Some Basic Homotopy Theory

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My goal in this talk is to provide a short introduction to some methods of modern homotopy theory - modern meaning since Quillen (1967) [3].

1 Preliminaries

An object A of a category $\mathcal{E}$ is said to be a *retract* of an object B of $\mathcal{E}$ if there are maps $i : A \to B$ and $r : B \to A$ such that $ri = id_A$. A map $u : A \to B$ is a retract of a map $v : C \to D$ if u is a retract of v in $\mathcal{E}^2$ - the category of arrows of $\mathcal{E}$ - i.e. if there is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \xrightarrow{r} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[j]{} & D & \xrightarrow[s]{} & B \end{array}$$

such that $ri = id_A$ and $sj = id_B$.

A class of maps $\mathcal{M}$ in $\mathcal{E}$ is said to be *closed under retracts* if whenever $v \in \mathcal{M}$ and u is a retract of v then $u \in \mathcal{M}$.

A class of maps $\mathcal{M}$ in $\mathcal{E}$ is said to be *closed under pushouts* if whenever $u : A \to B$ is in $\mathcal{M}$ and $f : A \to C$ is in $\mathcal{E}$ then in the pushout (when it exists)

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B +_A C \end{array}$$

the map u' is in $\mathcal{M}$. There is a dual concept of a class *closed under pullbacks*.

If $i : A \to B$ and $f : X \to Y$ are maps in $\mathcal{E}$, the notation $i \pitchfork f$ means that any commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[b]{} & Y \end{array}$$

has a *diagonal filler* $d : B \to X$, with $di = a$ and $fd = b$. We say i has *the left lifting property* (LLP) with respect to f, and f has *the right lifting property* (RLP) with respect to i, We denote by ${}^{\pitchfork}\mathcal{M}$ (respectively $\mathcal{M}^{\pitchfork}$) the class of all maps which have the LLP (respectively RLP) with respect to each map in $\mathcal{M}$.

Proposition 1.1 *$\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition, retracts, and pullbacks. Dually, ${}^{\pitchfork}\mathcal{M}$ contains the isomorphisms and is closed under composition, retracts, and pushouts.*

Proof: Excercise.

If $\mathcal{E}$ is cocomplete and $\alpha = \{i < \alpha\}$ is an ordinal, a functor

$$C : \alpha \to \mathcal{E}$$

is called an α-*sequence* if the canonical map

$$\varinjlim_{i<j} C_i \to C_j$$

is an isomorphism for every limit ordinal $j < \alpha$. The *composite* of an α-sequence C is the canonical map

$$C_0 \to \varinjlim_{i<\alpha} C_i$$

A subcategory $\mathcal{C}$ of $\mathcal{E}$ is *closed under transfinite composition* if it contains the composite of any α-sequence $C : \alpha \to \mathcal{C} \subseteq \mathcal{E}$.

We call a class of maps $\mathcal{M}$ of $\mathcal{E}$ *saturated* if it contains the isomorphisms, and is closed under composition, transfinite composition, pushouts and retracts. For example, ${}^{\pitchfork}\mathcal{M}$ is saturated. Any class $\mathcal{M}$ of maps is contained in a smallest saturated class $\overline{\mathcal{M}} \subseteq \mathcal{E}$ called the saturated class *generated* by $\mathcal{M}$.

Proposition 1.2 *A saturated class $\mathcal{M}$ is closed under arbitrary coproducts.*

Proof: Let I be a set and $A_i \to B_i$ a map in $\mathcal{M}$ for $i \in I$. We want to show that the map

$$\sum_i A_i \to \sum_i B_i$$

is in $\mathcal{M}$. We may assume I is an ordinal α since any set is isomorphic to an ordinal. Then define an α-sequence $C : \alpha \to \mathcal{E}$ by $C_0 = \sum_i A_i$. C_{i+1} is the pushout

$$\begin{array}{ccc} A_i & \longrightarrow & C_i \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & C_{i+1} \end{array}$$

and $C_j = \varinjlim_{i<j} C_i$ for limit ordinals $j < \alpha$. Then the composite $C_0 \to \varinjlim_{i<\alpha} C_i$ is isomorphic to $\sum_i A_i \to \sum_i B_i$. ■

2 Weak factorization Systems

Definition 2.1 *(Joyal [2]) A weak factorization system in a category $\mathcal{E}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ satisfying*

(i)$\mathcal{B} = \mathcal{A}^{\pitchfork}$ and $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$

(ii)Every map $f : X \to Y$ in $\mathcal{E}$ may be factored as $f = pi$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$.

Notice that each class of a weak factorization system determines the other. Also, $\mathcal{A} \bigcap \mathcal{B} = Iso(\mathcal{E})$ since a map having the LLP with respect to itself is an isomorphism.

Proposition 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of maps in a category $\mathcal{E}$. Suppose the following conditions are satisfied*

(i) $\mathcal{A} \pitchfork \mathcal{B}$

(ii) Every map $f : X \to Y$ in $\mathcal{E}$ may be factored as $f = pi$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$.

(iii) $\mathcal{A}$ and $\mathcal{B}$ are closed under retracts.

Then $(\mathcal{A}, \mathcal{B})$ is a weak factorization system in $\mathcal{E}$.

Proof: We use an important technique known as "the retract argument" to show, say, ${}^{\pitchfork}\mathcal{B} \subseteq \mathcal{A}$. So let $u \in {}^{\pitchfork}\mathcal{B}$ and factor u as $u = pv$ with $p \in \mathcal{B}$, and $v \in \mathcal{A}$. Then the square

$$\begin{array}{ccc} A & \xrightarrow{v} & C \\ {\scriptstyle u}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[id]{} & B \end{array}$$

has a diagonal filler d. Thus, u is a retract of $v \in \mathcal{A}$, so $u \in \mathcal{A}$. $\mathcal{A}^{\pitchfork} \subseteq \mathcal{B}$ is similar. ■

Definition 2.2 *Let κ be a cardinal. An ordinal α is said to be κ-filtered if it is a limit ordinal, and if $S \subseteq \alpha$ and $\mid S \mid \leq \kappa$ then $sup(S) < \alpha$.*

Definition 2.3 *Let $\mathcal{E}$ be cocomplete and $A \in \mathcal{E}$. A is called κ-small if for all κ-filtered ordinals α and all α-sequences $X : \alpha \to \mathcal{E}$, the canonical map*

$$\varinjlim_{i<\alpha} \mathcal{E}(A, X_i) \to \mathcal{E}(A, \varinjlim_{i<\alpha} X_i)$$

is a bijection. A is called small if it is κ-small for some κ, and finite if it is κ-small for a finite cardinal κ, in which case $\mathcal{E}(A, \ \ )$ commutes with colimits over any limit ordinal.

Exercise: Show any set is small.

The technique used in the following result is called the "small object argument", so named by Quillen. It is ubiquitous in homotopy theory, see Hovey [1] or Joyal [2].

Theorem 2.1 *Let $\mathcal{E}$ be cocomplete and $\mathcal{M}$ a set of maps in $\mathcal{E}$ with small domains. Then $(\overline{\mathcal{M}}, \mathcal{M}^{\pitchfork})$ is a weak factorization system in $\mathcal{E}$. Furthermore, the factorization of a map f can be chosen to be functorial in f.*

Proof: Choose a cardinal κ such that the domain of each element of $\mathcal{M}$ is κ-small, and let α be a κ-filtered ordinal. Let $f : X \to Y$ be a map in $\mathcal{E}$. We will define, by transfinite induction, a functorial α-sequence $E : \alpha \to \mathcal{E}$ and a natural transformation $p : E \to Y$ factoring f. We begin by setting $E_0 = X$, and $p_0 = f$. If we have defined E_i and p_i for all $i < j$ where j is a limit ordinal, set $E_j = \varinjlim_{i<j} E_i$ and let p_j be the map induced by the p_i. Given E_i and p_i, we define E_{i+1} and p_{i+1} as follows. Let S be the set of all commutative squares of the form

$$\begin{array}{ccc} A & \longrightarrow & E_i \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p_i} \\ B & \longrightarrow & Y \end{array}$$

with u in $\mathcal{M}$. For $s \in S$ denote by $u_s : A_s \to B_s$ the corresponding map in $\mathcal{M}$ and define E_{i+1} as a pushout

$$\begin{array}{ccc} \sum_{s\in S} A_s & \longrightarrow & E_i \\ \downarrow & & \downarrow \\ \sum_{s\in S} B_s & \longrightarrow & E_{i+1} \end{array}$$

Set $p_{i+1} : E_{i+1} \to Y$ equal to the map induced by the maps $B_s \to Y$ and $p_i : E_i \to Y$. Let $\overline{E} = \varinjlim_{i<\alpha} E_i$ and let $\overline{p} : \overline{E} \to Y$ be the map induced by the p_i. The canonical map $X \to \overline{E}$ is in $\overline{\mathcal{M}}$, and we want to show $p \in \mathcal{M}^{\pitchfork}$, so let

$$\begin{array}{ccc} A & \xrightarrow{a} & \overline{E} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ B & \xrightarrow[b]{} & Y \end{array}$$

be commutative with u in $\mathcal{M}$. Since A is κ-small, a factors as $A \to E_i \to \overline{E}$ for some $a_i : A \to E_i$. By construction then there is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{a_i} & E_i \\ {\scriptstyle u}\downarrow & & \downarrow \\ B & \xrightarrow[b_{i+1}]{} & E_{i+1} \end{array}$$

such that $p_{i+1}b_{i+1} = b$. Then the composite of b_{i+1} and the canonical map $E_{i+1} \to \overline{E}$ gives the required diagonal filler. ∎

3 Some Examples

The *simplicial category* $\mathbf{\Delta}$ has objects $[n] = \{0, \ldots, n\}$ for $n \geq 0$ a nonnegative integer. A map $\alpha : [n] \to [m]$ is an order preserving function. A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to Set$. To conform with traditional notation, when $\alpha : [n] \to [m]$ we write $\alpha^* : X_m \to X_n$ instead of $X_\alpha : X[m] \to X[n]$. The elements of X_n are called the $n - simplices$ of X.

Remark: An $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ can be decomposed uniquely as $\alpha = \varepsilon\eta$, where $\varepsilon : [p] \to [m]$ is injective, and $\eta : [n] \to [p]$ is surjective. Moreover, if $\varepsilon^i : [n-1] \to [n]$ is the injection which skips the value $i \in [n]$, and $\eta^j : [n+1] \to [n]$ is the surjection covering $j \in [n]$ twice, then $\varepsilon = \varepsilon^{i_s} \ldots \varepsilon^{i_1}$ and $\eta = \eta^{j_t} \ldots \eta^{j_1}$ where $m \geq i_s > \ldots > i_1 \geq 0$, and $0 \leq j_t < \ldots < j_1 < n$ and $m = n - t + s$. The decomposition is unique, the i's in $[m]$ being the values not taken by α, and the j's being the elements of $[m]$ such that $\alpha(j) = \alpha(j+1)$. The ε^i and η^j satisfy the following relations:

$$\begin{array}{rcll} \varepsilon^j\varepsilon^i & = & \varepsilon^i\varepsilon^{j-1} & i < j \\ \eta^j\eta^i & = & \eta^i\eta^{j+1} & i \leq j \\ \eta^j\varepsilon^i & = & \begin{cases} \varepsilon^i\eta^{j-1} & i < j \\ id & i = j \text{ or } i = j+1 \\ \varepsilon^{i-1}\eta^j & i > j+1 \end{cases} & \end{array}$$

Thus, a simplicial set X can be considered to be a graded set $(X_n)_{n\geq 0}$ together with functions $d^i = \varepsilon^{i*}$ and $s^j = \eta^{j*}$ satisfying relations dual to those satisfied by the ε^i's and the η^j's. Namely,

$$\begin{array}{rcll} d^i d^j & = & d^{j-1} d^i & i < j \\ s^i s^j & = & s^{j+1} s^i & i \leq j \\ d^i s^j & = & \begin{cases} s^{j-1} d^i & i < j \\ id & i = j \text{ or } i = j+1 \\ s^j d^{i-1} & i > j+1 \end{cases} & \end{array}$$

This point of view is frequently adopted in the literature.

The category of simplicial sets is $[\mathbf{\Delta}^{op}, Set]$, which we often denote simply by $\mathbf{S}$. Again for traditional reasons, the representable functor $\mathbf{\Delta}(\ \ , [n])$ is written $\Delta[n]$ and is called the *standard (combinatorial) n-simplex*. Conforming to this usage, we use $\Delta : \mathbf{\Delta} \to \mathbf{S}$ for the Yoneda functor, though if $\alpha : [n] \to [m]$, we write simply $\alpha : \Delta[n] \to \Delta[m]$ instead of $\Delta\alpha$.

The *boundary* of $\Delta[n]$ is a simplicial (n-1)-sphere $\partial\Delta[n]$ defined by

$$\partial\Delta[n]_m = \{\alpha : [m] \to [n] | \alpha \text{ is not surjective}\}$$

$\partial\Delta[n]$ can also be described as the union of the (n-1)-faces of $\Delta[n]$. That is,

$$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial^i \Delta[n]$$

where $\Delta\partial^i[n] = im(\varepsilon^i : \Delta[n-1] \to \Delta[n])$. Recall that the union is calculated pointwise, as is any colimit (or limit) in $[\mathbf{\Delta}^{op}, Set]$. By convention, we set $\partial\Delta[0] = 0$.

The k^{th} *horn* of $\Delta[n]$ is

$$\Lambda^k[n] = \bigcup_{i \neq k} \partial^i \Delta[n]$$

$\Delta[n]$, $\partial\Delta[n]$ and $\Lambda^k[n]$ are finite, and any simplicial set is small. We write $\mathcal{S}$ for the set of sphere inclusions $\{\partial\Delta[n] \to \Delta[n] \mid n \geq 0\}$, and $\mathcal{H}$ for the set of horn inclusions $\{\Lambda^k[n] \to \Delta[n] \mid 0 \leq k \leq n, n \geq 1\}$. Then $\overline{\mathcal{S}}$ is the class of all monomorphisms of simplicial sets, and $(\overline{\mathcal{S}}, \mathcal{S}^{\pitchfork})$ is a weak factorization system in $\mathbf{S}$. A member of $\mathcal{S}^{\pitchfork}$ is called a *trivial fibration.* $\overline{\mathcal{H}}$ is called the class of *anodyne extensions* and $(\overline{\mathcal{H}}, \mathcal{H}^{\pitchfork})$ is another weak factorization system in $\mathbf{S}$. The members of $\mathcal{H}^{\pitchfork}$ are called *Kan fibrations.* Notice that the first weak factorization system is easy to obtain in any topos, since a map which has the RLP with respect to the monomorphisms is just an injective over its base, and any map can be embedded in an injective. It is, however, also the case that the monomorphisms in any Grothendieck topos have a set of generators, so this weak factorization system can also be obtained in any Grothendieck topos as above.

4 Quillen Model Structures

Let $\mathcal{E}$ be a finitely complete and cocomplete category.

Definition 4.1 *A Quillen model structure on $\mathcal{E}$ consists of three classes of maps $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ in $\mathcal{E}$ called fibrations, cofibrations, and weak equivalences respectively. These are required to satisfy*

(i) $\mathcal{W}$ has the "three for two" property: if any two of f, g or fg is in $\mathcal{W}$, so is the third.

(ii) $(\mathcal{C} \bigcap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \bigcap \mathcal{W})$ are weak factorization systems in $\mathcal{E}$.

The usual way of expressing the conditions on $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is

(i) $\mathcal{W}$ satisfies "three for two".

(ii) $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ are closed under retracts.

(iii) $\mathcal{C} \bigcap \mathcal{W} \pitchfork \mathcal{F}$ and $\mathcal{C} \pitchfork \mathcal{F} \bigcap \mathcal{W}$.

(iv) Every map f in $\mathcal{E}$ can be factored as $f = pi$ in two ways: one in which $i \in \mathcal{C} \bigcap \mathcal{W}$ and $p \in \mathcal{F}$ and one in which $i \in \mathcal{C}$ and $p \in \mathcal{F} \bigcap \mathcal{W}$.

As we have seen, these are the same except for $\mathcal{W}$ closed under retracts, and this follows if $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ satisfy (i) and (ii) above.

An important aspect of the axioms for a Quillen model structure is that they are self dual. That is, if $\mathcal{E}$ carrys a Quillen model structure then so does $\mathcal{E}^{op}$. The cofibrations of $\mathcal{E}^{op}$ are the fibrations of $\mathcal{E}$, and the fibrations of $\mathcal{E}^{op}$ are the cofibrations of $\mathcal{E}$. The weak equivalences of $\mathcal{E}^{op}$ are the weak equivalences of $\mathcal{E}$. Thus, we need only prove results about cofibrations, say, since the dual results are automatic.

If $\mathcal{E}$ is a Quillen model category, we say an object $X \in \mathcal{E}$ is *cofibrant* if $0 \to X$ is a cofibration, and *fibrant* if $X \to 1$ is a fibration.

Examples:
(1) Three trivial ones. For any $\mathcal{E}$, take for $\mathcal{F}$, $\mathcal{C}$ or $\mathcal{W}$ the isomorphisms and let the other two classes be all maps.

(2) Take $\mathcal{E} = \mathbf{S}$. Let $X \in \mathbf{S}$, and define the *set of connected components* of X to be the set $\pi_0(X)$ in the coequalizer

$$X_1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X)$$

For $X, Y \in \mathbf{S}$, put $\mathbf{S}^{\pi_0}(X, Y) = \pi_0(Y^X)$. It is easy to see that π_0 preserves finite products, so we obtain a composition law

$$\mathbf{S}^{\pi_0}(Y, Z) \times \mathbf{S}^{\pi_0}(X, Y) \to \mathbf{S}^{\pi_0}(X, Z)$$

by applying π_0 to the composition map $Z^Y \times Y^X \to Z^X$. The resulting category $\mathbf{S}^{\pi_0}$ is called the *category of simplicial sets and homotopy classes of maps.* A map $f : X \to Y$ is called a *homotopy equivalence* if it becomes an isomorphism in $\mathbf{S}^{\pi_0}$. f is called a *weak homotopy equivalence* if

$$\mathbf{S}^{\pi_0}(f, Z) : \mathbf{S}^{\pi_0}(Y, Z) \to \mathbf{S}^{\pi_0}(X, Z)$$

is a bijection for each Kan complex Z. (Z is a Kan complex when $Z \to 1$ is a Kan fibration.) Let $\mathcal{W}$ be the class of weak homotopy equivalences. Take $\mathcal{C}$ to be the class of all monomorphisms, and let $\mathcal{F}$ be the class of Kan fibrations. Then $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ is a Quillen model structure on $\mathbf{S}$. We call it the *classical* model structure on $\mathbf{S}$.

This is probably the most central example, and it is difficult to establish. (i) is easy, and we have the two weak factorization systems. *But*, we do not know that $\mathcal{C} \bigcap \mathcal{W}$ is the class of anodyne extensions, and we do not know $\mathcal{F} \bigcap \mathcal{W}$ is the class of trivial fibrations. If we knew, say, the latter, then the former is easy. So that leaves us only one thing to show. This takes roughly the first four chapters of the book André Joyal and I are writing.

(3) Take $\mathcal{E} = Top_c$, the category of compactly generated spaces. The class $\mathcal{W}$ of weak equivalences consists of maps $f : X \to Y$ such that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for $n \geq 1$ and $x \in X$, $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism. The class $\mathcal{F}$ is the class of Serre fibrations, i.e. maps $p : E \to X$ with the *covering homotopy property* (CHP) for each n-simplex Δ_n, $n \geq 0$. This means that if $h : \Delta_n \times I \to X$ is a homotopy ($I = [0, 1]$), and $f : \Delta_n \to E$ is such that $pf = h_0$, then there is a "covering homotopy" $\bar{h} : \Delta_n \times I \to E$ such that $\bar{h}_0 = f$, and $p\bar{h} = h$. $\mathcal{C} = {}^{\pitchfork}(\mathcal{F} \bigcap \mathcal{W})$.

Once example (2) is established, this one is fairly easy.

(4)(Joyal 1984) Let $\mathcal{E}$ be a Grothendieck topos, and consider $\mathbf{S}(\mathcal{E})$ - the category of simplicial sheaves. Given $X \in \mathbf{S}(\mathcal{E})$ we can construct homotopy sheaves $\pi_n(X) \to X_0$ and we say $f : X \to Y$ is a weak equivalence if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\pi_n(f)} & \pi_n(Y) \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array}$$

is a pullback for $n \geq 1$. $\mathcal{C}$ is the class of monomorphisms, and $\mathcal{F} = (\mathcal{C} \bigcap \mathcal{W})^{\pitchfork}$. This example is very fundamental. It was the beginning of the homotopy theory of sheaves. The main difficulty in establishing it is showing that $\mathcal{C} \bigcap \mathcal{W}$ has a set of generators.

(5) (Joyal - T) Again, let $\mathcal{E}$ be a Grothendieck topos, and denote by $Gpd(\mathcal{E})$ the category of groupoids in $\mathcal{E}$. $\mathcal{W}$ is the class of *categorical equicalences*, where a

categorical equivalence is a functor $f : \boldsymbol{G} \to \boldsymbol{H}$ which is full and faithful and essentially surjective in the internal sense, i.e.

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow[f_0 \times f_0]{} & H_0 \times H_0 \end{array}$$

is a pullback, and in

$$\begin{array}{ccc} H^* & \longrightarrow & H_1 \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times H_0 & \xrightarrow[f_0 \times id]{} & H_0 \times H_0 \\ {\scriptstyle \pi_2}\downarrow & & \\ H_0 & & \end{array}$$

the composite $H^* \to H_0$ is surjective. $\mathcal{C}$ is the class of functors injective on objects, and $\mathcal{F} = (\mathcal{C} \bigcap \mathcal{W})^{\pitchfork}$. The fibrant objects in this structure are called *strong stacks*, since they represent a strenghtening of the notion of stack due to Grothendieck and Giraud.

Definition 4.2 *Let $(\mathcal{F}, \mathcal{C}, \mathcal{W})$ be a Quillen model structure on a category $\mathcal{E}$. We say the model structure is left proper if the pushout of a weak equivalence along a cofibration is a weak equivalence, and right proper if the pullback of a weak equivalence along a fibration is a weak equivalence. The model structure is called proper if it is left proper and right proper.*

All of the examples above are proper.

In some sense the most basic ingredient in a model structure is the class of weak equivalences. The cofibrations and fibrations are sort of auxilary notions which enable us to keep the weak equivalences under control. In fact it is not unusual for a category to have several model structures with the same weak equivalences, but different fibrations and cofibrations. For example, if $\boldsymbol{A}$ is a small category there are (at least) two model structures on $\mathbf{S}^{\boldsymbol{A}}$ for the obvious notion of pointwise weak equivalence. One in which the fibrations are pointwise, and one in which the cofibrations are pointwise.

Since the weak equivalences are "the isomorphisms of homotopy theory", we would like to "pass to homotopy" and have them become real isomorphisms. Thus, let $\mathcal{E}$ be a finitely complete and cocomplete category equiped with a model srtucture $(\mathcal{F}, \mathcal{C}, \mathcal{W})$. The *homotopy category* of $\mathcal{E}$ is

$$Ho(\mathcal{E}) = \mathcal{E}[\mathcal{W}^{-1}]$$

The problem with this is that, apriori, $Ho(\mathcal{E})$ may not be locally small, i.e. the maps between two objects may not form a set. In fact, in the presence of a model structure, this problem does not arise. We sketch how this works, refering to, say, Hovey [1] for details.

Definition 4.3 *Let $f, g : X \to Y$ be two maps of $\mathcal{E}$.*

(i) A cylinder object for X is a factorization of the codiagonal $\nabla : X + X \to X$ into a cofibration $(i_0, i_1) : X+X \to X'$ followed by a weak equivalence $s : X' \to X$.

(ii) A path object for Y is a factorization of the diagonal $\Delta : Y \to Y \times Y$ as a weak equivalence $t : Y \to Y'$ followed by a fibration $(p_0, p_1) : Y' \to Y \times Y$.

(iii) A left homotopy from f to g is a map $h : X' \to Y$ for some cylinder object X' for X such that $hi_0 = f$ and $hi_1 = g$. We say f and g are left homotopic if there is a left homotopy from f to g, and we write $f \overset{l}{\sim} g$ for this relation.

(iv) A right homotopy from f to g is a map $k : X \to Y'$ for some path object Y' for Y such that $p_0 k = f$ and $p_1 k = g$. We say f and g are right homotopic if there is a right homotopy from f to g, and we write $f \overset{r}{\sim} g$ for this relation.

(v) We say f and g are homotopic if they are both left homotopic and right homotopic, and we write $f \sim g$ for this relation.

(vi) We say f is a homotopy equivalence if there is a map $f' : Y \to X$ such that $ff' \sim id_Y$ and $f'f \sim id_X$.

We denote by $\mathcal{E}_c$, $\mathcal{E}_f$ and $\mathcal{E}_{cf}$ the full subcategories of cofibrant, fibrant, or cofibrant and fibrant objects, respectively, of $\mathcal{E}$.

Proposition 4.1 *Let A be a cofibrant object of $\mathcal{E}$ and X a fibrant object. Then on $\mathcal{E}(A, X)$ the left and right homotopy relations coincide, and they are equivalence relations which are compatible with composition.*

Proposition 4.2 *A map of $\mathcal{E}_{cf}$ is a weak equivalence iff it is a homotopy equivalence.*

Thus the canonical functor $\mathcal{E}_{cf} \to \mathcal{E}_{cf}/\sim$ inverts the weak equivalences.

Proposition 4.3 *The canonical functor $\mathcal{E}_{cf} \to \mathcal{E}_{cf}/\sim$ has the same universal property as the functor $\mathcal{E}_{cf} \to Ho(\mathcal{E}_{cf}) = \mathcal{E}_{cf}[\mathcal{W}_{cf}^{-1}]$, so there is an isomorphism of categories $\mathcal{E}_{cf}/\sim \to Ho(\mathcal{E}_{cf})$. In particular, $Ho(\mathcal{E}_{cf})$ is locally small.*

Proposition 4.4 *The inclusion $\mathcal{E}_{cf} \to \mathcal{E}$ induces an equivalence of categories $Ho(\mathcal{E}_{cf}) \to Ho(\mathcal{E})$.*

Proposition 4.5 *The canonical functor $\mathcal{E} \to Ho(\mathcal{E})$ identifies left or right homotopic maps, and if a map f becomes an isomorphism in $Ho(\mathcal{E})$ then f is a weak equivalence.*

5 Quillen Functors and Equivalences

Let $\mathcal{U}$ and $\mathcal{V}$ be model categories. A functor $F : \mathcal{U} \to \mathcal{V}$ is called a *left Quillen functor* if F preserves cofibrations and acyclic cofibrations, i.e. cofibration weak equivalences. By a result of Ken Brown's, F then preserves any weak equivalence between cofibrant objects. Dually, $G : \mathcal{V} \to \mathcal{U}$ is called a *right Quillen functor* if G preserves fibrations and acyclic fibrations. G then preserves any weak equivalence between fibrant objects. If $F : \mathcal{U} \longleftrightarrow \mathcal{V} : G$, then F is a left Quillen functor iff G is a right Quillen functor, in which case we call the adjoint pair (F, G) a *Quillen pair.*

A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ has a *left derived functor*

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

computed as follows: For $A \in \mathcal{U}$ choose a *cofibrant replacement* $\lambda_A : LA \to A$, i.e. an acyclic fibration λ_A with LA cofibrant. Then if $u : A \to B$, we can choose a map $L(u) : LA \to LB$ such that

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_A} & A \\ {\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u} \\ LB & \xrightarrow[\lambda_B]{} & B \end{array}$$

commutes. Then

$$F^L([u]) = [FL(u)] : FLA \to FLB$$

Dually, a right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ has a *right derived functor*

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

Furthermore, if $F : \mathcal{U} \longleftrightarrow \mathcal{V} : G$ is a Quillen pair, then

$$F^L : Ho(\mathcal{U}) \longleftrightarrow Ho(\mathcal{V}) : G^R$$

If A is cofibrant, the unit $A \to G^R F^L A$ of the above adjunction is given by the composite $A \to GFA \to GRFA$ where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the counit $F^L G^R X \to X$ is given by the composite $FLGX \to FGX \to X$ where $LGX \to GX$ is a cofibrant replacement of GX.

A Quillen pair (F, G) is said to be a *Quillen equivalence* if (F^L, G^R) is an equivalence of categories.

Theorem 5.1 *(Hovey [1]) Suppose $F : \mathcal{U} \longleftrightarrow \mathcal{V} : G$ is a Quillen pair. Then the following are equivalent.*

(i)(F, G) is a Quillen equivalence

(ii)F reflects weak equivalences between cofibrant objects, and for each fibrant $X \in \mathcal{V}$ the counit $F^L G^R X \to X$ is a weak equivalence.

(iii))G reflects weak equivalences between fibrant objects, and for each cofibrant $A \in \mathcal{U}$ the unit $A \to G^R F^L A$ is a weak equivalence.

The existence of a Quillen equivalence between two model categories means that the two model categories should be viewed as alternate models of the *same* homotopy theory.

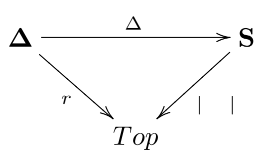
Example:
Geometrically, an *n-simplex* is the convex closure of $n+1$ points in general position in a euclidean space of dimension at least n. The *standard, geometric n-simplex* Δ_n is the convex closure of the standard basis $e_0, \ldots, e_n$ of $\mathbf{R}^{n+1}$. Thus, the points of Δ_n consists of all combinations

$$p = \sum_{i=0}^{n} t_i e_i$$

with $t_i \geq 0$, and $\sum_{i=0}^{n} t_i = 1$. We can identify the elements of $[n]$ with the vertices $e_0, \ldots, e_n$ of Δ_n. In this way a map $\alpha : [n] \to [m]$ can be linearly extended to a map $\Delta_\alpha : \Delta_n \to \Delta_m$. That is,

$$\Delta_\alpha(p) = \sum_{i=0}^{n} t_i e_{\alpha(i)}$$

Clearly, this defines a functor $r : \mathbf{\Delta} \to Top$. The functor r can be extended to a functor $|\quad| : \mathbf{S} \to Top$, called the *geometric realization*. The functor $|\quad|$ is determined by the commutative triangle



where

$$|X| = \varinjlim_{\Delta[n] \to X} \Delta_n$$

$|\quad|$ has a right adjoint $\mathcal{S}$. For any topological space T, $\mathcal{S}T$ is the *singular complex* of T

$$(\mathcal{S}T)_n = Top(\Delta_n, T)$$

Since a left adjoint preserves colimits , we see that the geometric realization $|\quad| : \mathbf{S} \to Top$ is colimit preserving. A consequence of this is that $|X|$ is a CW-complex. Furthermore, if Top is replaced by Top_c - the category of compactly generated spaces - then $|\quad|$ is also left-exact, i.e. preserves all finite limits.

The adjoint pair $| \quad | : \mathbf{S} \longleftrightarrow Top : \mathcal{S}$ is a Quillen equivalence. Thus we should think of Top and $\mathbf{S}$ as being alternate models of the same homotopy theory, since any homotopy theoretic result in one model gives rise to a similar result in the other.

References

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[3] D.G. Quillen, *Homotopical Algebra*, Springer Lecture Notes in Mathematics 43 (1967).