oplus F^n$, $R \subseteq F^n \oplus F^m$. Two special cases in LinRel_F are *states*, for which $m = 0$, and *effects*, for which $n = 0$.

A special case of a linear relation is a linear function, in which each $x \in F^m$ is related to exactly one $y \in F^n$. A linear function from F^m to F^n can be written as an $n \times m$ matrix A such that $y = Ax$. Linear functions form a prop by themselves, which is given by $\iota : \mathsf{Mat}_F \to \mathsf{LinRel}_F$.

Unlike linear functions, a general linear relation between $R : F^m \to F^n$ cannot be represented as an $n \times m$ matrix. However, it is possible to represent an arbitrary linear relation using larger matrices with $n + m$ columns, as is now reviewed.

Any linear subspace $\mathcal{R} : m \to n$ can be expressed as the *span* of a set of k independent vectors, that is as the set of all $(x, y) \in F^m \oplus F^n$ that

$$\begin{pmatrix} x \\ y \end{pmatrix} = G^T t \tag{3.1}$$

for some $t \in F^k$, where $G^T : k \to m + n$ has rank k. In the terminology of linear error-correcting codes, G is a **generator matrix** of the linear relation $\mathcal{R}$ [52].

Alternatively, we can specify a linear relation as a set of vectors $(x, y) \in F^m \oplus F^n$ satisfying a linear equation of the form:

$$H \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{3.2}$$

where $H : n+m \to n+m-k$ has entries in F, and rank $(n+m-k)$. In other words, $\mathcal{R}$ is the kernel of H. Adopting terminology from the theory of linear error-correcting codes, H is referred to as the **parity-check matrix** of the linear relation [52]. Its important to emphasize again that *any* linear relation can be expressed using a parity-check matrix or a generator matrix – and one does not require the relation to be a function.

Example 3.2.1. Consider the relation $\mathcal{R}$ between F^2 and F^2, defined by $\{((x,x),(x,x)) \in F^2 \oplus F^2 \mid x \in F\}$, (which can be represented by a copy spider with 2 inputs and 2 outputs, as shown in Figure 3.2.) This can alternatively be specified via a parity-check matrix as the set $((x_1, x_2), (y_1, y_2)) \in F^2 \oplus F^2$ that satisfies

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = 0.$$

It can also be specified via a generator matrix as

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} t \end{pmatrix}.$$

From these expressions one can see that the dimension of $\mathcal{R}$ is 1.

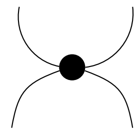
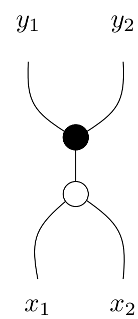


Figure 3.1: Black spider with 2 inputs and 2 outputs

Example 3.2.2. Consider a function from $F^2 \to F^2$ shown below:



(3.3)

This diagram encodes the following relation $\{(x_1, x_2), (y_1, y_2) | x_1 + x_2 = y_1 \ , \ y_1 = y_2\}$ A parity-check matrix H for this function is given below:

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the generator matrix G^T is:

$$G^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

One can also check that $HG^T = 0$ is satisfied using the above matrix expressions.

Note that H is not uniquely specified. Any vector $x \oplus y$ that satisfies equation (3.2), also satisfies the equation (3.2) with H replaced by $H' = SH$, where S is any invertible matrix with appropriate dimensions. Similarly, one can replace G^T by $G^T S$ in equation (3.1). More precisely if H and H' are parity-check matrices for R then there exists a unique isomorphism α such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, shift left, "\subseteq"] \arrow[r, shift right, "0"'] & F^m \oplus F^n \arrow[r, "H"] \arrow[dr, "H'"'] & F^{n+m-k} \arrow[d, "\alpha"] \\
 & & F^{n+m-k}
\end{tikzcd}$$

This is because, in Mat_F, a parity-check matrix for $\mathcal{R}$ is a universal coequalizer with kernel $\mathcal{R}$ and all such coequalizers are uniquely comparable as above.

Finally, note that H is a parity-check matrix and G is a generator matrix for the same subspace if and only if $HG^T = 0$ and this is an *exact* composite in the sense that the image of G^T is the kernel of H (both of which are the subspace being classified). Exactness of the composite is assured provided $\mathsf{Rank}(H) + \mathsf{Rank}(G) = m + n$, which is always the case for generator and parity-check matrices for the same subspace.

As mentioned above, the parity-check and generator matrices for a relation are not unique. The rows of G are linearly independent and form a basis for the linear subspace $\mathcal{R}$. Changing the order of rows, or making elementary row operations, such as adding one row to another row, does not change the linear subspace specified by G or H. One can see this by starting with the equation $Hx = 0$ and multiplying this equation by a matrix e on both sides of the equation, this corresponds to the row operations performed. One then gets the equation $eHx = e0$ now since e is invertible we can multiply by e^{-1} on both sides to get back to $Hx = 0$, this shows that under elementary row operations given by a matrix e the parity-check matrix does not change. Using this freedom one gets the following standard forms for H and G:

$$H = \begin{pmatrix} 1_{m+n-k} & A \end{pmatrix} \sigma : m + n \to m + n - k, \tag{3.4}$$

where σ is a permutation matrix. Corresponding to this is a generator matrix in the standard form:

$$G = \begin{pmatrix} -A^T & 1_k \end{pmatrix} \sigma : m + n \to k \tag{3.5}$$

where the same matrix A and permutation σ appear in equations (3.4) and (3.5). One can see that this G is a generator matrix as (using the fact that $\sigma^T = \sigma^{-1}$):

$$HG^T = \begin{pmatrix} 1_{m+n-k} & A \end{pmatrix} \sigma\sigma^T \begin{pmatrix} -A \\ 1_k \end{pmatrix} = \begin{pmatrix} 1_{m+n-k} & A \end{pmatrix} \begin{pmatrix} -A \\ 1_k \end{pmatrix} = A - A = 0$$

and the ranks make this composite exact. The standard form for a generator parity-check pair is, thus determined by (A, σ).

While standard forms for characterizing a given $\mathcal{R}$ are not unique, the choice of σ completely determines the pair (A, σ). This can be seen from the commuting diagram:

$$
\begin{tikzcd}
& m+n-k \arrow[d, "P"'] \arrow[dr, "1_{m+n-k}"] & \\
& m+n \arrow[r, "H"] & m+n-k \arrow[dd, "\alpha"] \\
N \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & & \\
& m+n \arrow[r, "H'"'] & m+n-k
\end{tikzcd}
$$

where $P := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This shows that the isomorphism α between parity-check matrices σH and $\sigma' H'$ for a relation $\mathcal{R}$ is $\alpha = H'\sigma'\sigma^{-1}P$. Thus, for a fixed σH, an α which produces $\sigma' H'$ is completely determined by the permutation σ'. Suppose two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ between the same vector spaces are described by the parity-check and generator matrices H_1, G_1 and H_2, G_2 respectively, then it is easy to see that $\mathcal{R}_1 \subset \mathcal{R}_2$ if and only if $H_2 G_1 = 0$.

An **affine relation** between the vector spaces F^m and F^n can be specified as the subset of vectors $(x, y) \in F^m \oplus F^n$ satisfying a linear equation of the form:

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \chi.$$

Note that a linear relation always contains the zero vector; so the empty set is not a linear relation. However, the empty set is an affine relation because Equation (3.2) may not have any solution, e.g. if H has rank zero, but $\chi \neq 0$. For the affine relation to be non-empty, one requires that the length of χ equals the rank of H. Borrowing terminology from the theory of error-correcting codes again, χ is called the **syndrome** of the affine relation. If the affine relation is not empty, it can alternatively be expressed as

$$H \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = 0,$$

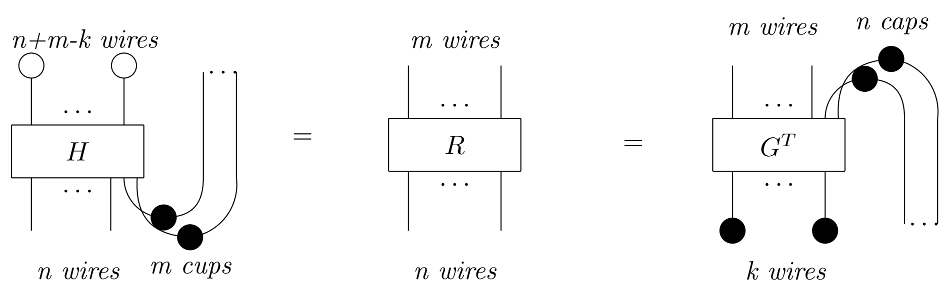
or via an analogous formula in terms of the generator matrix G, for some constant vectors x_0 and y_0, which are non-uniquely determined from the syndrome χ such that:

$$H \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \chi$$

Two relations which can be used to convert inputs into outputs, and vice versa are respectively the **cup** and the **cap**:

$$cap := \{(0, (x, x)) \mid x \in F)\} \quad \text{and} \quad cup := \{((x, x), 0) \mid x \in F\}$$

One may use cups and caps to convert any relation from F^m to F^n into a *state* (where $m = 0$) or an effect (where $n = 0$), or indeed into a relation between $F^{m'}$ and $F^{n'}$, where $m' + n' = m + n$. In particular, these relations allows one to recover a relation from its parity matrix and from its generator matrix as shown below:



The monoidal sum of two linear relations described by generator matrices G and G' respectively can be described by a generator matrix G_{sum} which is the direct sum of generator matrices:

$$G_{sum} = \begin{pmatrix} G & 0 \\ 0 & G' \end{pmatrix} \; : \; n + m \to k_1 + k_2$$

Similarly for a parity-check matrix for the monoidal sum of relations is the direct sum of the parity-check matrices for the relation.

$$H_{sum} = \begin{pmatrix} H & 0 \\ 0 & H' \end{pmatrix} \; : \; n + m \to n - k_1 + m - k_2$$

However, the generator matrix for the composition of two linear relations, is *not* the matrix product of their generator matrices. Nor is the parity-check matrix for the composition of the two linear relations, the matrix product of their parity-check matrices.

3.3 Orthogonal Complement and Parity-Check Matrices

In the previous chapter the functor $(-)^{\perp} : \mathsf{LinRel}_F \to \mathsf{LinRel}_F$ is described. Here the behaviour of the functor will be captured using parity-check matrices:

Theorem 3.3.1. Let the orthogonal complement of a state $\mathcal{R}$ be denoted as $\mathcal{R}^{\perp}$. The parity-check matrix of $\mathcal{R}^{\perp}$ can be chosen to be the generator matrix of $\mathcal{R}$; and the generator matrix of $\mathcal{R}^{\perp}$ can be chosen to be the parity-check matrix of $\mathcal{R}$.

Proof. Let $\mathcal{R}$ be a rank k state described by generator matrix G. Then, for all $x \in \mathcal{R}^{\perp}$, and for all $u \in F^k$ one have $x^T G^T u = 0$, by definition of $\mathcal{R}^{\perp}$. This means $x \in \mathcal{R}^{\perp}$ iff $Gx = 0$. This result means $R^{\perp}$ is the nullspace of G – which is another way of saying that G serves as a parity-check matrix for $\mathcal{R}^{\perp}$. □

This is a well known result in theory of linear error-correcting codes [52]. An immediate corollary of the above theorem is as follows. Suppose a state $\mathcal{R}$ is described by parity-check matrix $H = \begin{pmatrix} 1 & A \end{pmatrix} \sigma$. Then the state $\mathcal{R}^{\perp}$ is described by the parity-check matrix $H^{\perp} = \begin{pmatrix} -A^T & 1 \end{pmatrix} \sigma$. A special case of the above Theorem 3.3 is the Lemma 2.9.2.

3.4 Lagrangian Relations and Parity-Check Matrices

In this section Lagrangian relations and their properties are studied in terms of parity-check matrices, recalling the definition of a symplectic dual:

Definition 3.4.1. The **symplectic dual** of a linear subspace $U \subseteq V$ of a symplectic vector space V is the linear subspace $U' := \{u' \in V \mid \forall u \in U, \langle u', u \rangle = 0\} \subseteq V$.

To see what this means in terms of parity-check and generator matrices, Suppose a linear subspace $U \subseteq V$ is specified by the generator matrix G then a vector $u' \in V$ is in U' if and only if it satisfies $u'SGt = 0$ for all $t \in F^k$. This means that U' can be specified using the parity-check matrix $H' := GS^T$, which means $H'S = G$. Similarly, letting H be a parity-check matrix matrix for U then one obtains a generator matrix G' for U' as $G' := HS$. Thus, similar to the situation for orthogonal subspaces, for symplectic duals one can flip between generating and parity-checking.

Definition 3.4.2. A **Lagrangian subspace** U of a symplectic vector space V is linear subspace which is its own symplectic dual, so that $U = U'$.

When a subspace $U \subseteq V$ of a symplectic vector space is Lagrangian one can obtain from a parity-check matrix H a generator matrix G, using the above observations, as $G = HS$. In fact, it is clear that a subspace is Lagrangian if and only if a parity-check and generator matrices can be chosen to satisfy $G = HS$. This has the consequence that the codomain of G and H must have the same dimension, so if V has dimension $2n$ a Lagrangian relation must have dimension n. If one specializes to $J = S$, when the canonical basis is a Darboux basis one can write,

$$H = \begin{pmatrix} H_q \mid H_p \end{pmatrix}$$

and the generator matrix G as:

$$G = \begin{pmatrix} G_q \mid G_p \end{pmatrix}$$

so that:

$$\begin{pmatrix} H_q \mid H_p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} G_q \\ G_p \end{pmatrix}$$

implies that

$$H_p = -G_q, \quad H_q = G_p. \tag{3.6}$$

Note that an arbitrary permutation of the coordinates viewed as a relation will not in general be a Lagrangian relation. For a permutation to be a Lagrangian relation it must preserve the symplectic form and so be a *symplectomorphism* or a *symplectic permutation*. Thus, it must not only respect the grading but also permute the two grades in the same manner. This means a symplectic permutation can be written as:

$$\sigma_S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Earlier it was shown that parity-check and generator matrices of a linear relation can be brought into a standard form. One can attempt to do the same for Lagrangian states, however, one will also want the permutation matrix σ in (3.4) and (3.5) that permutes the columns of G and H to be Lagrangian. Taking this into account one can state the theorem:

Theorem 3.4.3. The parity-check matrix H for a Lagrangian subspace $\mathcal{R} \subseteq (F^n)^2$ can be put into the following standard form:

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S \tag{3.7}$$

where $n_p + n_q = n$ (the dimension of $\mathcal{R}$), σ_S is a symplectic permutation, A has dimensions $n_q \times n_p$, and

$Y = Y^T$.

Proof. First note that $H = (H_q \mid H_p)$ and $HJH^T = 0$ this implies that

$$0 = HJH^T = \begin{pmatrix} H_q \mid H_p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_q^T \\ H_p^T \end{pmatrix} = -H_p H_q^T + H_q H_p^T \implies H_p H_q^T = H_q H_p^T$$

This implies that $H_p H_q^T$ is self-transpose. Consider $H = (H_q \mid H_p)$ and putting H_q in standard form that is:

$$H_q = \begin{pmatrix} 1 & A^T \\ 0 & 0 \end{pmatrix} \sigma_q$$

Adjusting H_p to be $H_p = H'_p \sigma_q$ where $H'_p = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ Hence $H_p H_q^T$ is:

$$H_p H_q^T = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \sigma_q \sigma_q^T \begin{pmatrix} 1 & 0 \\ A & 0 \end{pmatrix}$$

Now since $\sigma_q^T = \sigma_q^{-1}$ one has the following:

$$H_p H_q^T = \begin{pmatrix} B_1 + B_2 A & 0 \\ B_3 + B_4 A & 0 \end{pmatrix}$$

Now since $H_p H_q^T$ is self-transpose, it follows that:

$$B_3 + B_4 A = 0$$

and

$$B_1 + B_2 A = B_1^T + A^T B_2^T$$

But note that

$$(H_q | H_p) = \begin{pmatrix} 1 & A & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \end{pmatrix} \sigma_S$$

has rank n and so

$$rank \begin{pmatrix} 0 & 0 & B_2 & B_4 \end{pmatrix} = n_p$$

Note that B_2 depends on B_4 so B_4 has rank n_p and hence is invertible. This means one can use row operations to reduce B_4 to the identity which results in the following form for H_p

$$H_p = \begin{pmatrix} B_1' & B_2' \\ B_3' & B_4' \end{pmatrix} := \begin{pmatrix} Y & 0 \\ A' & 1 \end{pmatrix}$$

where the same equations hold so $A' = -1A = -A$ and $Y = Y + 0A = Y^T + A^T 0 = Y^T$. Hence, finally, one gets the desired form:

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S$$

where σ_S is a symplectic permutation. □

Again, this standard form is not unique but is determined by (Y, A, σ) where $\sigma_S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. This standard form is for states, for Lagrangian relations one can convert the relation to a state using cups and then this standard form can apply.

In terms of parity-check matrices one can define affine Lagrangian relation as follows:

Definition 3.4.4. An affine Lagrangian relation between two symplectic vector spaces F^m and F^n is either the empty set, or a subset of $F^m \oplus F^n$ specified by an equation of the form:

$$H(u - U) = 0$$

where $HU = \chi$

also H is the parity-check matrix of a Lagrangian relation between F^m and F^n.

Note that an affine Lagrangian relation does not satisfy $\mathcal{R} = \mathcal{R}'$, as this would force U to be zero. This condition can also be written as $Hu = \chi$, where $\chi = HU$. Note that, for a given choice of H, different choices of U could give rise to the same affine relation. However, for a given H, different choices for χ give rise to different affine relations.

As defined in the previous chapter there is functor L, from the category of linear relations to the category of Lagrangian relations. Here one views this functor as follows:

Definition 3.4.5. If $\mathcal{R}$ is a linear subspace of F^n, then $L(\mathcal{R})$ can be defined as follows:

$$L(\mathcal{R}) := \mathcal{R} \oplus_{q,p} \mathcal{R}^{\perp}$$

Here, $\mathcal{R}^\perp$ denotes the orthogonal complement of $\mathcal{R}$.

Consider $L(\mathcal{R})$ where $\mathcal{R}$ is a state: to see that $L(\mathcal{R})$ is a Lagrangian subspace, observe that one can choose the generator matrix of $L(\mathcal{R})$ to be of the form

$$G_{L(\mathcal{R})} = \begin{pmatrix} G & 0 \\ 0 & -H \end{pmatrix}, \quad H_{L(\mathcal{R})} = \begin{pmatrix} H & 0 \\ 0 & G \end{pmatrix}$$

where G and H are generator and parity-check matrices of $\mathcal{R}$. Then observe that $G_{L(\mathcal{R})} = H_{L(\mathcal{R})}J$ verifying this is a (non-standard) parity-check generator pair for a Lagrangian relation.

A useful lemma, which follows immediately from the definition of L is the following.

Lemma 3.4.6. Let $M : m \to n$ be a matrix. Then $L(\iota(M)) : F^{2m} \to F^{2n}$ is the following Lagrangian relation:

$$L(\iota(M)) = \{(q,p),(q',p') \in F^{2n} \oplus F^m \mid q' = Mq \;\; p = M^T p'\}$$

Proof. This follows immediately from Lemma 2.9.2 in the previous section. □

3.5 Kirchhoff Relations

In this section the category of Kirchhoff relations is described as a subcategory of LagRel. While similar categorical structures have been investigated in [21,53], Our approach of using parity-check matrices, besides being novel, brings some new subcategories to the fore. As remarked earlier objects in LagRel are graded vector spaces with the grading in classic physics being interpreted as position and momentum. However, we wish now to switching interpretation so the grading represents voltage and current: to achieve this some additional structure seems necessary. The word "current" implies the flow of some conserved quantity, while the word "voltage" implies a sort of gauge-invariance, which gives the freedom to shift all voltages by a constant. Kirchhoff relations, which is now introduced, interprets these intuitions.

Definition 3.5.1. A Lagrangian relation $\mathcal{R} : m \to n$ in LagRel satisfies:

(i) **Kirchhoff's Current Law** if, for all $((q,p),(q',p')) \in \mathcal{R}$, the following equality holds:

$$\sum_{j=1}^{m} p_j = \sum_{k=1}^{n} p'_k$$

(ii) **Translation invariance** if, whenever $\lambda \in F$ and $((q,p),(q',p')) \in \mathcal{R}$, then $((q+\vec{\lambda}_m, p),(q'+\vec{\lambda}_n, p')) \in \mathcal{R}$, where $\vec{\lambda}_n$ is a vector of dimension n all of whose components are the same $\lambda \in F$.

Hence one can define Kirchhoff relation as follows:

Definition 3.5.2. A **Kirchhoff relation** is a Lagrangian relation that obeys Kirchhoff's Current Law.

Lemma 3.5.3. Resistors, Spiders, Current Dividers are Kirchoff relations.

Proof. To see this observe we can interpret the graphical representation of each shown in Figure 2.2, as linear relations (see Tables 4.1, 4.2, 4.3, 4.4 in Chapter 4). One then checks that sum of input currents equals the sum of output currents which is KCL. □

Affine Kirchhoff relations are defined as below:

Remark 3.5.4. An **affine Kirchhoff relation** is an affine Lagrangian relation that obeys Kirchhoff's Current Law.

From the perspective of classical physics, Kirchhoff's Current Law asks for conservation of momentum. Translation invariance is the sort of gauge-invariance expected in electrical circuits: it provides the freedom to shift all voltages by a constant reference voltage.

Kirchhoff's Current Law may be equivalently expressed as the requirement that $p^T\vec{1}_n = {p'}^T\vec{1}_m$ while translation invariance amounts to insisting that $((\vec{1}_m, 0),(\vec{1}_n, 0))$ is a member of the relation.

The following lemma shows that Kirchhoff's current law implies translation invariance for Lagrangian relations. This result is essentially a version of Noether's theorem that applies even when F is an arbitrary field.

Lemma 3.5.5. For a state $\mathcal{R}$ in LagRel_F the following are equivalent:

(i) $\mathcal{R}$ satisfies the Kirchhoff current law;

(ii) $\mathcal{R}$ satisfies translation invariance.

Proof. Let $\mathcal{R}$ be specified by a generator matrix G and a parity-check matrix H which satisfy $G = HJ$.

(ii) ⇒ (i): If $\mathcal{R}$ is translationally invariant then if $u \in \mathcal{R}$ this implies $(u + \lambda\epsilon) \in \mathcal{R}$ where $u := \begin{pmatrix} q \\ p \end{pmatrix}$ and $\epsilon_n := \begin{pmatrix} \vec{1}_n \\ 0 \end{pmatrix}$. This implies that $H(u + \lambda\epsilon_n) = H(u) + \lambda H(\epsilon) = 0$ as $H(u) = 0$ this means, in turn that

$H(\epsilon_n) = 0$, and so $\epsilon_n \in \mathcal{R}$ by substituting $\lambda = 1$. Now the symplectic product of ϵ_n with any $u \in \mathcal{R}$ must be zero so

$$0 = \langle u, \epsilon_n \rangle = u^T J \epsilon = \begin{pmatrix} q^T & p^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix} = -p^T \vec{1}$$

implying that $p^T \vec{1} = 0$ which is Kirchhoff current law.

(i) $\Rightarrow$ (ii): Conversely suppose one has conservation of momentum then $0 = p^T \vec{1} = u^T J \epsilon_n \; \forall u \in \mathcal{R}$ however $u^T J \epsilon = 0$ implies $\epsilon \in \mathcal{R}$ which in turn implies translational invariance, thereby concluding the proof.

□

For affine Lagrangian relations the situation is a bit more complicated. Suppose an affine Lagrangian relation $\mathcal{R}$ is invariant under $u \to u + \lambda\epsilon$, i.e., $H(u - U) = 0 \implies H(u - U + \lambda\epsilon) = 0$ This again implies $H\epsilon = 0$. Then, using the generator matrix to calculate $\epsilon^T \cdot Su$, one finds that:

$$\epsilon^T Su = \epsilon^T SU$$

Thus, one must also require that $\epsilon^T SU$ to arrive at the result $\epsilon^T Su = 0$. Translation invariance, therefore, does not imply conservation of momentum for affine Kirchhoff relations. However, it is easy to modify the above argument to show that momentum conservation $\epsilon^T Su = 0$ does imply translation invariance for affine Lagrangian relations.

When $\epsilon = \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}$ then $\epsilon^T SU$ translates into the requirement that

$$U = \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \text{where } \vec{1}^T P = 0, \text{or}, \; \sum_i P_i = 0$$

In other words, affine Kirchhoff relations are Linear Kirchhoff relations, composed with a constant affine translation. The affine translation can involve an arbitrary constant shift in the positions of particles, but shifts in momenta must conserve total momentum.

In the language of electrical circuits, this tells us that two-terminal devices which increment the voltage by a constant (i.e., voltage sources) are allowed, but two-terminal devices that increment the current by a constant are not allowed in our category. Note that a device with increments the current is not what is conventionally defined as current source. Current sources are defined as devices that fix the input and output current to a specified value, and these are present in Affine (but not linear) Kirchhoff Relations.

A linear relation $\mathcal{L} : m \to n$ in LinRel_R is a **quasi-stochastic** whenever it relates $\vec{1}_m$ to $\vec{1}_n$. Similarly, a matrix $A : m \to n$ in Mat_R is **quasi-stochastic matrix** if $A\vec{1}_m = \vec{1}_n$. Thus, for a quasi-stochastic relation, the most disordered input quasi-probabilistic distribution must be related to the most disordered output distribution.

The following lemma deals with the structure of the parity-check matrix for a Kirchhoff relation:

Lemma 3.5.6. A Lagrangian relation is translation invariant that is a **Kirchhoff relation** if and only if every standard parity-check matrix has A quasi-stochastic and $Y\vec{1} = 0$.

Proof.

($\Rightarrow$) As $\epsilon_n \in \mathcal{R}$ one has $0 = H_{\mathcal{R}}\epsilon_n = \begin{pmatrix} Y & 0 & 1 & A^T \\ -A & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \vec{1}_n \\ 0 \end{pmatrix} = \begin{pmatrix} Y\vec{1} \\ -A\vec{1}_{n_q} + \vec{1}_{n_p} \end{pmatrix}$ So $A^T\vec{1}_{n_q} = \vec{1}_{n_p}$ (thus A^T is quasi-stochastic) and $Y\vec{1} = 0$.

($\Leftarrow$) Conversely, if A is quasi-stochastic and $Y\vec{1} = 0$ then $H_{\mathcal{R}}\epsilon_n = 0$ and $\epsilon_n \in \mathcal{R}$ showing translation invariance.

□

Kirchhoff relations form a prop, this is encapsulated in the following lemma:

Proposition 3.5.7. The category of all Kirchhoff relations forms a prop which is denoted as KirRel.

To complete the proof one needs to show that Kirchhoff relations are closed under tensor product and sequential composition, this is shown in the following lemma:

Lemma 3.5.8. Let $\mathcal{R}_1 : a \to b$ and let $\mathcal{R}_2 : c \to d$ be Lagrangian relations which satisfy KCL then:

(i) $\mathcal{R}_2 \oplus \mathcal{R}_1 : a + c \to b + d$ also satisfies KCL.

(ii) $\mathcal{R}_2 \circ \mathcal{R}_1 : a \to c$ satisfies KCL.

Proof.

(i): If $((q_a, p_a), (q_b, p_b)) \in \mathcal{R}_1$ and $((q_c, p_c), (q_d, p_d)) \in \mathcal{R}_2$ then $((q_a, q_c, p_a, p_c), (q_b, q_d, p_b, p_d)) \in \mathcal{R}_1 \oplus \mathcal{R}_2$. To show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ satisfies KCL one has:

$$\begin{pmatrix} p_a^T & p_c^T \end{pmatrix} \vec{1}_{a+c} = p_a^T\vec{1} + p_c^T\vec{1} = p_b^T\vec{1} + p_d^T\vec{1} = \begin{pmatrix} p_b^T & p_d^T \end{pmatrix} \vec{1}_{b+d}$$

where the middle step follows as the individual relations satisfy KCL.

(ii): Suppose $((q_a, p_a), (q_b, p_b)) \in \mathcal{R}_1$ and $((q_b, p_b), (q_c, p_c)) \in \mathcal{R}_2$ one needs to show that the composite $((q_a, p_a), (q_c, p_c)) \in \mathcal{R}_2 \circ \mathcal{R}_1$ obeys KCL. To show this first note that since $\mathcal{R}_1$ satisfies KCL one has $p_a^T \vec{1} = p_b^T \vec{1}$ and since $\mathcal{R}_2$ satisfies KCL one has $p_b^T \vec{1} = p_c^T \vec{1}$ so that $p_a^T \vec{1} = p_c^T \vec{1}$ which shows that $\mathcal{R}_2 \circ \mathcal{R}_1$ satisfies KCL.

□

This shows that KirRel_F forms a subprop of LagRel_F. Significantly, while Kirchhoff relations include resistor circuits, they also allow additional new components: namely ideal current dividers as shown in lemma 3.5.3.

3.6 Deterministic Relations

Consider Lagrangian relations which have a parity-check matrix determined by (Y, A, σ) such that A is not only quasi-stochastic but also is **deterministic**, in the sense that each row of A has only one non-zero entry (which must be 1). Of course, this is not a well-defined notion unless when one parity-check matrix has this form all will have this form:

Lemma 3.6.1. If a Lagrangian relation $\mathcal{R}$ has a standard parity-check matrix, (Y, A, σ), with A deterministic then every standard parity-check matrix of $\mathcal{R}$, (Y', A', σ') has A' deterministic.

Proof. Consider the submatrix $\begin{pmatrix} -A & 1_{n_q} \end{pmatrix}$ of the standard parity-check matrix which is deterministic. This matrix has the property that each row has exactly one 1 and one -1 to complete the proof one needs to show that this property is invariant under the action of any permutation σ which produces an alternative standard parity-check matrix. The action of a σ on this matrix maintains the property as σ only permutes the columns. However, one needs to show that when such a shuffled parity check-matrix is put back into standard form using Gaussian elimination that this deterministic property is still maintained. To show this consider the following cases of Gaussian elimination:

Permute rows: In this case the property is maintained since changing the rows does not change the entries in the rows at all.

Negating a row: This switches the -1 to a 1 and a 1 to a -1 and hence the property is still maintained.

Eliminating an entry: This replaces a row by adding another rows to it with 1, -1 in the same column. In this case P is maintained only if the off-elimination non-zero entries are in different columns. However, notice if these off-elimination values are in the same column the rows are linearly dependent but this is not the case since the matrix has full rank.

$\square$

This shows that Lagrangian relations which have a parity-check matrix (Y, A, σ), in which A is deterministic is a well-defined notion so one can make the following definition:

Definition 3.6.2. A Lagrangian relation is **deterministic** in case any or all of its standard parity-check matrices, determined by (Y, A, σ), has A deterministic.

Our objective, in this section is to explore these deterministic relations. To do so one starts with an alternative rather different description of them which allows us to develop their properties.

Definition 3.6.3. A Lagrangian relation $\mathcal{R} : n \to m$ is **position-partitioned** if there is an equivalence relation $\_ \sim \_$ on $\underline{n+m} = \{1, ..., n+m\}$ such that

- For every $\begin{pmatrix} q \\ p \end{pmatrix} \in \mathcal{R}$ if $i \sim j$ then $q_i = q_j$ (the relation is position-constant over equivalence classes);
- For each $i \in \underline{n+m}$, there is an $\begin{pmatrix} q \\ p \end{pmatrix} \in \mathcal{R}$ with $q_j = 1$ when $i \sim j$ and $q_j = 0$ when $j \not\sim i$ (thus, the equivalence classes are position-separated).

Observe that a Lagrangian relation cannot be position-partitioned in two different ways as, on the one hand, it must be position-constant over equivalence classes and, on the other hand, position-separated on non-equivalent components. One way to think about these position-partitioned relations is that they are capturing the voltage behaviour at the nodes of the electrical circuit. So if different wires are incident on the same node they have the same voltage and hence are in the same equivalence class, if this is not the case then the equivalence classes are different. This is shown in the picture below:

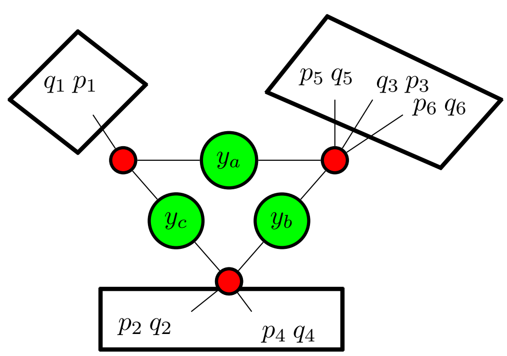


Figure 3.2: An example of a position-partitioned electrical network relation. The wires within the same black boxes belong to an equivalence class.

Our first observation on position-paritioned relations is:

Lemma 3.6.4. Position-partitioned relations form a subprop of LagRel

Proof. One must show these relations are closed to composition and to direct sums. For the latter, observe that when one takes a direct sum one takes the disjoint union of the entries: this means the partitions induced by the equivalence relation on each component induce a partition on the disjoint union. This is clearly position-constant on partitions and can be separated by pairing the separating elements in each component with a zero in the other component.

For the composite $\mathcal{R}_2 \circ \mathcal{R}_1$ of two position-partitioned relations consider the composition of jointly epic cospans induced by the equivalence relations. Two entries are related, $i_1 \sim i_n$ if and only if there is a sequence $i_i \sim_{\delta_1} \ldots \sim_{\delta_{n-1}} i_n$ where $\delta_k = 1$ or 2. If $\sim_1$ and $\sim_2$ are position-constant then for each $\begin{pmatrix} q \\ p \end{pmatrix} \in \mathcal{R}_2 \circ \mathcal{R}_1$ one has $q_j = q_{j+1}$ so $q_1 = q_n$ and $\mathcal{R}_2 \circ \mathcal{R}_1$ is position-constant over $\_ \sim \_$.

Separation is given by taking the sum of the separators for all the partitions amalgamated by the composite relation. $\square$

Using this definition one can prove the following lemma:

Lemma 3.6.5. A Lagrangian relation $\mathcal{R}$ is deterministic if and only if it is position-partitioned.

Proof.

($\Rightarrow$) If one parity check matrix has A deterministic then every parity check matrix has A is deterministic.

Consider H to be a parity check matrix for a deterministic Kirchhoff relation recall:

$$\begin{pmatrix} Y & 0 & 1 & A^T \\ -A & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} Yq_0 + p_0 + A^T p \\ -Aq_0 + q_1 \end{pmatrix} = 0$$

which happens if and only if $q_1 = Aq_0$ and $p_0 = -Yq_0 - A^T p_1$. But this means q_0 (and p_1) can be freely chosen.

As A is deterministic, there is a surjective function $f : n_{q_1} \to n_{q_0}$ where $f : n_q \to n_{q_0}$ where $n_{q_1} + n_{q_0} = n$ such that $q_i = q_{f(i)}$ this defines an equivalence relation defined by $i \sim f(i)$ and $i \sim i'$ when $f(i) = f(i')$ for which the relation is position-constant. The equation $Aq_0 = q_1$ allows for separation of the equivalence, as one can then choose p_1 freely to determine p_0. Thus, one has position-separation as required.

($\Leftarrow$) Assuming the relation is position-partitoned, and that one has a parity-check matrix for the relation $H = \begin{pmatrix} Y & 0 & 1 & A^T \\ -A & 1 & 0 & 0 \end{pmatrix}$. Then, by assumption exists an equivalence relation $\_ \sim \_$ (on n), for which $\mathcal{R}$ is position-constant and separating. Notice that q_0 can be arbitrary but this means each component of q_0^i is in a separate equivalence class. The characteristic relation of the equivalence class determined by q_0^i provides a column in A. The fact that one has an equivalence relation now guarantees each row has at most one 1 and so A is deterministic.

$\square$

In Lagrangian relations, whenever one has a position oriented notion, one expects a corresponding momentum oriented notion. This is described for states as follows:

Definition 3.6.6. A Lagrangian state $\mathcal{R} : 0 \to n$ is **momentum-grouped** if there is an equivalence relation, $\_ \sim \_$, on $\underline{n}$ such that:

- For every $S \subseteq \underline{n}$, which is $\sim$-closed (that is if $i \in S$ and $i \sim j$ then $j \in S$), every $\begin{pmatrix} 0 \\ p \end{pmatrix} \in \mathcal{R}$ has $\sum_{i \in S} p_i = 0$ (i.e. has zero average momentum at position zero on every set closed to the equivalence);
- For each $S \subseteq \underline{n}$ for which there are $i, j \in \underline{n+m}$ with $i \sim j$ and $i \in S$ and $j \notin S$ there is a $\begin{pmatrix} 0 \\ p \end{pmatrix} \in \mathcal{R}$ such that $\sum_{j \in S} p_j \neq 0$ (i.e. has non-zero average momentum at position zero on every set which is not closed to the equivalence).

Lemma 3.6.7. A Langrangian state is position-partitioned if and only if it is momentum-grouped.

Proof. It suffices show that for any parity-check matrix, determined by (Y, A, σ), of a relation $\mathcal{R}$ which is momentum grouped has A is deterministic. The parity-check matrix implies the equation $Yq_0 + p_0 + A^T p_1 = 0$. However, one is interested in the case when $q_0 = 0$ this gives the equation $p_0 = -A^T p_1$. Asking for A to be deterministic is precisely to demand that $\mathcal{R}$ is momentum grouped. Note that p_1 can be arbitrarily chosen, when A is deterministic composition with A^T causes equivalent entries of q_1 to be summed to determine the entry of p_0 (which is an entry in the same equivalence class). Notice that all the entries of q_0 are, therefore, in distinct equivalence classes. This guarantees the first condition of being momentum grouped. The second condition is guaranteed as one may choose q_1 freely. $\square$

This shows that the deterministic Kirchhoff relations form a prop which will be denote by ResRel_F. This will be identified in the next chapter as the prop studied in [20, 21, 53] of resistor circuits. Using the above properties of deterministic relations and matrices one can make the following remark about the structure of parity-check matrices for deterministic Kirchhoff relations:

Corollary 3.6.8. A deterministic Kirchhoff relation over F, can be described by a parity-check matrix in standard Lagrangian form, equation (3.7), where Y and A must also satisfy the additional constraints:

$$Y\vec{1}_{k_p} = 0, \quad A\vec{1}_{k_p} = \vec{1}_{k_q}. \tag{3.8}$$

and in addition, A is deterministic.

3.7 Graph States

Although standard forms for both linear relations and Lagrangian relations were defined, these standard forms are not unique. There is a special class of Kirchhoff relations, for which one can define a canonical form, and this will be useful in demonstrating universality.

Definition 3.7.1. Any Kirchhoff relation $\mathcal{R}$ with $k_p = k$ (i.e "no extra wires") is known as a graph state.

A graph state is specified by a parity-check matrix of the following form:

$$H = \begin{pmatrix} Y & 1_{k\times k} \end{pmatrix} \sigma_S$$

The name "graph state" is given since this coincides with the notion of a quantum graph state. The following theorem will show that the σ_S can be removed making the parity-check matrix form in the case of graph states unique:

Theorem 3.7.2. A graph state is uniquely specified by a parity-check matrix of the following canonical form:

$$H = \begin{pmatrix} Y & 1_{k\times k} \end{pmatrix}$$

where the matrix Y satisfies: $Y = Y^T$ and $Y\vec{1} = 0$.

Proof. First, observe that any symplectic permutation leaves $\begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}$ invariant:

$$\sigma_S \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}$$

Let us apply the constraint

$$H \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix} = 0,$$

to a parity-check matrix H that is in standard form for a Lagrangian relation. (This is a necessary and sufficient condition for the relation defined by H to be Kirchhoff.) By assumption, $k_q = 0$. Then $Y\vec{1} = 0$, and so one has:

$$H = \begin{pmatrix} Y & 1_{k\times k} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Note that one can choose $\sigma = 1_{k\times k}$. Absorbing σ_S into the normal form for a Lagrangian relation one has the following has:

$$H = \begin{pmatrix} Y\sigma & \sigma. \end{pmatrix}$$

By row operations alone, one can convert σ into $1_{k\times k}$. This is equivalent to left-multiplying by $\sigma^{-1} = \sigma^T$. The final expression is:

$$H = \begin{pmatrix} \sigma^T Y\sigma & 1_{k\times k} \end{pmatrix}.$$

Note that $\tilde{Y} = \sigma^T Y \sigma$ satisfies the same conditions as Y. These conditions allow us to interpret the off-diagonal parts of Y and $\tilde{Y}$ as adjacency matrices of weighted graphs, with weights in F. (The diagonal parts are determined by $Y\vec{1} = 0$.) The weighted graphs described by Y and $\tilde{Y}$ differ only by a relabeling of vertices. Hence the graph state is uniquely determined by the weighted graph whose adjacency matrix is the off-diagonal part of Y. □

Using that standard form for parity-check matrices in the equation $Hu = 0$, one has:

$$p = -Yq$$

Therefore note that any graph relation defines a function from the space of positions to momenta. Since it is a well known fact that canonical forms exist for linear functions, this gives another explanation of Theorem 3.7.2.

3.8 Power input

Let us define the concept of the **power input** to a Lagrangian relation. Power input is usually used to distinguish between passive and active components in electrical networks. The power input, [20, 21, 53], of an electrical circuit is determined by the product of voltage (difference) times the current. Similarly the product of momentum and velocity determines the energy of a physical system. In the generalization to finite fields, the distinction between passive and active can no longer be made. However, one can restrict to the *lossless* case when the power input is zero. This give a series of subprops given by intersecting with lossless Lagrangian relations.

$$\begin{tikzcd}
\mathsf{LosLagRel}_F \arrow[r, hook] & \mathsf{LagRel}_F \\
\mathsf{LosKirRel}_F \arrow[r, hook] \arrow[u, hook] & \mathsf{KirRel}_F \arrow[u, hook] \\
\mathsf{Spiders} \arrow[r, hook] \arrow[u, hook] & \mathsf{ResRel}_F \arrow[u, hook]
\end{tikzcd}$$

Power input is formally defined as:

Definition 3.8.1. Given any Lagrangian relation, $\mathcal{R} : m \to n$,the associated **power input** is a function

$$P_{\mathcal{R}} : \mathcal{R} \subseteq (F^m)^2 \otimes (F^n)^2 \to F; ((q,p),(q',p')) \mapsto \sum_{j=1}^{n} q_j p_j - \sum_{k=1}^{m} q'_k p'_k.$$

A relation, $\mathcal{R}$ is said to be **lossless** in case $P_{\mathcal{R}}$ is everywhere zero.

Spiders, symplectic permutations, and (ideal) current dividers are all lossless. Notice, using spiders one can convert any relation to a state; because the cup changes the sign of output positions, a relation in LagRel has the same power input function as its corresponding state.

The physical motivation behind this definition is most clear if q is replaced by a voltage V or a velocity $\dot{q}$, and p is replaced by a current I, or a force, $\dot{p}$, in which case the above quantity has the dimensions of power. An example of a power calculation of a spider with 2 inputs and 2 outputs is shown below:

Example 3.8.2. The power input of any spider is identically zero.

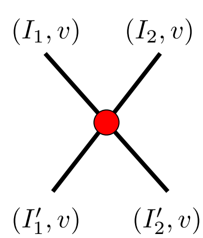


Figure 3.3: Spider

The above relation can be encoded:

$$\left\{ \begin{pmatrix} I_1 \\ I_2 \\ I_1' \\ I_2' \\ v \\ v \\ v \\ v \end{pmatrix} \mid I_1 + I_2 = I_1' + I_2' \right\}$$

The expression for power for the above spider is as follows:

$$P = (I_1 + I_2)v - (I_1' + I_2')v$$

One can simplify to show $P = 0$ as $I_1 + I_2 = I_1' + I_2'$.

Notice also that $P_{\mathcal{R}}(\alpha q, \alpha p) = \alpha^2 P_{\mathcal{R}}(q, p)$. Using this concept of power input one can define the following lossless relations:

Definition 3.8.3. The prop of **lossless Lagrangian relations** over a field F, denoted as $\mathsf{LosLagRel}_F$, consists of those Lagrangian relations for which the power input is exactly zero.

Definition 3.8.4. The prop of **lossless Kirchhoff relations** over a field F, denoted as $\mathsf{LosKirRel}_F$, consists of those Kirchhoff relations for which the power input is exactly zero.

An explicit expression for the power input for the state $\mathcal{R}$ can be calculated as follows: Consider the standard form for a parity-check matrix for a Lagrangian state:

$$\begin{pmatrix} Y & 0 & I & A^T \\ -A & I & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ p_0 \\ p_1 \end{pmatrix} = 0$$

This gives the following relations:

$$q_1 = Aq_0 \quad \text{and} \quad p_0 = -Yq_0 - A^T p_1 \tag{3.9}$$

The expression for power input using the above relations is then:

$$q_0^T p_0 + q_1^T p_1 = -q_0^T (Yq_0 + A^T p_1) + (Aq_0)^T p_1 = -q_0^T Y q_0$$

For $\mathcal{R}$ to be lossless $q_0^T Y q_0$ must vanish. To derive the conditions for $\mathcal{R}$ to be lossless one has the following lemma:

Lemma 3.8.5. In any field F with $char(F) \neq 2$ when $Y = Y^T$ then $q^T Y q = 0 \; \forall \; q$ if and only if $Y = 0$.

Proof.

($\Rightarrow$) Use the standard basis e_i then $0 = e_i^T Y e_i = Y_{i,i}$ this shows that the diagonal is all zeros. Now consider:

$$0 = (e_i + e_j)^T Y (e_i + e_j) = e_i^T Y e_i + e_i^T Y e_j + e_j^T Y e_i + e_j^T Y e_j$$

This is further equal to the following equation because the diagonal elements are zero $Y_{ii} = 0$

$$0 = Y_{i,j} + Y_{j,i} = 2Y_{i,j}$$

as $Y = Y^T$. The fields of interest in our case are those of characteristic odd prime and hence division by 2 is well defined which implies that $Y_{i,j} = 0$

($\Leftarrow$) For the other direction If $Y = 0$ this implies that $q^T Y q = 0$, for all q this completes the proof.

□

This implies that if $\mathcal{R}$ is lossless then Y must be equal to zero, this gives us the following lemma:

Corollary 3.8.6. Any state in $\mathsf{LosLagRel}$ over F, can be described by a parity-check matrix in standard Lagrangian form with $Y = 0$.

Thus lossless Lagrangian relations are specified by

$$p_0 = -A^T p_1 \qquad q_1 = Aq_0 \tag{3.10}$$

for some matrix A. This gives:

Lemma 3.8.7. The category of Lossless Lagrangian Relations is isomorphic to the subcategory of linear relations determined by $L : \mathsf{LinRel}_F \to \mathsf{LagRel}_F$.

Proof. Consider the triangle shown below:

$$\begin{tikzcd}
\mathsf{LosLagRel} \arrow[r, "\subset"] \arrow[dr, shift left, "(2)"] & \mathsf{LagRel} \\
& \mathsf{LinRel} \arrow[u, "L"'] \arrow[ul, shift left, "(1)"]
\end{tikzcd}$$

Recall that a functor $L : \mathsf{LinRel} \to \mathsf{LagRel}$ is faithful and bijective on objects. For direction (1) one needs to show that $L(\mathcal{L})$ is lossless. To show this note that because of (3.4):

$$\mathcal{L} = \{(q, q') \mid q = Aq'\}$$

Now by applying the functor L to this relation one gets the following:

$$L(\mathcal{L}) = \{(q, p)\ (q', p') \mid q' = Aq,\ p = A^T p'\}$$

but this is lossless as:

$$q^T p - q'^T p'^T = q^T A^T p' - (Aq)^T p' = q^T A^T p' - q^T A^T p' = 0$$

For direction (2) one projects onto the q coordinate. This gives the condition $q' = Aq$ which is the required condition on q as shown in Equation 3.10 for the relation to be lossless. □

Turning now to $\mathsf{LosKirRel}$, the category of lossless Kirchhoff relations one has the following remark has the following characterization in terms of parity-check matrices:

Corollary 3.8.8. Any state in $\mathsf{LosKirRel}$ can be described by a parity-check matrix in standard Lagrangian form (3.7) such that $Y = 0$ and $A\vec{1}_{n_p} = \vec{1}_{n_q}$.

Corollary 3.8.9. A lossless deterministic Kirchhoff relation is a spider.

Proof. If the relation is deterministic Kirchhoff this constrains A to be deterministic and hence the relation contains no current dividers, furthermore the relation is lossless, this implies that no resistors are allowed. This only leaves $\mathsf{Spiders}$ as elements of $\mathsf{LosKirRel}$ if it is deterministic, thereby completeing the proof. □

Remark 3.8.10. From this result, it follows that GLA forms a universal, sound and complete graphical calculus for $\mathsf{LosLagRel}$. This is a corresponds to a fragment of the graphical calculus for lagrangian relations, generated by L(copy spiders), L(addition spiders), and L(multiply by k), with the same equations as those of GLA.

Remark 3.8.11. The definition of power input makes sense for an arbitrary Lagrangian relation. So one can also define the categories of passive Lagrangian relations over $\mathbb{R}$, denoted as $\mathsf{PasLagRel}$ which is defined as those Lagrangian relations $\mathcal{R}$ for which the power input $P_{\mathcal{R}}(q,p) \geq 0$ for $(q,p) \in \mathcal{R}$. Similarly one can define the prop of **passive Kirchhoff relations** over $\mathbb{R}$, denoted as $\mathsf{PasKirRel(R)}$, are those relations $\mathcal{R}$ for which the power input $P_{\mathcal{R}}(q,p) \geq 0$ for $(q,p) \in \mathcal{R}$.

Physically, these correspond to elements that dissipate or conserve, but do not generate energy.

Remark 3.8.12. Resistors with negative resistances are not present in the prop of passive Kirchhoff relations over $\mathbb{R}$, but they are present in Kirchhoff relations over $\mathbb{R}$.

Chapter 4

Universality for Kirchhoff Relations

In this chapter, we give universal sets of generators for several of the subcategories discussed in the previous chapter, these universality proofs provide the interpretation of different subcategories of Kirchhoff relations as categories of electrical networks. The generators for Kirchhoff relations are built up in a series of steps. First the "spiders" which give the basic hyper-graphical structure are discussed, to these resistors are added, and then it is shown that graph states can be obtained by a mesh of resistors. Finally current dividers are added to give a set of generators for an arbitrary map in KirRel. Also considered are the generators of $\mathsf{LosKirRel}$ and ResRel.

4.1 A Universal Set of Generators for $\mathsf{Spiders}$

The generators for the category of spiders Spider are shown in Table 4.1. These are obtained by applying the functor $L : \mathsf{LinRel}_{\mathsf{F}} \to \mathsf{LagRel}_{\mathsf{F}}$ to the generators of LinRel associated with the copy spider.

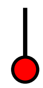
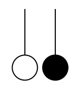
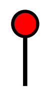
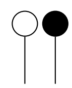
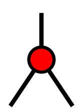
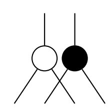
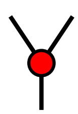
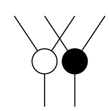
Name	Generator	Definition in terms of GAA	Relation: $p_i,\ q_i \in F$ s.t.
Unit		$p_1\ q_1$ 	$p_1 = 0$
Counit		 $p_1\ q_1$	$p_1 = 0$
Monoid		$p_3\ q_3$  $p_1\ q_1\ p_2\ q_2$	$p_3 = p_1 + p_2$ $q_3 = q_1 = q_2$
Comonoid		$p_3\ q_3\ p_3\ q_3$  $p_1\ q_1$	$p_1 = p_2 + p_3$ $q_3 = q_1 = q_2$

Table 4.1: These are a universal set of generators for Spiders.

Note the category of Lagrangian relations contains two inequivalent spiders, L(copy spider) and L(addition spider). Only one of these spiders is in the category of Kirchhoff relations, L(copy spider) – which copies positions and adds momenta. We represent it by a red spider as shown in Table 4.1. That these generators are universal follows immediately from the fact that L is a functor and copy spiders form a universal set of generators for the sub-prop copy of LinRel where the copy is the category generated by the copy spiders in LinRel_F. One can use these spiders to define cups and caps for the category of Kirchhoff relations, this allows relation in Kirchhoff relations to be converted into a state (or an effect). Borrowing terminology from electrical engineering, we sometimes refer to any relation $\mathcal{R} : F^{2n} \to F^{2m}$ as an $(n+m)$-terminal device.

4.2 A Universal Set of Generators for KirRel

For Kirchhoff relations, we supplemented the generators in Tables 4.1 with resistors shown in Table 4.2, and ideal current dividers as shown in Table 4.3.

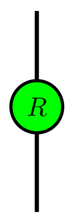
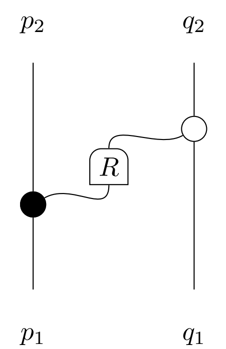
Name	Generator	Definition in terms of GAA	Relation: $p_i,\ q_i \in F$ s.t.
Resistor			$p_1 = p_2$ $p_1 R = (q_2 - q_1)$

Table 4.2: **Resistor**: In addition to the generators for Spiders, we must include *resistors*, defined above, to obtain a universal set of generators for ResRel.

A resistor defines the following relationship between current and voltage:

$$V_2 - V_1 = I_1 R,\ I_1 = I_2$$

This is sometimes known as the Ohm's law.

The power input of a resistor is given by:

$$P = I^2 R$$

A positive resistor (in the case $F = \mathbb{R}$) is usually considered to be a power dissipating (passive) electrical element, but if the resistor takes on negative values then it is an active element. Sometimes its more convenient to use **conductance** instead of resistance the conductance is just the inverse of resistance, that

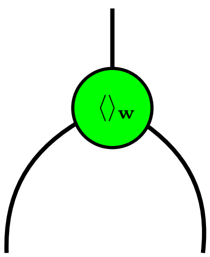
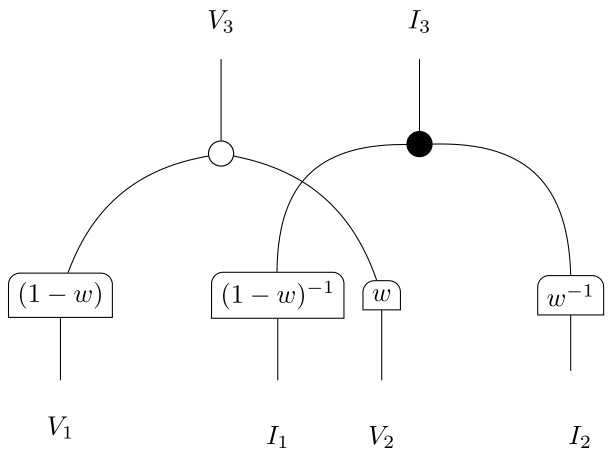
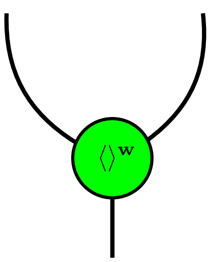
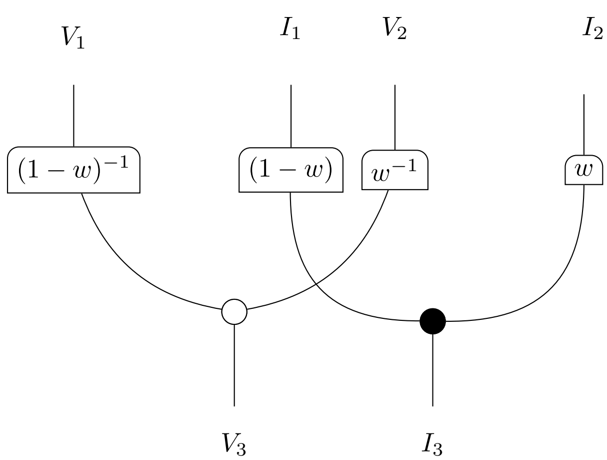
<table>
<tr><th>Generator</th><th>Definition in terms of GAA generators</th><th>Interpretation as a Kirchhoff Relation</th></tr>
<tr><td></td><td></td><td>$V_3 = V_1(1-w) + V_2 w$
$I_3 = (1-w)^{-1} I_1 = w^{-1} I_2$</td></tr>
<tr><td></td><td></td><td>$V_1(1-w)^{-1} + V_2 w^{-1} = V_3$
$I_3 = I_1(1-w) = I_2 w$</td></tr>
</table>

Table 4.3: **Ideal Current Dividers** A universal set of generators for LosKirRel require, in addition to the spiders in Table 4.1, the above "ideal current divider" as defined in this table.

is:

$$y = \frac{1}{R}.$$

Note that when the resistance across a wire is 0 it is said to be "short circuited", in that case the conductance is not defined, although we can informally regard it as $y = \infty$. In the case when conductance $y = 0$ then the value of resistance is undefined, although we can be informally regard it as $R = \infty$, and the relation is a disconnection. In electrical engineering this is said to be an "open circuit": $I_1 = I_2 = 0$, with V_1 and V_2 arbitrary.

Ideal Current Dividers

While the resistors and junctions of ideal wires are familiar from electrical engineering, the ideal current divider may seem unfamiliar. Let us discuss the physical interpretation of this generator.

One can view an ideal current divider as a limiting case of the following relation in Kirchhoff Relations over $\mathbb{R}$. Consider a three-terminal device, constructed from three resistors, $R_{12} = -\frac{1}{w}R$, $R_{13} = R$, and $R_{23} = \frac{1-w}{w}R$. Then, in the limit $R \to 0$, one can check that this describes the relation:

$$\begin{aligned} I_1 &= wI_3 \\ I_2 &= (1-w)I_3 \\ V_3 &= wV_1 + (1-w)V_2 \end{aligned}$$

which is that of a ideal current divider. Thus in practice current dividers are constructed using resistors, note at least one of these resistances must be negative, so in practice, a ideal divider requires active components to be constructed; although the power input can be made arbitrarily close to zero.

By virtue of the above construction, we could choose not to regard the ideal current divider as an independent generator for the theory of Kirchhoff relations over $\mathbb{R}$. However, for Kirchhoff relations over an arbitrary, possibly-finite field, the above limiting procedure does not make sense, so we have no choice but to consider the ideal current divider as an independent generator.

4.3 A Universal Set of Generators for LosKirRel

The category of LosKirRel generators include the generators of Spiders in Table 4.1, supplemented with *ideal current dividers* as shown in Table 4.3.

4.4 A universal set of generators for AffKirRel

To obtain the category of Affine Kirchhoff relations, we need two additional generators, shown in Table 4.4.

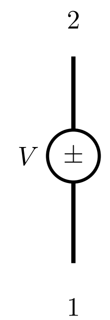
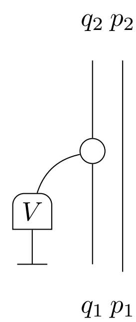
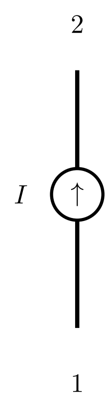
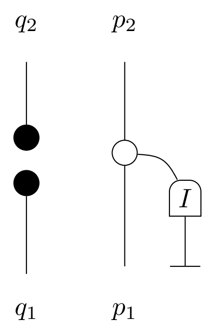
Name	Generator	Definition in terms of GAA	Relation: $p_i,\ q_i \in F$ s.t.
Voltage Source:			$q_2 = q_1 + V$ $p_2 = p_1$
Current Source:			$p_2 = I + p_1$

Table 4.4: **Affine Kirchhoff Relations**: In addition to the generators of KirRel we must add the voltage and current source generators to obtain a universal set of generators for AffKirRel

The Tables 4.1, 4.2, 4.3 and 4.4 give the generators for affine Kirchhoff relations AffKirRel. This is discussed briefly in section 4.7.

Voltage Sources and Current Sources

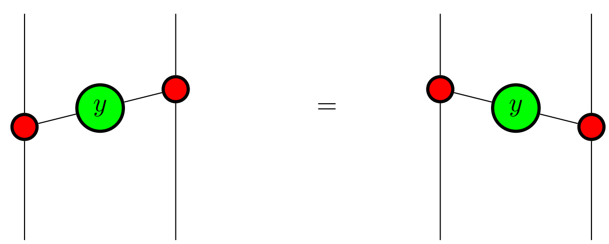
The current source "forces" the current to take on a certain value, while the voltage source, forces a specific voltage difference across two terminals.

Any affine shift in position and momenta that obeys KCL can clearly be written in terms of these generators as shown in section 4.7. Therefore, if one has a universal set of generators for KirRel, then one can obtain any relation in AffKirRel, by precomposing or post-composing with a direct sum of current sources, and a direct sum of voltage sources.

4.5 Universality for Graph States

We now argue that the resistor and the junction, allow one to obtain any graph state as defined in section 3.7. Observe that a resistor is not "directional", in the sense that the diagrams of lemma 4.5.1 are equal and so can be thought of as a weighted connection between two wires: we refer to the component as being "horizontal". The following lemma shows that a resistor is horizontal in KirRel and discusses the matrix representation:

Lemma 4.5.1.



Proof. The proof is by direct calculation, as shown below the constraints of the horizontal resistor are given as follows:

$$I_1 + yV_1 - yV_2 - I_3 = 0 \quad -yV_1 + I_2 + yV_2 = I_4$$

$$V_1 = V_3 \quad V_2 = V_4$$

This matrix can be written functionally as shown below, note that this relation is actually an isomorphism $M(y) : (F^2)^2 \to (F^2)^2$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ y & -y & 1 & 0 \\ -y & y & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_1 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_3 \\ V_4 \\ I_3 \\ I_4 \end{pmatrix}$$

□

The concept of a "mesh of resistors" is defined as follows:

Definition 4.5.2. A **mesh of resistors** is defined to be a weighted non-reflexive undirected graph where the weights are given by the values of the conductance $y \in F$ between pair of nodes. Each node is connected by convention to a unique output. This makes a mesh a state.

An example for a mesh with 3 conductance is given below:

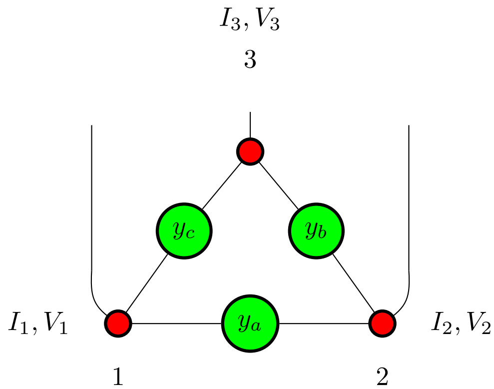


Figure 4.1: A mesh with 3 resistors.

Theorem 4.5.3. A graph state with the following parity-check matrix:

$$\begin{pmatrix} Y & I \end{pmatrix} \begin{pmatrix} V_{in} \\ I_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

can be realized by a mesh of resistors.

Proof. To prove this theorem consider a mesh of n resistors. For each layer in the mesh we define a function $C_{ij}(y) : F^{2n} \to F^{2n}$, and any graph state can be then drawn as a composition of resistors. For the case of 3 resistors this is shown below:

To define $C_{ij}(y)$ we first define the matrix $Y_{ij}(y)$ to be an $n \times n$ matrix whose entries are all zeroes except:

$$(Y_{ij}(y))_{ii} = y \quad (Y_{ij}(y))_{ij} = -y = (Y_{ij}(y))_{ji} \quad (Y_{ij}(y))_{jj} = y$$

Using this structure one can write down $C_{ij}(y)$ as follows:

$$C_{ij}(y) = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y) & 1 \end{pmatrix}$$

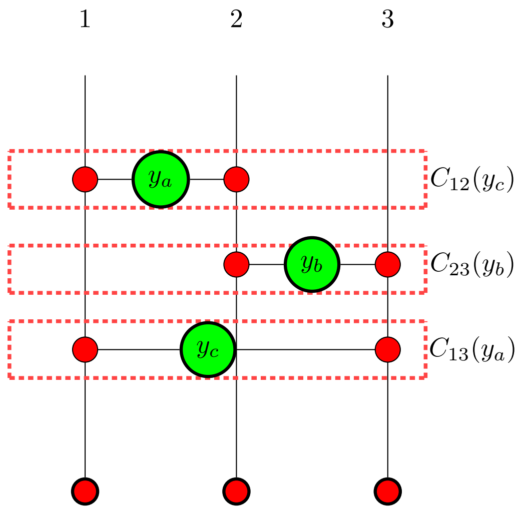


Figure 4.2: A graph state expressed as a composition of resistors

The properties of $C_{ij}(y)$ are captured in the following lemma:

Lemma 4.5.4. The matrices $C_{ij}(y)$ satisfy the following properties:

(i) $C_{ij}(y_1)C_{kl}(y_2) = C_{kl}(y_2)C_{ij}(y_1)$, these matrices commute.

(ii) $Y_{ij}(y)$ is a symmetric matrix and $Y_{ij}^T(y)\vec{1} = 0$, that is the rows sum to zero

(iii) $\prod_{i\neq j} C_{ij}(y) = \begin{pmatrix} 1 & 0 \\ \sum_{i\neq j} Y_{ij}(y) & 1 \end{pmatrix}$

(iv) $(\sum_{i\neq j} Y_{ij}(y))$ is symmetric and $(\sum_{i\neq j} Y_{ij}(y))^T\vec{1} = 0$

Proof.

(i) To show that the C's commutes one first writes the matrix structure explicitly as follows:

$$C_{ij}(y_1) = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y_1) & 1 \end{pmatrix} \qquad C_{kl}(y_2) = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y_2) & 1 \end{pmatrix}$$

The product $C_{ij}(y_1)C_{kl}(y_2)$ is:

$$C_{ij}(y_1)C_{kl}(y_2) = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y_1) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y_{ij}(y_2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y_1) + Y_{kl}(y_2) & 1 \end{pmatrix} = C_{kl}(y_2)C_{ij}(y_1)$$

which completes the proof.

(ii) Recall, the structure of $Y_{ij}(y)$ is:

$$(Y_{ij})(y))_{ii} = y \quad (Y_{ij}(y))_{ij} = -y = (Y_{ij}(y))_{ji} \quad (Y_{ij}(y))_{jj} = y$$

and therefore its symmetric. $Y_{ij}^T(y)\vec{1}$ sums the entries in each row and hence is always equal to zero given the structure defined above.

(iii) One can show this by direct calculation, this essentially relies on the fact that when lower triangular matrices of the following form are multiplied, the entries in the lower block add so we get the following result:

$$\prod_{i\neq j} C_{ij}(y_{ij}) = \begin{pmatrix} 1 & 0 \\ \sum_{i\neq j} Y_{ij}(y_{ij}) & 1 \end{pmatrix}$$

(iv) $\sum_{i\neq j} Y_{ij}(y_{ij})$ is a symmetric matrix since its a sum of symmetric matrices. To show this $(\sum_{i\neq j} Y_{ij}(y))^T\vec{1} = 0$, note that $(\sum_{i\neq j} Y_{ij}(y)))^T\vec{1} = \sum_{i\neq j} Y_{ij}^T(y)\vec{1} = 0$, where the last equality follows from (ii).

□

The resistors for each layer are composed by multiplying the matrices C_{ij}, as shown in the lemma above $C_{ij}(y)$'s commute with each other and the product of $C_{ij}(y)$'s can then be written:

$$\prod_{i\neq j} C_{ij}(y_{ij}) = \begin{pmatrix} 1 & 0 \\ \sum_{i\neq j} Y_{ij}(y_{ij}) & 1 \end{pmatrix}$$

where $Y = \sum_{i\neq j} Y_{ij}(y_{ij})$ is a symmetric matrix, whose off-diagonal components are the adjacency matrix of a graph with weights defined by (minus) of the value of the conductance between wires i and j, $Y_{ij} = -y_{ij}$, and whose diagonal components are defined via $Y_{ii} = -\sum_{j\neq i} Y_{ij}(y_{ij})$. The function encoded the composition of the horizontal resistances, then has the following form:

$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} V_{in} \\ I_{in} \end{pmatrix} = \begin{pmatrix} V_{out} \\ I_{out} \end{pmatrix}$$

The last layer consists of the effect, to V_{out} and I_{out}. The resulting relation allows each component of V_{out} can be anything although each component of I_{out} must be zero. The parity-check matrix can then be written in the following form:

$$\begin{pmatrix} Y & 1 \end{pmatrix} \begin{pmatrix} V_{in} \\ I_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice that every graph state can be obtained from a mesh of resistors by choosing appropriate conductance values and hence this completes the proof. □

To give an intuitive understanding of the steps used in the proof above, consider the following example:

Example 4.5.5. Consider a weighted graph, with weights in F. We convert this into a relation by associating each vertex of the graph with a spider, and each edge of the graph with a resistor, with non-zero resistance. The conductance associated with each resistance is the weight of the edge y_{ij}, which could be zero. An example, for a graph 3 vertices is shown in Figure 4.3.

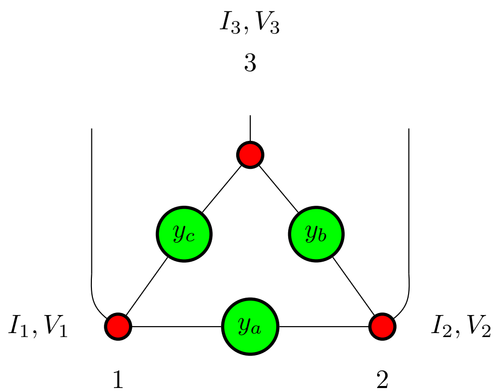


Figure 4.3: Graph State with 3 resistors.

This graph state, can be redrawn as a sequence of isomorphisms, each consisting of a single horizontal resistor acting on each pair of wires. For the example the Figure 4.3 can be redrawn as Figure 4.4.

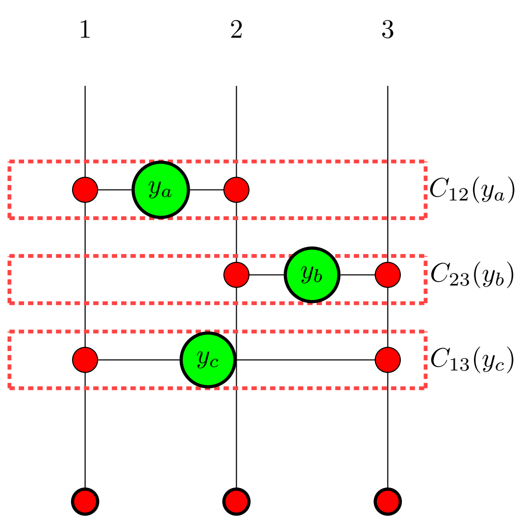


Figure 4.4: The diagram of Figure 4.3 expressed as the composition of a series of resistors, followed by a direct sum of units.

Each layer, highlighted in a red box in the example in Figure 4.4, defines a function $C_{ij}(y) : F^{2n} \to F^{2n}$:

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = C_{ij}(y) \begin{pmatrix} V \\ I \end{pmatrix}.$$

where:

$$C_{ij}(y) = \begin{pmatrix} 1 & 0 \\ Y_{ij}(y) & 1 \end{pmatrix}.$$

Explicitly, for the example in Figure 4.3, $n = 3$ and

$$C_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ y_a & -y_a & 0 & 1 & 0 & 0 \\ -y_a & y_a & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & y_b & -y_b & 0 & 1 & 0 \\ 0 & -y_b & y_b & 0 & 0 & 1 \end{pmatrix},$$

and,

$$C_{31} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ y_c & 0 & -y_c & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -y_c & 0 & y_c & 0 & 0 & 1 \end{pmatrix}.$$

The product of the C_{ij}, can be written as

$$C_{12}C_{23}C_{31} = \begin{pmatrix} 1 & 0 \\ Y_{12}(y_A) + Y_{23}(y_B) + Y_{31}(y_C) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ y_a + y_c & -y_a & -y_c & 1 & 0 & 0 \\ -y_a & y_a + y_b & -y_b & 0 & 1 & 0 \\ -y_c & -y_b & y_b + y_c & 0 & 0 & 1 \end{pmatrix}$$

The above matrix can be written as a block matrix in the following way:

$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} V_{in} \\ I_{in} \end{pmatrix} = \begin{pmatrix} V_{out} \\ I_{out} \end{pmatrix}$$

We now apply the unit to V_{out} and I_{out}, as shown in the last layer of Figure 4.4, this results in the following parity-check matrix

$$\begin{pmatrix} Y & 1 \end{pmatrix} \begin{pmatrix} V_{in} \\ I_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

4.6 Universality for Kirchhoff Relations

In this section we wish to show that *any* Kirchhoff relation can be specified as a composition or direct sum of the generators described in Tables 4.1, 4.2, 4.3 and 4.4. Since any relation can be converted to a state using spiders, it suffices to show that any state can be expressed in terms of the generators listed above. Furthermore, the parity-check matrix of any Kirchhoff state can be written in standard form, so, to prove universality, we only need to provide an explicit construction of this standard form using our generators.

Theorem 4.6.1. Kirchhoff spiders, resistances and ideal current dividers form a universal set of generators for KirRel.

Proof. We first note that any Kirchhoff relation can be converted into a state using spiders. We showed that the parity-check matrix for any state can be put into the standard form described in the previous chapter. This depends on two matrices, the matrix Y which satisfies $Y = Y^T$ and $Y\vec{1} = \vec{1}$, and the matrix A which is quasi-stochastic matrix. Our aim is to show one realizes a state with a parity-check matrix in this standard form using spiders, resistances and ideal current dividers. Let us choose M to be the following matrix:

$$M = \begin{pmatrix} 1 \\ A \end{pmatrix}$$

where A is the quasti-stochastic matrix appearing in the standard form of the state. Then M is clearly also quasi-stochastic. The relation $L(M)$ can be realized using a layer of current-dividers composed with a layer of spiders.

Consider the composition of a graph state with $L(M)$ as shown in Figure 4.5. The relation encoded by

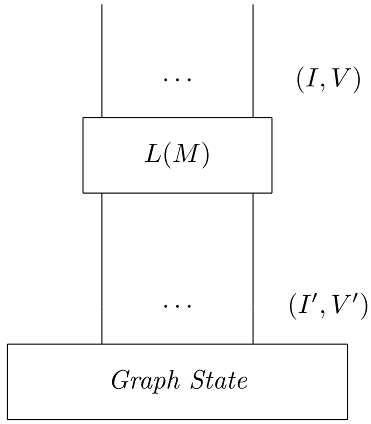


Figure 4.5: Any state in KirRel can be realized as a graph state, composed with a relation of the form $L(M)$ where M is an appropriately chosen quasi-stochastic matrix.

the graph state is:

$$\mathcal{W} = \{(V', I') \mid \begin{pmatrix} Y & 1 \end{pmatrix} \begin{pmatrix} V' \\ I' \end{pmatrix} = 0\} = \{(V', I') \mid YV' = -I'\}$$

The relation $L(M)$ is the following:

$$L(M) = \{((V, V')\ (I, I')) \mid V = MV' \textit{ and } I' = M^T I\}$$

We compose these relations as follows:

$$\begin{aligned} L(M)(1 \oplus \mathcal{W}) &= \{(V, I) \mid \exists V',\ I' \ .\ V = MV',\ I' = M^T I,\ YV' = -I'\} \\ &= \{\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \mid \exists\ V',\ I' \ .\ V_1 = V',\ V_2 = AV',\ I' = I_1 + A^T I_2,\ YV' = -I'\} \\ &= \{\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} | V_2 = AV_1,\ YV' = -I_1 + A^T I_2\} \end{aligned}$$

Where the last step removes the existential quantification We claim to have the following parity-check matrix for this relation:

$$\begin{pmatrix} Y & 0 & M^T \\ -A & 1 & 0 \end{pmatrix}$$

As for any $\left(\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}\right) \in L(M)(1 \oplus \mathcal{W})$ we have:

$$\begin{pmatrix} Y & 0 & I & A^T \\ -A & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} YV_1 + I_1 - A^T I_2 \\ -AV_1 + V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is the standard form for a parity-check matrix for a state in KirRel. This concludes the proof. □

Corollary 4.6.2. If A is deterministic as shown in section 3.6 corollary 3.6.8 then only resistors and spiders are sufficient to generate all the maps in the category ResRel.

Example 4.6.3. Consider the relation encoded by Figure 4.6. This is an example of a state in ResRel and we will now study how to write a parity-check matrix for such a relation:

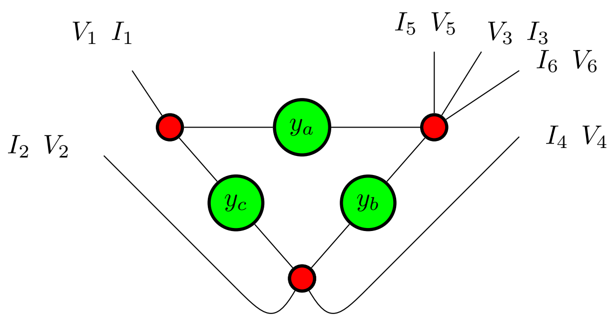


Figure 4.6: An example of a **state in** ResRel. This state can be thought of as a graph state, composed with $L(M)$, where M is a deterministic matrix.

The relation is easily seen to be encoded by the following equations, relating I_i and V_i for $i = 1$ to 6:

$$\begin{pmatrix} Y & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ I_1 \\ I_2 + I_4 \\ I_3 + I_5 + I_6 \end{pmatrix} = \vec{0} \quad V_4 = V_2 \quad V_5 = V_3 \quad V_6 = V_3$$

We can encode these equations into the following parity-check matrix:

$$\begin{pmatrix} Y_{3\times 3} & 0_{3\times 3} & I_{3\times 3} & A_{3\times 3} \\ -A^T_{3\times 3} & I_{3\times 3} & 0_{m\times n} & 0_{3\times 3} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

where V is the vector of 6 voltages and I is the vector of 6 currents in the above figure 4.6 where A is the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad -A^T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Y is adjacency matrix of the network shown in 4.6.

4.7 Kirchhoff Relations to Affine Kirchhoff Relations

We first observe that any affine Kirchhoff transformation can be obtained from a linear Kirchhoff relation, by composing voltage and current sources to the inputs and outputs. Assume without loss of generality that the linear relation is a state. Then we only need to show that any affine relation of the form

$$\forall \vec{q}, \vec{p} \in F^n, \qquad \begin{pmatrix} q \\ p \end{pmatrix} \sim \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}, \quad \text{where } q_0, p_0 \in F^k, \text{ and } \vec{1}^T \cdot p_0 = 0$$

can be realized using our generators.

We first see that any shift in position can be realized using the voltage sources and any shift in currents that satisfies conservation of momentum can be obtained from current sources. This is true since the current and voltage sources obey KCL as shown in Table 4.4.

Notice that the current source can only realize shifts in momentum that satisfy conservation of momentum. Using these properties of current and voltage sources we have the following theorem:

Theorem 4.7.1. Spiders, resistances, ideal current divders, voltage and current sources form a universal set of generators for AffKirRel.

Proof. To extend the earlier proof for the universality of KirRel to AffKirRel, we compose the diagram shown in figure 4.5 with voltage and current sources to realize general affine shifts as shown in the figure below:

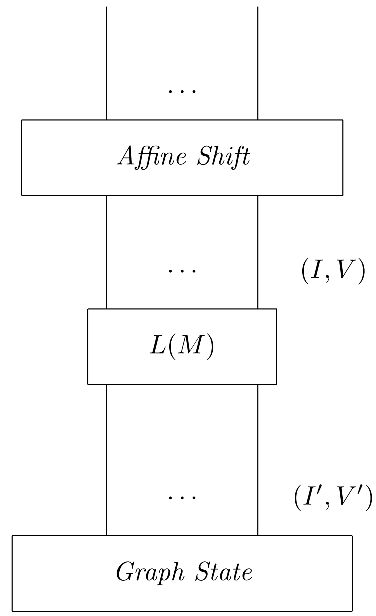


Figure 4.7: Universality for AffKirRel

Since any arbitrary affine shift in AffKirRel can be realized using a voltage and a current source, this completes the proof of universality for AffKirRel. $\square$

Chapter 5

Discussions and Conclusion

5.1 Discussions

In this section the relationship of the category of Lagrangian relations and Kirchhoff relations to qudit quantum mechanics is discussed. We begin with a review of qudit quantum mechanics and its relationship to Lagrangian relations.

5.2 Qudit Quantum Mechanics

Here we review the basic definitions and properties of the Pauli and Clifford group for qudits, for more detail the reader can refer to [3, 54–57]. Recall that the X and Z gates for qudits are defined as follows:

Definition 5.2.1. (See definition 4.8 in [3]) The p-dimensional qudit X and Z gates in $\mathsf{Mat}_{\mathbb{C}}$ are defined as follows:

$$X = \sum_{k=0}^{p-1} |k+1\rangle \langle k| \qquad Z = \sum_{k=0}^{p-1} \omega^k |k\rangle \langle k|$$

where $\omega = e^{2\pi i/p}$, p is a prime number and k is a natural number mod p.

For $p = 3$ (qutrits), for example the X and Z matrices are given as follows:

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Definition 5.2.2. (See definition 4.11 in [3] and definition 1 in [58]) The p dimensional Pauli group on n

qudits $\mathcal{P}_p^n$ is generated by X, Z, scalar $e^{\pi i/p}$ and identity I under matrix multiplication and tensor product.

Observe that $ZX = \omega XZ$. Thus $\mathcal{P}_p^n$ upto global phase consists of elements $X^n Z^m$. Thus $\mathcal{P}_p^n$ consists of $p_1 \otimes p_2 \ldots \otimes p_n$ where $p_i \in \mathcal{P}_p$. Elements of the Pauli group are sometimes referred to as Pauli operators.

Definition 5.2.3. (See definition 4.1.1 in [3]) The p dimensional Clifford group on n qudit $\mathcal{C}_p^n$ is defined to be the group of unitary operators that map p-dimensional Pauli operators to p-dimensional Pauli operators. That is:

$$\mathcal{C}_n^p = \{C \mid \forall p \in \mathcal{P}_p^n.\ \ CpC^\dagger \in \mathcal{P}_p^n\}$$

Definition 5.2.4. (See definition 4.1.1 in [3]) A stabilizer state on n-qudits is state of the form $C\left|0\right\rangle^{\otimes n}$ where C is an element of the Clifford group.

Definition 5.2.5. (See definition 4.1.1 in [3]) Given an n qudit stabilizer state $\left|\phi\right\rangle$ the group of Pauli operators that satisfy $U\left|\phi\right\rangle = \left|\phi\right\rangle$ forms a group called the stabilizer group of $\left|\phi\right\rangle$. This is an Abelian subgroup of the Pauli group.

Lemma 5.2.6. (See lemma 4.13 in [3]) The odd prime dimensional Clifford group $\mathcal{C}_n^p$ upto scalars is generated by the following operators:

$$C = \sum_{k,l=0}^{p-1} \left|k, k+l\right\rangle \left\langle k, l\right| \qquad H = \frac{1}{\sqrt{p}} \sum_{k,l=0}^{p-1} \left|l\right\rangle \left\langle k\right| \qquad S = \sum_{k=0}^{p-1} e^{\pi i k(k+p)/p} \left|k\right\rangle \left\langle k\right|$$

Definition 5.2.7. (See definition 4.14 in [3]) The Stab_p is a generated upto scalars by $\left|0\right\rangle$, $\left\langle 0\right|$ and the Clifford group in prime dimensions.

In [3] it was shown that the generators for the category LagRel_F are given as follows:

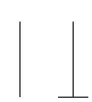
Lemma 5.2.8. (See theorem 3.1 in [3]) The generators of LagRel_F are given by $L(\mathsf{LinRel_F})$, F and R_a.

where:

$$\mathcal{F} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{R}_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

To establish the isomorphism $G : \mathsf{AffLagRel}_{F_p} \to \mathsf{Stab}_p$. First note that:

Lemma 5.2.9. (See definition 4.4 in [3]) The category of $\mathsf{AffLagRel}_F$ is generated by Lagrangian relations and an additional generator given as follows:

$\mathcal{X} \;=$ 

Then following lemma proven in [3] determines the isomorphism:

Lemma 5.2.10. (See lemma 4.15 in [3]) There is an isomorphism $G : \mathsf{AffLagRel}_{F_p} \to \mathsf{Stab}_p$ for every odd prime p characterized the following mapping of the generators upto scalars:

$$\mathcal{X} \leftrightarrow X \quad \mathcal{F} \leftrightarrow H \quad \mathcal{R}_a \leftrightarrow S \quad \mathcal{C} \leftrightarrow C$$

where:

$$\mathcal{C} = \begin{pmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix}$$

5.3 Kirchhoff Quantum Mechanics

Using the isomorphism $G : \mathsf{AffLagRel}_{F_p} \to \mathsf{Stab}_p$ described above one can interpret the generators of $\mathsf{AffKirRel}$ as quantum maps on a Hilbert space as shown in the Tables 5.1 and 5.2. Note that we denote the eigenstates of X with an overbar:

$$X\,|\tilde{k}\rangle = \omega^k\,|\tilde{k}\rangle\,,$$

These formulas are essentially derived using the phase space formalism. One can show that a pure density matrix ρ is a pure state (see [25]) – and therefore must equal $|\psi_n\rangle\,\langle\psi_n|$. We can interpret this correspondence as follows: A Lagrangian subspace can be thought of as a *possibility distribution*, the set of points in the subspace correspond to events that are possible, and the set of points not in the subspace correspond to events that are impossible. In quantum mechanics, we convert this into a quasi-probability distribution, given by a Wigner function in which all possible events are equally probable.

A general quantum map $\varepsilon : \mathcal{H}_d^n \to \mathcal{H}_d^m$ can be converted to a state ρ_ε via the cups and caps. If the resulting state is a pure stabilizer state, then we consider ε to be a stabilizer map. The Lagrangian subspace corresponding to ρ_ε can be converted to a Lagrangian relation using the appropriate spider resulting in a Lagrangian relation associated with ε.

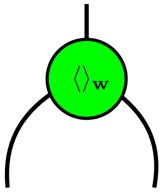
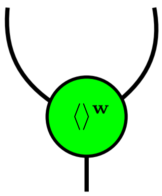
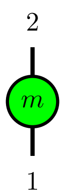
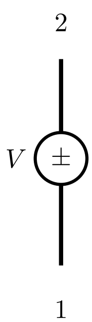
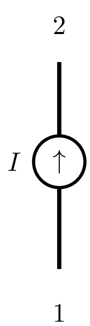
Generator	Qudit Quantum Mechanics Interpretation
	$\sum_k \lvert k\rangle \langle w^{-1}k\rvert \langle (1-w)^{-1}k\rvert$
	$\sum_k \lvert w^{-1}k\rangle \lvert (1-w)^{-1}k\rangle \langle k\rvert$
	$\sum_k \omega^{mk^2} \lvert k\rangle \langle k\rvert$
	$\sum_k \omega^{Vk} \lvert k\rangle \langle k\rvert$
	$\lvert I\rangle \langle I\rvert$

Table 5.1: Qudit Gates

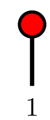
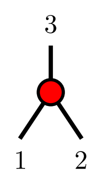
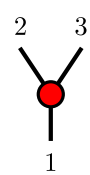
Name	Generator	Quantum Interpretation
Unit		$\lvert 0 \rangle$
Counit		$\langle 0 \rvert$
Monoid		$\sum_{\tilde{k}} \lvert \tilde{k} \rangle \langle \tilde{k} \rvert \langle \tilde{k} \rvert$
Kirchhoff Co-monoid		$\sum_{\tilde{k}} \lvert \tilde{k} \rangle \lvert \tilde{k} \rangle \langle \tilde{k} \rvert$

Table 5.2: Representation of **Spiders** as quantum maps

.

$$
\begin{tikzcd}
\mathrm{LossLagRel} \arrow[r, hook] & \mathrm{LagRel} \arrow[r, hook] & \mathrm{AffLagRel} \\
\mathrm{LossKirRel} \arrow[u, hook] \arrow[r, hook] & \mathrm{KirRel} \arrow[u, hook] \arrow[r, hook] & \mathrm{AffKirRel} \arrow[u, hook] \\
\mathrm{Spiders} \arrow[u, hook] \arrow[r, hook] & \mathrm{ResRel} \arrow[u, hook] \arrow[r, hook] & \mathrm{AffResRel} \arrow[u, hook]
\end{tikzcd}
$$

Figure 5.1: A web of inter-relations of the categories involved in the thesis

5.4 Conclusion

The goal of the thesis was to investigate the connection between electrical circuits and quantum circuits. Stabilizer quantum circuits correspond to affine Lagrangian relations over a field of odd prime characteristic. These relations have components that do not have a direct electrical analogue: in particular the Fourier/Hadamard transform does not have a electrical analogue. To capture just the electrical circuits, Lagrangian relations satisfying the special property of Kirchhoff's current law were considered. However even this allowed, in addition to resistor circuits, (ideal) current dividers. To isolate exactly the subcategory of resistors the additional constraint of being deterministic was applied.

Parity check matrices were used to characterize KirRel and its subcategories. First a standard form for Lagrangian relations was derived. Using this standard form a characterization of Kirchhoff relations was given by adding the effect of Kirchhoff's current law to the Lagrangian parity-check standard form. Using these standard forms deterministic Lagrangian and Kirchhoff relations were defined and their properties studied. Using the standard form for Kirchhoff relations a special class of states called graph states were also isolated and their parity-check matrices studied. Further subcategories of LagRel and KirRel namely $\mathsf{LosLagRel}$ and $\mathsf{LosKirRel}$ were classified using the property of "power input".

These categories and subcategories were then interpreted in terms of electrical networks of different types, explicitly the category of KirRel corresponded to those circuits with resistors, junctions and current dividers, the category of ResRel corresponded to electrical networks built from resistors and junctions, while graph states were interpreted as resistive networks with "no extra wires" and lastly the category of $\mathsf{AffKirRel}$ corresponded to electrical networks built from current, voltage sources, junctions and resistors.

The intersection of categories and subcategories studied in the thesis and their interrelationships are pictorially shown in 5.1.

The second half of the thesis explicitly proves that the maps in the categories KirRel are generated by electrical elements namely, resistors, current dividers and spiders and the maps in $\mathsf{AffKirRel}$ are generated by these elements supplemented by voltage and current sources. Furthermore the maps in the category of lossless Kirchhoff relations LosKirRel were generated by spiders and current dividers and the maps in ResRel were resistor circuits generated by resistors and junctions. In the process of these proofs we also discuss how to write parity-check matrices for different types of electrical networks. The generators of these categories and their relationship with one another is shown in 5.2.

$$\begin{tikzcd}
\text{Current Dividers and Junctions} \arrow[r, hook] & \text{Current Divider and Resistor Circuits} \arrow[r, hook] & \text{Current Divider, Resistor Circuits and Sources} \\
\text{Junctions} \arrow[r, hook] \arrow[u, hook] & \text{Resistor Circuits} \arrow[r, hook] \arrow[u, hook] & \text{Resistor Circuits and Sources} \arrow[u, hook]
\end{tikzcd}$$

Figure 5.2: The first row in 5.1 has no natural interpretation in terms of electrical networks. This figure explains the interpretation of row 2 and row 3 in 5.1 as electrical networks.

In the end of the thesis a connection to qudit quantum mechanics was discussed and the generators of Kirchhoff relations were interpreted in terms of quantum maps on Hilbert spaces.

5.5 Future Work

There are several potential directions of future work. The various forms of the parity-check matrices which we investigated suggested a relation to normal forms for their corresponding electrical networks. It would be interesting to make this relation precise. From the point of view of rewriting theory it would be useful to design a minimal set of rewrite rules for the category of KirRel and ResRel, this would essentially consist of understanding the rules that govern the interaction of current dividers, resistors and sources with each other. This could in principal help in simplifying circuits designed using the generators of KirRel. In the context of the ResRel one would like to precisely answer the question if the series, parallel and star-mesh rule allows one to convert an arbitary resistor network to a mesh of resistor networks in all case. It would be interesting if this work leads to a ZX calculus styled calculus for electrical circuits using the normal forms suggested by the various forms of parity-check matrices studied in the thesis. This would lead to another graphical calculus to understand the behaviour of electrical circuits.

In the current formalism to show composition of maps is well-defined one always has to go to the underline definition of the category and have a special argument. In our results we do not have an explicit parity-check formula for composition. It would have been useful to have had a formula for composition at the level of parity-check matrices as this would have provided a characterization of the subcategories of Lagrangian relations, purely using the formalism of parity-check matrices.

On the qudit quantum computation front it would be interesting to see if simplification strategies used in electrical circuits could be imported to qudit quantum circuits via the formalism of the category of Kirchhoff relations. So for example is there some equivalent of well known network theorems like Thevenin and Norton's theorems for Clifford circuits? This would help in simplifying Clifford circuits and understanding stabilizer circuits using connections to electrical network theory. As discussed earlier there is an equivalence functor between $\mathsf{AffLagRel}_{F_p}$ and $\mathsf{Stab_p}$ which implies that a quantum computer built out of generators of Lagrangian relations is as powerful as one built from the stabilizer fragment of qudit quantum mechanics, one can think of a similar question for the $\mathsf{AffKirRel}$ and ask what is the power of a quantum computer built solely out of the generators of the category $\mathsf{AffKirRel}$? The first step towards answering this question has been discussed in the thesis where we interpreted the generators of $\mathsf{AffKirRel}$ in terms of quantum maps on a Hilbert spaces.

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