AMPLE AND LEFT AMPLE SEMIGROUPS
Extended Abstract

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Introduction

Let E be a commutative subsemigroup of idempotents, that is, a subsemilattice, of a semigroup S, and let $^\dagger : S \to E$ be a unary operation. We say that S satisfies the *left ample condition with respect to* E, if

(AL) $$ae = (ae)^\dagger a \text{ for all } a \in S \text{ and } e \in E.$$

For the dual property, we usually denote the unary operation by * and so the *right ample condition with respect to* E is

(AR) $$ea = a(ea)^* \text{ for all } a \in S \text{ and } e \in E.$$

When E is actually the set of all idempotents of S, we drop the phrase "with respect to" and refer simply to the left or right ample condition.

Recall that a *partial permutation* of a nonempty set X is a bijection $\sigma : Y \to Z$ for some subsets Y, Z of X, and that the set of all such partial permutations (denoted by I_X) is a monoid under the usual composition of partial functions. Note that if $\sigma : Y \to Z$ is a member of I_X, then so is its inverse $\sigma^{-1} : Z \to Y$ so that on I_X, we can define three unary operations $^{-1}$, † and * as follows: for $\sigma \in I_X$,

$$\sigma^{-1} \text{ is the inverse of } \sigma;\ \sigma^\dagger = \sigma\sigma^{-1} \text{ and } \sigma^* = \sigma^{-1}\sigma.$$

Suppose that the semigroup S is isomorphic to a subemigroup of I_X via an isomorphism θ. We say that S is *inverse* if $S\theta$ is closed under $^{-1}$, that S is *left ample* if $S\theta$ is closed under † and that S is *right ample* if $S\theta$ is closed under *. An *ample* semigroup is one which is both left and right ample. Note that any inverse semigroup is ample. Left ample semigroups used to be known as *left type A* semigroups.

As usual, $E(S)$ denotes the set of all idempotents of a semigroup S. It is immediate from the definition that, in a left ample semigroup S, the idempotents commute with each other, and so $E(S)$ is a subsemilattice of S. A left ample semigroup satisfies the left ample condition, and similarly, a right ample semigroup the left ample condition. The two ample conditions are the properties that underly much of the structure theory for inverse semigroups, and so it is reasonable to expect that analogous structure results hold for left ample semigroups.

Classes of semigroups more general than (left) ample semigroups but which satisfy one or both of the ample conditions have been widely studied. We mention type SL2 γ-semigroups [2], weakly (left) ample and weakly (left) E-ample semigroups [4, 18, 19, 10, 11, 12], twisted LC-semigroups [15, 16] and guarded semigroups [26]. In fact, twisted LC-semigroups are the same things as weakly left $C(S)$-semigroups and type SL2 γ-semigroups.

In the talk we will concentrate on approaches to structural properties of left ample and ample semigroups, with just brief mentions of some generalisations.

In the theory of left ample semigroups, a generalisation of Green's relation $\mathscr{R}$ which we now describe plays a role which is to some extent analogous to that of $\mathscr{R}$ in the theory of inverse semigroups. On any semigroup S, the relation $\mathscr{R}^*$ is defined by the rule that $(a, b) \in \mathscr{R}^*$ if and only if the elements a, b of S are related by Green's relation $\mathscr{R}$ in some oversemigroup of S. The relation $\mathscr{R}^*$ seems to have been introduced by Lyapin [22], and, as observed by Pastijn [28] and McAlister [25], $a\mathscr{R}^*b$ is equivalent to the condition that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$. Given this, it is easy to see that $\mathscr{R}^*$ is a left congruence on S.

One can define the left-right dual $\mathscr{L}^*$ of $\mathscr{R}^*$ in a similar way, and then we put $\mathscr{H}^* = \mathscr{R}^* \cap \mathscr{L}^*$ and $\mathscr{D}^* = \mathscr{R}^* \vee \mathscr{L}^*$. If S is a regular semigroup, that is, for every element $a \in S$, there is an element $b \in S$ such that $aba = a$, then $\mathscr{R}^* = \mathscr{R}$ and $\mathscr{L}^* = \mathscr{L}$ etc.

We can now give an abstract characterisation of left ample semigroups (see [6]). A semigroup S is left ample if and only if every $\mathscr{R}^*$-class contains an idempotent, $E(S)$ is a subsemilattice of S and the *left ample* condition is satisfied:

$$ae = (ae)^{\dagger}a \text{ for all } a \in S \text{ and } e \in E(S).$$

Of course, there is a similar characterisation of right ample semigroups using the relation $\mathscr{L}^*$ and the *right ample* condition:

$$ea = a(ea)^{*} \text{ for all } a \in S \text{ and } e \in E(S).$$

It is clear from these descriptions that right cancellative monoids are left ample, and that cancellative monoids are ample.

The original impetus for investigating (left) ample semigroups came from the study of monoids via their actions on sets by analogy with the study of rings via their actions on modules. The only monoid S over which every S-set is projective is the one element monoid, but we get a more interesting class when we assume only that every principal left ideal is projective. A principal left ideal Sa of S is projective if and only if for some idempotent e there is a bijection $\theta : Sa \to Se$ which preserves the action of S and maps a to e. Note that this is equivalent to saying that $a\mathscr{R}^*e$. Such monoids were called *left PP monoids*. Examples are furnished by the multiplicative monoids of left PP rings, and by regular semigroups. The latter observation inspired the study of the structure of such monoids, the first result being obtained by Kilp [17] for commutative monoids; this was subsequently generalised in [5] to monoids with central idempotents where it is shown that a monoid is left PP with central idempotents if and only if it is a semilattice (with greates element) of right cancellative monoids. This compares with the description of regular semigroups with central idempotents (which are necessarily inverse) as semilattices of groups.

The left ample condition is easily seen to hold in right PP monoids with central idempotents; indeed, they are just left ample monoids with central idempotents, and so these results suggest that the structure theory for inverse semigroups should inform the study left ample semigroups. There are three main approaches for investigating the structure of inverse semigroups. One to be discussed in Mark Kambites' talk is via the theory of Schein [29] relating inverse semigroups and inductive groupoids which is studied in depth in Lawson's book [21]. Another which was initiated by Munn in [27] is via the fundamental representation of an inverse semigroup in what is now known as the Munn semigroup of the semilattice $E(S)$. The third approach is via McAlister's theory of E-unitary covers and the P-theorem [23, 24].

The first extension of the groupoid approach was due to Armstrong in [1] where she showed that the category of ample semigroups and (2,1,1)-homomorphisms is isomorphic to the category of inductive cancellative categories and order-preserving functors. This was generalised to weakly ample semigroups and inductive categories by Lawson in his thesis [18]. In [20], he gave a further generalisation which showed that ampleness is not the crucial property for this kind of result. One-sided results along these lines have recently been obtained by Hollins, a research student of Gould at York.

Given a semilattice E, the *Munn semigroup* T_E is the inverse subsemigroup of I_E consisting of all isomorphisms between principal ideals of E. The semilattice of idemotents of T_E is isomorphic to E, and Munn showed that for any inverse semigroup S there is a homomorphism from S into $T_{E(S)}$ which maps $E(S)$ isomorphically onto $E(T_{E(S)}$ and induces the maximum idempotent separating congruence on S. The maximum idempotent separating congruence on an inverse semigroup is the largest congruence contained in Green's relation H. An ample semigroup does not necessarily have a maximum idempotent separating congruence, but there is a largest congruence in $\mathscr{H}^*$. Munn's result is extended to an ample semigroup S to give a homomorphism from S into $T_{E(S)}$ mapping $E(S)$ isomorphically onto $E(T_{E(S)}$ and inducing the largest congruence contained in $\mathscr{H}^*$ [7]. This result was generalised to weakly abundant semigroups independently by El Qallali [4] and Lawson [18].

On a left ample semigroup S, there is a least congruence σ such that S/σ is a right cancellative monoid. We say that S is *proper* if for all elements a and b of S such that $a^+ = b^+$ and $a\sigma b$, we have $a = b$. It is well known that an inverse semigroup is proper if and only if it is E-unitary (see, for example, [14, Proposition 5.9.1]). Example 3 of [6] shows that the corresponding statement does not hold for left ample semigroups, but it proper semigroups that we work with for the analogues of McAlister's covering theorem and P-theorem.

Let S be a left ample semigroup and T be a right cancellative monoid. We say that a left ample semigroup P is a *proper cover* of S (*over* T) if P is proper and there is a surjective $+$-homomorphism α from P onto S which maps $E(P)$ isomorphically onto $E(S)$ (and is such that $P/\sigma \cong T$). The existence of proper covers for left ample monoids and an analogue of the P-theorem were given in [6]; the results are easily extended to the semigroup case. Two sided versions of these theorems can be found in [19]. The results have been refined in, for example, [8, 9] and extended to the weakly left ample case in [10, 11, 12].

References

[1] S. Armstrong, The Structure of Type A Semigroups, *Semigroup Forum*, **29** (1984), 319–336.

[2] A. Batbedat, γ-demi-groupes, Demi-modules, Produit Demi-direct, in *Semigroups Proceedings, Oberwolfach 1978*, Lecture Notes in Mathematics **855**, (Springer-Verlag, 1981), 1–18.

[3] J. R. B. Cockett and S. Lack, Restriction Categories I: Categories of Partial Maps, *Theoret. Comput. Sci. 270*, 223–259.

[4] A. El Qallali, *Structure Theory for Abundant and Related Semigroups*, D.Phil. thesis, York, 1980.

[5] J. Fountain, Right PP Monoids with Central Idempotents, *Semigroup Forum* **13** (1977), 229–237.

[6] J. Fountain, A Class of Right PP Monoids, *Quart. J. Math. Oxford* **28** (1977), 285–300.

[7] J. Fountain, Adequate Semigroups, *Proc. Edinburgh Math. Soc.*, **22** (1979), 113–125.

[8] J. Fountain and G. M. S. Gomes

[9] J. Fountain and G. M. S. Gomes

[10] G. M. S. Gomes and V. Gould, Proper Weakly Left Ample Semigroups, *Int. J. Algebra Comput.*, **9** (1999), 721–739.

[11] G. M. S. Gomes and V. Gould, Graph Expansions of Unipotent Monoids, *Comm. Algebra*, **28** (2000), 447–463.

[12] G. M. S. Gomes and V. Gould, Finite Proper Covers in a Class of Finite Semigroups with Commuting Idempotents, *Semigroup Forum* **66** (2003), 433–454.

[13] V. Gould, Graph Expansions of Right Cancellative Monoids, *Int. J. Algebra Comput.*, **6** (1996), 713–733.

[14] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, 1995.

[15] M. Jackson and T. Stokes, An Invitation to C-semigroups, *Semigroup Forum*, **62** (2001), 279–310.

[16] M. Jackson and T. Stokes, Agreeable Semigroups, *J. Algebra 266* (2003), 392–417.

[17] M. Kilp, Commutative Monoids all of whose Principal Ideals are Projective, *Semigroup Forum*, **6** (1973), 334–339.

[18] M.V. Lawson, *The Structure Theory of Abundant Semigroups*, D.Phil. thesis, York, 1985.

[19] M. V. Lawson, The Structure of Type A Semigroups, *Quart. J. Math. Oxford* **37** (1986), 279–298.

[20] M. V. Lawson, Semigroups and Ordered Categories I. The Reduced Case, *J. Algebra 141* (1991), 422–462.

[21] M. V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Scientific, 1998.

[22] E. S. Lyapin, *Semigroups*, Amer. Math. Soc., 1963.

[23] D. B. McAlister, Groups, Semilattices and Inverse Semigroups, *Trans. Amer. Math. Soc.* **192** (1974), 227–244.

[24] D. B. McAlister, Groups, Semilattices and Inverse Semigroups, II, *Trans. Amer. Math. Soc.* **196** (1974), 351–370.

[25] D. B. McAlister, One-to-one Partial Right Translations of a Right Cancellative Semigroup. *J. Algebra* **43** (1976), 231–251.

[26] E. Manes, Guarded and Banded Semigroups, *Semigroup Forum*, (2006),

[27] W. D. Munn, Fundamental Inverse Semigroups, *Quart. J. Math. Oxford* **21** (1970), 157–170.

[28] F. Pastijn, A Representation of a Semigroup by a Semigroup of Matrices over a Group with Zero, *Semigroup Forum* **10** (1975), 238–239.

[29] B. M. Schein, On the Theory of Inverse Semigroups and Generalised Grouds, *Amer. Math. Soc. Transl.* **113** (1979), 89–122.