Some Notes on (Double) Glueing

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1 Introduction

Glueing is a technique frequently applied in categorical logic and topos theory. We look at it from the point of view of finding new models for linear logic. Having reviewed the standard construction we introduce a more sophisticated version which is well-behaved with respect to involutions such as those of $*$-autonomous categories. The material covered here is contained in [HS03].

Definition 1 *A symmetric monoidal closed category* $\mathbf{C}$ *is* $*$**-autonomous** *if has an object* $\perp$ *such that the canonical arrows* $A \longrightarrow (A \multimap \perp) \multimap \perp$ *(the transposes of the evaluation maps) are isomorphisms.*

Note that every $*$-autonomous category is symmetric monoidal closed since for every object A, the functor $- \otimes A \colon \mathbf{C} \longrightarrow \mathbf{C}$ is left adjoint to $(A \otimes -^{\perp})^{\perp} \colon \mathbf{C} \longrightarrow \mathbf{C}$

Definition 2 *A* categorical model of intuitionistic linear logic *consists of a category which*

- *is symmetric monoidal closed;*
- *has finite products;*
- *is equipped with a linear exponential comonad*[1]*. .*

A model of classical linear logic is a model of intuitionistic logic with a strong duality.

Definition 3 *A* model for classical linear logic *consists of a category which*

- *is $*$-autonomous;*
- *has finite products and (so) finite coproducts;*
- *is equipped with a linear exponential comonad*[1] *and (so) a linear exponential monad.*

To generate a $*$-autonomous category from a symmetric monoidal closed one $\mathbf{C}$ one can take $\mathbf{C} \times \mathbf{C}^{\mathsf{op}}$ (sketched in [HS03]). More sophisticated versions are the *Chu* (see [Chu79, Bar91]) and *dialectica* (see [dP91, dP89]) constructions.

[1] For a definition of 'linear exponential comonad' see [Sch04]

2 Simple Glueing

2.1 In General

Suppose $L\colon \mathbf{C} \longrightarrow \mathbf{E}$ is a functor. The category $\mathbf{G} = \mathbf{G}(L)$ obtained by glueing along L is the comma category obtained from

$$\mathbf{E} \xrightarrow{\mathsf{id}} \mathbf{E} \xleftarrow{L} \mathbf{C}.$$

$\mathbf{G}$ can be described explicitly as follows.

- **Objects** $(R, U, (U \longrightarrow L(R))) = (U \longrightarrow L(R))$ of $\mathbf{G}$ consist of an object $R \in \mathbf{C}$, an object $U \in \mathbf{E}$ and a map $U \longrightarrow L(R)$ of $\mathbf{E}$.
- **Maps** from $(U \longrightarrow L(R))$ to $(V \longrightarrow L(S))$ in $\mathbf{G}$ are given by commuting diagrams in $\mathbf{E}$

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\phi"'] & L(R) \arrow[d, "L(f)"] \\ V \arrow[r] & L(S), \end{tikzcd}$$

  where $f\colon R \longrightarrow S$ in $\mathbf{C}$ and $\phi\colon U \longrightarrow V$ in $\mathbf{E}$.

Often we consider full subcategories of $\mathbf{G}$ which are obtained by restricting the structure maps $U \longrightarrow L(R)$ which occur, typically only allowing monomorphisms.

Suppose that $\mathbf{C}$ and $\mathbf{E}$ are symmetric monoidal and that L is a monoidal functor in the usual lax sense. In other words we have

$$I \xrightarrow{m_{\mathbf{I}}} L(\mathbf{I}) \text{ and } L(R) \otimes L(S) \xrightarrow{m_{R,S}} L(R \otimes S)$$

such that m is natural in R and S satisfying the usual unit, symmetry and associativity laws.

Proposition 2.1 *$\mathbf{G}$ is a symmetric monoidal category where the tensor of $(U \longrightarrow L(R))$ and $(V \longrightarrow L(S))$ is given by*

$$U \otimes V \longrightarrow L(R) \otimes L(S) \xrightarrow{m_{R,S}} L(R \otimes S)$$

and has unit $(\mathbf{I} \xrightarrow{m_{\mathbf{I}}} L(\mathbf{I}))$. The forgetful functors $\mathbf{G} \longrightarrow \mathbf{C}$ and $\mathbf{G} \longrightarrow \mathbf{E}$ then are strict monoidal functors.

Note that if $\mathbf{C}$ and $\mathbf{E}$ are symmetric monoidal closed then from m we obtain a natural transformation

$$L(S \multimap T) \longrightarrow L(S) \multimap L(T),$$

and we use m to denote this, too.

Proposition 2.2 *Suppose $\mathbf{C}$ and $\mathbf{E}$ are symmetric monoidal closed and $\mathbf{E}$ has pullbacks. Then $\mathbf{G}$ is symmetric monoidal closed and $\mathbf{G} \longrightarrow \mathbf{C}$ is a strict map of symmetric monoidal closed categories.*

Proposition 2.3 *(i) Suppose* $\mathbf{C}$ *has finite products and* $\mathbf{E}$ *has pullbacks. Then* $\mathbf{G}$ *has finite products and* $\mathbf{G} \longrightarrow \mathbf{C}$ *preserves them strictly. The terminal object is the identity* $L(\mathbf{1}) \xrightarrow{\mathsf{id}} L(\mathbf{1})$ *and the product of* $(U \longrightarrow L(R))$ *and* $(V \longrightarrow L(S))$ *is the morphism* $P \longrightarrow L(R \times S)$ *in the double pullback diagram*

$$\begin{tikzcd}
U \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d] \\
L(R) & L(R \times S) \arrow[l, "L(\pi_1)"] \arrow[r, "L(\pi_2)"'] & L(S).
\end{tikzcd}$$

(ii) Suppose $\mathbf{C}$ *and* $\mathbf{E}$ *have finite coproducts. Then so does* $\mathbf{G}$ *and both functors* $\mathbf{G} \longrightarrow \mathbf{C}$ *and* $\mathbf{G} \longrightarrow \mathbf{E}$ *preserve them strictly. The initial object is the unique* $(\mathbf{0} \longrightarrow L(\mathbf{0}))$*; and the coproduct of* $(U \longrightarrow L(R))$ *and* $(V \longrightarrow L(S))$ *is the obvious composite*

$$(U + V \longrightarrow L(R) + L(S) \xrightarrow[{[L(\mathsf{inl}), L(\mathsf{inr})]}]{} L(R + S).$$

Exponential structure. Provided both underlying categories have linear exponential comonads and the functor L is well-behaved with respect to those, $\mathbf{G}$ has a linear exponential comonad and $\mathbf{G} \longrightarrow \mathbf{C}$ preserves it strictly. We don't give details here.

Examples and applications: Logical relations are an example, and indecomposability and conservative extension results provide some applications. For another example take a category of vector spaces. Add a norm by specifying the unit ball with respect to it. Then a norm-non-increasing linear function is precisely a linear function that maps the unit ball of the source space into the unit ball of the target space. This gives rise to a symmetric monoidal closed category which includes a subcategory of normed spaces.

2.2 Along hom-functors

The general construction becomes more user-friendly when applied to a particularly useful example, glueing along hom-functors where we only allow monomorphisms as objects.

Suppose that $\mathbf{C}$ is a symmetric monoidal closed category. We consider glueing for the functor $\mathbf{C}(\mathbf{I}, -) \colon \mathbf{C} \longrightarrow \mathbf{Set}$, which is obviously monoidal. Restricting to objects which are monomorphisms $U \rightarrowtail \mathbf{C}(\mathbf{I}, R)$ we describe the glued category $\mathbf{G}(\mathbf{C})$ explicitly as follows:

- **Objects** $A = (R, U)$ of $\mathbf{G}(\mathbf{C})$ are given by an object R of $\mathbf{C}$ together with a set $U \subseteq \mathbf{C}(\mathbf{I}, R)$.
- **Maps** in $\mathbf{G}(\mathbf{C})$ from (R, U) to (S, V) are given by maps $f \colon R \longrightarrow S$ in $\mathbf{C}$ such that: for all $\mathbf{I} \xrightarrow{u} R$ in U, $\mathbf{I} \xrightarrow{u} R \xrightarrow{f} S$ is in V.

Proposition 2.4 *If* $\mathbf{C}$ *is a symmetric monoidal closed category then so is* $\mathbf{G}(\mathbf{C})$*, and the forgetful functor* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves the structure. The structure is defined as follows.*

- *The tensor unit is* $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_{\mathbf{I}}\})$*.*

- *The tensor product of* $A = (R, U)$ *and* $B = (S, V)$ *is*

$$\begin{array}{rl} & A \otimes B = (R \otimes S, U \otimes V) \\ \textit{where} & U \otimes V = \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \mid u \in U, v \in V\} \end{array}$$

- *The linear function space*

$$\begin{array}{rl} & A \multimap B = (R \multimap S, W) \\ \textit{where} & W = \{\mathbf{I} \xrightarrow{w} R \multimap S \mid \forall \mathbf{I} \xrightarrow{u} R \textit{ in } U.\ \mathbf{I} \xrightarrow{u} R \xrightarrow{\hat{w}} S \in V\} \end{array}$$

represents the hom-set $\mathbf{G}(\mathbf{C})(A, B)$ *and* $\hat{w}$ *is the obvious transpose of* w.

Proposition 2.5 *(i) If* $\mathbf{C}$ *has finite products then so has* $\mathbf{G}(\mathbf{C})$*; the functor* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$\begin{array}{l} A \times B = (R \times S, U \times V) \quad \textit{where} \\ U \times V = \{\langle u, v\rangle \colon \mathbf{I} \longrightarrow R \times S \mid u \in U, v \in V\}. \end{array}$$

The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}))$.

(ii) If $\mathbf{C}$ *has finite coproducts then so has* $\mathbf{G}(\mathbf{C})$*, and* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves them. They are given by*

$$\begin{array}{l} A + B = (R + S, U \oplus V) \quad \textit{where} \\ U \oplus V = \{\mathbf{I} \xrightarrow{u} R \xrightarrow{\mathsf{inl}} R + S \mid u \in U\} \cup{}^{2} \{\mathbf{I} \xrightarrow{v} S \xrightarrow{\mathsf{inr}} R + S \mid v \in V\} \end{array}$$

The unit for the coproduct is $(\mathbf{0}, \emptyset)$.

Exponential structure. If $\mathbf{C}$ has a linear exponential comonad then under some mild conditions we can define two linear exponential comonads on the glued category.

3 Double Glueing

Problem with 'simple' glueing: Even if $\mathbf{C}$ is $*$-autonomous there's no way to make $\mathbf{G}(\mathbf{C})$ $*$-autonomous—where would the duality come from? Here we only describe double glueing along hom-functors, but the construction works in general. See [HS03] for details.

Take $\mathbf{C}$ to be a category with an involution[3] $(-)^{\perp}$.

An object of the glued category (which by abuse of notation we again call) $\mathbf{G}(\mathbf{C})$ is given by an object R of $\mathbf{C}$ together with $U \subseteq \mathbf{C}(\mathbf{I}, R)$ and $X \subseteq \mathbf{C}(R, \perp) \cong \mathbf{C}(\mathbf{I}, R^{\perp})$. Making this identification of $\mathbf{C}(R, \perp)$ with $\mathbf{C}(\mathbf{I}, R^{\perp})$ enables us to define an involution

$$(R, U, X)^{\perp} = (R^{\perp}, X, U).$$

[2]We get union rather than sum here as in Proposition 2.3 because we have restricted to monomorphisms.

[3]In general, take a self-duality whose square is isomorphic to the identity functor, but here we prefer identities over isomorphisms for a clearer exhibition.

We make such identifications without further comment below.

The second component (which is co-variant) works just as it did for the simple case, and the third component (which is contra-variant) is determined by that and the intended involution.

We employ the following notation for generalized composition. If J is an arbitrary object in $\mathbf{C}$ then for $h\colon R \otimes S \longrightarrow J$ and $v\colon \mathbf{I} \longrightarrow S$, we define $\langle v|h\rangle_S\colon R \longrightarrow J$ to be

$$R \cong R \otimes \mathbf{I} \xrightarrow{\mathsf{id}_R \otimes v} R \otimes S \xrightarrow{h} J;$$

we can think of this as cutting on S. Provided with some $u\colon \mathbf{I} \longrightarrow R$ we can similarly define $\langle u|h\rangle_R\colon S \longrightarrow J$, this time cutting on R.

Proposition 3.1 *If $\mathbf{C}$ is $*$-autonomous then so is $\mathbf{G}(\mathbf{C})$, and the forgetful functor to $\mathbf{C}$ preserves the $*$-autonomous structure, which is as follows.*

- *The involution $(R, U, X)^\perp = (R^\perp, X, U)$.*
- *The tensor unit $\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \mathbf{C}(\mathbf{I}, \perp))$.*
- *the tensor product of $A = (R, U, X)$ and $B = (S, V, Y)$ is*

$$\begin{aligned}
A \otimes B &= (R \otimes S, U \otimes V, \mathbf{G}(\mathbf{C})(A, B^\perp)) \quad \textit{where}\\
U \otimes V &= \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S \mid u \in U, v \in V\} \quad \textit{and}\\
\mathbf{G}(\mathbf{C})(A, B^\perp) &= \{R \otimes S \xrightarrow{z} \perp \mid \forall \mathbf{I} \xrightarrow{u} R \in U.\ \langle u|z\rangle_R\colon S \longrightarrow \perp \in Y\\
&\qquad\qquad \forall \mathbf{I} \xrightarrow{v} S \in V.\ \langle v|z\rangle_S\colon R \longrightarrow \perp \in X\}.
\end{aligned}$$

Up to natural identification the last component is the set of maps in $\mathbf{G}(\mathbf{C})$ from (R, U, X) to $(S, V, Y)^\perp$, hence the notation.

Proposition 3.2 *If $\mathbf{C}$ has finite products then so has $\mathbf{G}(\mathbf{C})$, and the functor $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. They are given by*

$$\begin{aligned}
A \times B &= (R \times S, U \times V, X \oplus Y) \quad \textit{where}\\
U \times V &= \{\langle u, v\rangle\colon \mathbf{I} \longrightarrow R \times S \mid u \in U, v \in V\} \quad \textit{and}\\
X \oplus Y &= \{R \times S \xrightarrow{\pi_1} R \xrightarrow{x} \perp \mid x \in X\} \cup \{R \times S \xrightarrow{\pi_2} S \xrightarrow{y} \perp \mid y \in Y\}.
\end{aligned}$$

The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}), \emptyset)$.

If $\mathbf{C}$ has finite coproducts then so has $\mathbf{G}(\mathbf{C})$, and $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. They are given by

$$\begin{aligned}
A + B &= (R + S, U \oplus V, X + Y) \quad \textit{where}\\
X + Y &= \{[x, y]\colon R + S \longrightarrow \perp \mid x \in X, y \in Y\} \quad \textit{and}\\
U \oplus V &= \{\mathbf{I} \xrightarrow{u} R \xrightarrow{\mathsf{inl}} R + S \mid u \in U\} \cup \{\mathbf{I} \xrightarrow{v} S \xrightarrow{\mathsf{inr}} R + S \mid v \in V\}
\end{aligned}$$

The unit for the coproduct is $(\mathbf{0}, \emptyset, \mathbf{C}(\mathbf{0}, \perp))$.

Exponential structure. As before we can define a linear exponential comonad under some fairly mild conditions.

Examples and Applications. Double glueing allows us to define a notion of logical relation for $*$-autonomous categories, for example Loader's category of linear logical predicates [Loa94a, Loa94b] is a double glued category.

In the example of finite dimensional normed vector spaces we can obtain a $*$-autonomous category by double (rather than simple) glueing. Also double glueing allows the definition of fully complete models of fragments of linear logic ([Tan97] gives a number of examples for the multiplicative fragment, and [BHS] extends this technique to the additives). There is a formulation of 'process realizability' [Abr] which is based on double glueing, and there are plenty of other examples.

More sophistication. The double glueing construction allows us to cut down on morphisms by insisting that they preserve some extra structure that is put on objects. For some examples, the resulting category isn't quite as we'd like it to be.

For example, one might try to get Girard's coherence spaces [Gir87, Gir95]. A *coherence space* is given by a set R and a subset U of the finite powerset of R, $\mathcal{P}R$, the 'coherent sets'. Here every singleton is coherent, and every subset of a coherent set is coherent.

Given a set of coherent sets U we define another subset of $\mathcal{P}R$ by setting

$$U^\circ = \{x \subseteq_{\mathsf{f}} R \mid \forall u \in U, |u \cap x| \leq 1\}.$$

We find it convenient to make this part of the structure of a coherence space. Note that for $U \subseteq \mathcal{P}R$ U is a set of coherent sets if and only if $U = U^{\circ\circ}$. A morphism $(R, U, U^\circ) \longrightarrow (S, V, Y^\circ)$ is given by a relation

$$f\colon R \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; S$$

such that

$$\begin{array}{ccc}
u \in U & \qquad\qquad & f[y] = \{r \in R \mid \exists s \in v.\, (r, s) \in f\} \in U^\circ \\
\text{implies} & & \text{implies} \\
[u]f = \{s \in S \mid \exists r \in u.\, (r, s) \in f\} \in V & & y \in V^\circ.
\end{array}$$

This is really a double glued category, where we glue along the hom-functor for **Rel**. However, the categorical structure of the double glued category is not that for coherence spaces. Given (R, U, X) and (S, V, Y) their tensor product $(R \otimes S, U \otimes V, Z)$ has a second component,

$$U \otimes V \subseteq \mathcal{P}(R \otimes S)$$

which is not closed with respect to forming subsets, so this does not describe a coherence space. We need to apply a closure operation, forming $U \otimes V^{\circ\circ}$ instead of $U \otimes V$ in order to obtain a coherence space. There is a full subcategory of our glued category which is the category of coherence spaces—but note that the categorical structure is defined differently now.

In general one can use the notion of an **orthogonality** to create subcategories of a double glued category (when glueing along hom-functors) such that the chosen subsets (of the appropriate hom-sets) have various closure properties. Details are again found in [HS03]. Using this technique one can define such a subcategory of the double glued category sketched above which is equivalent to the standard category of finite dimensional normed vector spaces.

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