ON THE RELATION BETWEEN CHURCH-STYLE TYPING AND CURRY-STYLE TYPING

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Pure Type Systems (PTSs)

Church-style

Pseudoterms:

$$M \longrightarrow v|c|(MM)|(\lambda v : M \,.\, M)|(\Pi v : M \,.\, M)$$

Specification: A triple $\lambda S = \lambda(\mathcal{S}, \mathcal{A}, \mathcal{R})$, where

1. $\mathcal{S}$: a set of constants called *sorts*

2. $\mathcal{A}$: a set of *axioms* of the form $c : s$

3. $\mathcal{R}$: a set of *rules* of the form (s_1, s_2, s_3)

A rule (s_1, s_2) means (s_1, s_2, s_2)

Contexts: Γ is $x_1 : A_1, \ldots, x_n : A_n$; all x_i are distinct, and x_i not free in $A_1, \ldots, A_{i-1}$

$\mathsf{dom}(\Gamma) = \{x_1, \ldots, x_n\}$

Typing judgments: $\Gamma \vdash M : B$

Rules:

(axiom) If $x : s \in \mathcal{A}$

$$\vdash \; c : s$$

(start) If $s \in \mathcal{S}$ and $x \notin \mathrm{dom}(\Gamma)$

$$\frac{\Gamma \;\vdash\; A : s}{\Gamma, x : A \;\vdash\; x : A}$$

(weakening) If $s \in \mathcal{S}$ and $x \notin \mathrm{dom}(\Gamma)$

$$\frac{\Gamma \;\vdash\; M : B \quad \Gamma \;\vdash\; A : s}{\Gamma, x : A \;\vdash\; M : B}$$

(product) If $(s_1, s_2, s_3) \in \mathcal{R}$ and $x \notin \mathrm{dom}(\Gamma)$

$$\frac{\Gamma \;\vdash\; A : s_1 \quad \Gamma, x : A \;\vdash\; B : s_2}{\Gamma \;\vdash\; (\Pi x : A \,.\, B) : s_3}$$

(application)

$$\frac{\Gamma \;\vdash\; M : (\Pi x : A \,.\, B) \quad \Gamma \;\vdash\; N : A}{\Gamma \;\vdash\; MN : [N/x]B}$$

(abstraction$_{\mathsf{Ch}}$) If $s \in \mathcal{S}$ and $x \notin \mathsf{dom}(\Gamma)$

$$\frac{\Gamma, x : A \ \vdash \ M : B \qquad \Gamma \ \vdash \ (\Pi x : A \,.\, B) : s}{\Gamma \ \vdash \ (\lambda x : A \,.\, M) : (\Pi x : A \,.\, B)}$$

(conversion) If $x \in \mathcal{S}$

$$\frac{\Gamma \ \vdash \ M : B \qquad \Gamma \ \vdash \ B' : s \qquad B =_{*} B'}{\Gamma \ \vdash \ M : B'}$$

Curry-style

Same as Church-style except:

Pseudoterms

$$M \longrightarrow v|c|(MM)|(\lambda v \,.\, M)|(\Pi v : M \,.\, M)$$

Rule (abstraction$_{\mathsf{Ch}}$) replaced by

(abstraction$_{\mathsf{Cu}}$) If $s \in \mathcal{S}$ and $x \notin \mathrm{dom}(\Gamma)$

$$\frac{\Gamma, x : A \ \vdash \ M : B \qquad \Gamma \ \vdash \ (\Pi x : A \,.\, B) : s}{\Gamma \ \vdash \ (\lambda x \,.\, M) : (\Pi x : A \,.\, B)}$$

Called $CS = C(\mathcal{S}, \mathcal{A}, \mathcal{R})$

Also called *domain-free PTS*

Church-style also called *domain-full PTS*

To interpret Church-style in Curry-style:

Make

$$(\lambda x : A \,.\, M)$$

an abbreviation for

$$\mathsf{Label} A(\lambda x \,.\, M)$$

Want

$$\mathsf{Label} A(\lambda x \,.\, M) N \rhd [N/x]M$$

To get this, define

$$\mathsf{Label} \equiv \lambda xyz \,.\, yz$$

Define: $-^{\mathsf{Cu}}$ from λS to CS by:

1. If x is a variable, $x^{\mathsf{Cu}} \equiv x$
2. If c is a constant, $c^{\mathsf{Cu}} \equiv c$
3. $(MN)^{\mathsf{Cu}} \equiv M^{\mathsf{Cu}} N^{\mathsf{Cu}}$
4. $(\lambda x : A \,.\, M)^{\mathsf{Cu}} \equiv \mathsf{Label} A^{\mathsf{Cu}} (\lambda x \,.\, M^{\mathsf{Cu}})$
5. $(\Pi x : A \,.\, B)^{\mathsf{Cu}} \equiv (\Pi x : A^{\mathsf{Cu}} \,.\, B^{\mathsf{Cu}})$

This translation preserves free variables, substitution, and reduction

For a pseudocontext

$$\Gamma \equiv x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$$

define

$$\Gamma^{\mathsf{Cu}} \equiv x_1 : A_1^{\mathsf{Cu}}, x_2 : A_2^{\mathsf{Cu}}, \ldots, x_n : A_n^{\mathsf{Cu}}$$

A *topsort* of a PTS is a sort s for which there is no sort s' such that $s : s' \in \mathcal{A}$

Given a PTS $\lambda S \equiv \lambda(\mathcal{S}_\lambda, \mathcal{A}_\lambda, \mathcal{R}_\lambda)$ in the Church-style, let CS be defined as $C(\mathcal{S}_C, \mathcal{A}_C, \mathcal{R}_C)$ where $\mathcal{S}_C$ is obtained from $\mathcal{S}_\lambda$, $\mathcal{A}_C$ from $\mathcal{A}_\lambda$, and $\mathcal{R}_C$ from $\mathcal{R}_\lambda$ as follows:

1. If there is a topsort s which occurs in the first position of a rule in $\mathcal{R}_\lambda$, then a new sort s^T is added to $\mathcal{S}_C$ and for each such topsort s an axiom $s : s^T$ is added to $\mathcal{A}_C$

2. If there is a rule $(s_1, s_2, s) \in \mathcal{R}_\lambda$ of λS, then (s', s, s) and (s, s, s) are added to $\mathcal{R}_C$ if they are not already present, where $s_1 : s' \in \mathcal{A}_C$

Theorem 1 *If*

$$\Gamma \vdash M : A$$

in λS, then

$$\Gamma^{\mathsf{Cu}} \vdash M^{\mathsf{Cu}} : A^{\mathsf{Cu}}$$

in CS. Furthermore, if no new rules need to be added to $\mathcal{R}_C$, then the converse is true.

Note that in all the systems of the λ-cube, new rules need to be added to those in $\mathcal{R}_\lambda$ to get $\mathcal{R}_C$, so for none of these systems is the converse true.

An example for which the converse is true is Luo's *Extended Calculus of Constructions*. For this system, which is an extension of a PTS, we have

$$\begin{array}{rcl}
\mathcal{S} & = & \{*\} \cup \{\square_i | i \in \mathbb{N}\} \\
\mathcal{A} & = & \{* : \square_0\} \cup \{\square_i : \square_{i+1} | i \in \mathbb{N}\} \\
\mathcal{R} & = & \{(*, *)\} \cup \{(\square_i, \square_j, \square_{\max i,j}) | i, j \in \mathbb{N}\} \\
 & & \cup \{(*, \square_i) | i \in \mathbb{N}\} \cup \{(\square_i, *, \square_i) | i \in \mathbb{N}\}
\end{array}$$

The reason we do not have to add a rule $(\square_i, *)$ to $\mathcal{R}_C$ is that this system has the property that

$$\begin{array}{rcl}
\Gamma \vdash A : * & \rightarrow & \Gamma \vdash A : \square_i \ \ (i \geq 0), \\
\Gamma \vdash A : \square_i & \rightarrow & \Gamma \vdash A : \square_j \ \ (j \geq i)
\end{array}$$

The proof of the theorem is by induction on the derivation of $\Gamma \vdash M : A$. The interesting case is for (abstraction), and the additions specified in clauses 1 and 2 of the definition of CS are used to show that Label has the type needed for this clause of the proof, which assures that if

$$\Gamma \vdash (\lambda x \,.\, M) : (\Pi x : A \,.\, B) \tag{1}$$

in CS, then

$$\Gamma \vdash \mathsf{Label} A (\lambda x \,.\, M) : (\Pi x : A \,.\, B) \tag{2}$$

is also derivable in CS.

Note that (1) follows from (2) by reduction

To interpret Curry-style in Church-style:

Add new type A to the Church-style systax so that

$$(\lambda x : \mathsf{A} \,.\, M)$$

interprets Curry-style

$$(\lambda x \,.\, M)$$

We need rules:

(AI) If B is any legal type,

$$\frac{\Gamma \ \vdash \ M : B}{\Gamma \ \vdash \ M : \mathsf{A}}$$

(Aλ)

$$\frac{\Gamma \ \vdash \ (\lambda x : B \,.\, M) : (\Pi x : B \,.\, C)}{\Gamma \ \vdash \ (\lambda x : \mathsf{A} \,.\, M) : (\Pi x : B \,.\, C)}$$

The function $-^{\mathsf{Ch}}$ from the Curry-style syntax to the Church-style syntax is defined by induction on the structure of the pseudoterms:

1. $x^{\mathsf{Ch}} \equiv x$
2. $c^{\mathsf{Ch}} \equiv c$
3. $(MN)^{\mathsf{Ch}} \equiv M^{\mathsf{Ch}} N^{\mathsf{Ch}}$
4. $(\lambda x \,.\, M)^{\mathsf{Ch}} \equiv (\lambda x : \mathsf{A} \,.\, M^{\mathsf{Ch}})$
5. $(\Pi x : B \,.\, C)^{\mathsf{Ch}} \equiv (\Pi x : B^{\mathsf{Ch}} \,.\, C^{\mathsf{Ch}})$

This translation preserves free variables, substitutions, and reduction

Given a PTS CS in the Curry-style, define the system $\lambda S'$ by first defining λS to be the Church-style PTS with the same specification as CS. Then add the two typing rules (AI) and (Aλ) given above.

Note that the system $\lambda S'$ is not a PTS

Theorem 2 *If*

$$\Gamma \vdash M : B$$

in CS, then

$$\Gamma^{\mathsf{Ch}} \vdash M^{\mathsf{Ch}} : B^{\mathsf{Ch}}$$

in $\lambda S'$, and conversely.

Open problems:

1. The above interpretation of the Curry-style system in a Church-style system leaves one problem open: how to deal with η-reduction. This is no problem in the Curry-style semantics, but in the Church-style semantics, CR fails for η-reduction, as the following example due to Nederpelt shows: Let x, y, and z be distinct variables. Then

$$\lambda x : y \,.\, (\lambda x : z \,.\, x)x \rhd_{\beta} \lambda x : y \,.\, x$$

and

$$\lambda x : y \,.\, (\lambda x : z \,.\, x)x \rhd_{\eta} \lambda x : z \,.\, x,$$

and the terms $\lambda x : y \,.\, x$ and $\lambda x : z \,.\, x$ are distinct terms in normal form.

In the Church-style syntax, we might consider extending β_{Ch}-reduction by adding the contraction scheme

$$(\lambda x : B \,.\, M) \rhd (\lambda x : \mathsf{A} \,.\, M). \qquad (3)$$

In conjunction with this extended reduction, the typing rule (Aλ) would preserve the Subject-Reduction Theorem. Furthermore, the contraction scheme (3) is the analogue for the Church-style syntax of a valid reduction in the Curry-style syntax. But would the Church-Rosser property hold for this reduction?

2. It is possible to carry out the interpretation of the Curry-style system in a Church-style interpretation using the type ω of the intersection type discipline instead of A. Then instead of Rule (AI), we have

   (ωI) If M is any pseudoterm of the Church-style semantics,

   $$\overline{\Gamma \vdash M : \omega}$$

   The normal form theorem fails for a system with this typing rule. In intersection type systems with this type, it can be proved that any term that has a type in which ω does not occur has a normal form. Would this be true here?

3. Is it possible to find pairs of systems of the form $\lambda S'$ and CS such that the functions $-^{\mathsf{Cu}}$ and $-^{\mathsf{Ch}}$ interpret each system in the other without further change?