Restriction for Semigroups and Categories
First afternoon talk of Calgary workshop
Ernie Manes, June 2, 2006

This handout includes axiom systems useful for the talk, a very small set of definitions in category theory and a 12-step program to help semigroup theorists who have never studied category theory to get up to speed. The references at the end will be mentioned in the talk.

Slogan: *In the study of semigroups, it is sometimes useful to add some new objects to obtain a category associated with that semigroup.*

Tentative talk outline:

1. History of restriction in semigroups.
2. The unusual expressiveness of universal mapping properties.
3. Idempotents: monoids vs. categories.
4. The regular representation is the Yoneda embedding.
5. Boolean restriction.

I apologize in advance that gf means $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a category or in a semigroup.

Cockett-Lack axioms for restriction categories [3]. There is a unary **restriction** operator $f : X \to Y \quad \mapsto \quad \overline{f} : X \to X$ satisfying

(R.1) For $f : X \to Y$, $f\,\overline{f} = f$.

(R.2) For $f : X \to Y$, $g : X \to Z$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For $f : X \to Y$, $g : X \to Z$, $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\overline{g}\,f = f\,\overline{gf}$.

For **pre-restriction**, replace (R.4) with

(PR.1) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\overline{gf} = \overline{\overline{g}\,f}$.

Restriction $\Rightarrow$ pre-restriction.

Cockett-Guo axioms on range [2]. Starting with a restriction category, add another unary operator $f : X \to Y \quad \mapsto \quad \widehat{f} : Y \to Y$ subject to

(RR.1) For $f : X \to Y$, $\overline{\widehat{f}} = \widehat{f}$.

(RR.2) For $f : X \to Y$, $\widehat{f}\,f = f$.

(RR.3) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\widehat{\overline{g}\,f} = \overline{g}\,\widehat{f}$.

(RR.4) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\widehat{g\,\widehat{f}} = \widehat{gf}$.

Equivalent Schweizer-Sklar axioms for $\overline{f}$, $\widehat{f}$ on a semigroup (1967) [21]:

(SS.1) $\widehat{\overline{a}} = \overline{a}$, $\overline{\widehat{a}} = \widehat{a}$.

(SS.2) $\widehat{a}\, a = a = a\, \overline{a}$.

(SS.3) $\widehat{ab} = \widehat{a\, \widehat{b}}$, $\overline{ab} = \overline{\overline{a}\, b}$.

(SS.4) $\widehat{a}\, \overline{b} = \overline{b}\, \widehat{a}$.

(SS.5) $\overline{a}\, b = b\, \overline{ab}$.

Here, all but the fifth axiom is self-dual.

Freyd's axioms on an allegory [7, Chapter 2]

From [7, Page 195] "Allegories...are to binary relations between sets as categories are to functions between sets".

An **allegory** is a category $\mathcal{A}$ in which $\mathcal{A}(X,Y)$ is a meet semilattice for which composition on either side is monotone, together with order isomorphisms $\mathcal{A}(X,Y) \to \mathcal{A}(Y,X)$, $R \mapsto R^o$, subject to the **modular law** $(SR) \cap T \subset S(R \cap S^o T)$.

For $R : X \to Y$, define $\overline{R} : X \to X$ by $\overline{R} = 1 \cap (R^o R)$. R is a **map** if R is **total** ($\overline{R} = 1$) and **deterministic** ($RR^o \subset 1$). $\mathcal{A}$ is **tabular** if for every $R : X \to Y$ there exist maps $X \xleftarrow{f} Z \xrightarrow{g} Y$ with $R = gf^o$.

Axioms for a weakly left E-ample semigroup (see [8]). Given semigroup S with a semilattice E of (not necessarily all) idempotents, write $a\, \tilde{\mathcal{R}}\, b$ to mean $\forall\, e \in E \;\; ae = a \Leftrightarrow be = b$.

(WLEA.1) $\forall\, a \in S\; \exists\, e \in E \;\; a\tilde{\mathcal{R}}e$. Such e is provably unique; call it $\overline{a}$.

(WLEA.2) $\forall\, a,b,c \in S \;\; a\tilde{\mathcal{R}}b \;\Rightarrow\; ac\, \tilde{\mathcal{R}}\, bc$.

(WLEA.3) $\forall\, a,b \in S \;\; \overline{b}\, a = a\, \overline{\overline{b}\, a}$.

For a semigroup S, $E = \{\overline{x} : x \in S\}$ establishes a bijection between $\overline{x}$ satisfying (R.1,...,R.4) and E rendering S weakly left E-ample.

Some useful definitions in a category, universal mapping properties

The set of morphisms $X \to Y$ in a category $\mathcal{C}$ is denoted $\mathcal{C}(X,Y)$. The **dual** or **opposite** category $\mathcal{C}^{op}$ has the same objects but $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$. When $\mathcal{C}$ has one object, $\mathcal{C}$ amounts to a monoid and $\mathcal{C}^{op}$ is the opposite monoid.

$f : X \to Y$ is an **isomorphism** if there exists (necessarily unique) $g : Y \to X$ with $X \xrightarrow{f} Y \xrightarrow{g} X = 1_X$, $Y \xrightarrow{g} X \xrightarrow{f} Y = 1_Y$. Isomorphism is self-dual and forms an equivalence relation on objects. The next table establishes notation, and the definitions follow immediately after.

in $\mathcal{C}$	$\Leftrightarrow$ in $\mathcal{C}^{op}$
terminal object 1	**initial object** 0
product $X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y$	**coproduct** $X \xrightarrow{in_X} X + Y \xleftarrow{in_Y} Y$
equalizer $E \xrightarrow{i} X \xrightarrow{f,g} Y$ of f,g	**coequalizer** $X \xrightarrow{f,g} Y \xrightarrow{q} Q$ of f,g
pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$	**pushout** of $X \xleftarrow{f} Z \xrightarrow{g} Y$

1 is defined (uniquely up to isomorphism) by $\forall X\ \exists$ unique $X \to 1$. If $1'$ is also terminal there exist unique $f : 1 \to 1'$, $g : 1' \to 1$. As $1, gf : 1 \to 1$ whereas there exists but one map of this type, $gf = 1$ (context distinguishes between the terminal object and the identity morphism). Similarly, $fg = 1$ so f is an isomorphism. By the same argument, the other seven concepts are unique up to unique isomorphisms, as you should check.

The product is defined by the "universal property" that given $X \xleftarrow{t} W \xrightarrow{u} Y$ there exists unique $[t,u] : W \to X \times Y$ such that $pr_X[t,u] = t$ and $pr_Y[t,u] = u$.

We say i is the equalizer of f, g, written $i = eq(f,g)$, if i satisfies the universal property that $fi = gi$ and, whenever $t : W \to X$ with $ft = gt$, there exists a unique $v : W \to E$ with $iv = t$. Dually, we write $q = coeq(f,g)$.

The pullback is $X \xleftarrow{a} P \xrightarrow{b} Y$ with the universal property that $fa = gb$ and, whenever $X \xleftarrow{t} W \xrightarrow{u} Y$ with $ft = gu$, there exists unique $v : W \to P$ with $av = t$ and $bv = u$.

These universal constructions can be described in any monoid (a monoid is just a one-object category!) but rarely exist.

Cancellability is definable in a category. A right-cancellable $f : X \to Y$ (that is, $\forall\, t, u : W \to X$ if $ft = fu$ then $t = u$) is called a **monic**. With composition in the reverse order, semigroup theorists would call this left cancellable. A map $f : X \to Y$ **is an equalizer** if there exist g, h with $i = eq(g,h)$. A map $f : X \to Y$ is **split monic** if there exists $g : Y \to X$ with $gf = 1$. Dually: **epic**, **a coequalizer**, **split epic**.

Here is a small list of useful categories.

Set	Sets and total functions
Pfn	Sets and partial functions
Rel	Sets and relations
Sem	Semigroups and semigroup homomorphisms
Mon	Monoids and monoid homomorphisms
Grp	Groups and group homomorphisms
Top	Topological spaces and continuous maps
T2	Hausdorff spaces and continuous maps
Ban	Banach spaces and norm-decreasing linear maps

The Twelve Steps

1. Show that isomorphism is an equivalence relation on the objects of a category. Determine isomorphism in the categories listed above. What are the isomorphisms of the category of Banach spaces and continuous linear maps?

2. Explain what is meant by "products, equalizers, pullbacks are unique up to isomorphism". By duality, the same thing holds for initial objects, coproducts, coequalizers and pushouts.

3. Show that a split epic is a coequalizer and that an coequalizer is epic. State the dual result.

4. Show that a category with a terminal object and pullbacks also has products and equalizers.

5. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, show that f, g monic $\Rightarrow gf$ monic, and gf monic $\Rightarrow f$ monic. State the dual results.

6. In **Set**, $X \times Y = \{(x,y) : x \in X,\ y \in Y\}$ with $pr_X(x,y) = x$, $pr_Y(x,y) = y$. Verify the universal property. Show the following: In **Pfn**, the product of X, Y exists and is $X + (X \times Y) + Y$ where $\times$ is the product in **Set** and $+$ is disjoint union. In **Rel** the product exists and is $X + Y$ (hint:

show first that this is the coproduct and then observe that mapping a relation to its converse is an isomorphism of categories **Rel** $\cong$ **Rel**^{op}). The usual products work as the categorical product for **Sem**, **Mon**, **Grp**, **Top** and **T2**. The product in **Ban** is the usual vector space product with norm $\|(x, y)\| = \|x\| \wedge \|y\|$.

7. Show the following. Disjoint union provides coproduct in **Set**, **Pfn** and **Rel**. In **Set**, **Sem**, **Mon**, **Grp**, for $f, g : X \to Y$, $eq(f, g)$ is the subset of X (a subalgebra!) on which f, g agree and $coeq(f, g)$ obtains by dividing out by the smallest congruence containing $\{(fx, gx) : x \in X\}$.

8. An object that is simultaneously terminal and initial is a **zero object**. In that case there exists, for each X, Y a distinguished **zero map** $0 : X \to Y$, namely $X \to 0 \to Y$. As for semigroups, such zero maps are unique and satisfy $f0 = 0 = 0g$. Verify the following table, where 1 denotes a one-element set:

Category	Terminal Object	Initial Object	Zero Object
Set	1	$\emptyset$	
Pfn			$\emptyset$
Rel			$\emptyset$
Sem	1	$\emptyset$	
Mon			1
Grp			1
Top	1	$\emptyset$	
T2	1	$\emptyset$	
Ban			1

Show that the category of rings with unit (with $0 \neq 1$) has $\boldsymbol{Z}_2$ as initial object and that this remains true even if $0 = 1$ is allowed. Combining exercises 4, 6, 7 construct equalizers, pullbacks and coequalizers in the example categories.

9. Show that, in **Top**, equalizers are constructed as in **Set**, giving the subset the subspace topology whereas coequalizers are constructed as in **Set**, giving the quotient set the quotient topology. In **Top**, epics are the continuous surjections and monics are the continuous injections. Show, however, that in **T2**, $f : X \to Y$ is epic if and only if $f(X)$ is dense in Y. In **Sem**, show that the inclusion of the nonnegative integers in the group of additive integers is epic, though not surjective. In any category, show that a morphism which is both monic and a coequalizer is an isomorphism.

10. If you work the following, you will see it is the type of calculation common in semigroup theory, even though its conclusion is of a different sort. In any category, consider the commutative square

$$\begin{array}{ccc} P & \xrightarrow{i} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow[m]{} & X \end{array}$$

("commutative" means that $mj = ni$). Write k for the common value $mj = k = ni$ and assume that n, m, k are split monic, $pm = 1$, $qn = 1$, $rk = 1$. Assume, further, that $kr = mpnq$. Show that the square is a pullback.

11. Label a square as in 10 and assume that it is a pullback. Show that if m is monic then i is monic. Show that, in general, i need not be split monic if m is, even in **Set**.

12. Read the book *Categories, Allegories* referenced below as follows: Read in Chapter 1 until you get perplexed, then jump to Chapter 2 and read until you need more help from Chapter 1. See

how many cycles you can go, then contrast with Schein's survey on relation algebras referenced below. Then read the Cockett and Lack paper referenced below, ignoring all parts requiring category theory that goes beyond that discussed here. You'll see it's a lot like semigroup theory!

Axioms for abelian and Boolean categories

In a category with zero maps, for $f : X \to Y$, $eq(f, 0)$, if it exists, is called the **kernel of** f. Dually, $coeq(f, 0)$ is the **cokernel of** f. Say that $i : P \to X$ is a **coproduct injection** if there exists a coproduct $P \xrightarrow{i} X \xleftarrow{i'} P'$.

Abelian Category Axiom	Boolean Category Axiom
$\exists$ zero object 0	$\exists$ initial object 0
$\forall\, X, Y\ X + Y$, $X \times Y$ exist	$\forall\, X, Y$, $X + Y$ exists
Every map has a kernel and a cokernel	Every $X \xrightarrow{f} Y \xleftarrow{j} Q$ with j a coproduct injection has a pullback $X \xleftarrow{i} P \longrightarrow Q$ with i a coproduct injection; moreover, if f is a coproduct injection, it pulls back a coproduct $Q \xrightarrow{j} Y \xleftarrow{j'} Q'$ to a coproduct $P \xrightarrow{i} X \xleftarrow{i'} P'$.
Every monic is a kernel Every epic is a cokernel	if $X \xrightarrow{1} X \xleftarrow{1} X$ is a coproduct, $X = 0$

In both cases, the third axiom is about special types of pullback, noting that a kernel is the same thing as the pullback of the zero object. Notice that the abelian category axioms are self-dual.

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