# A complete equational theory for Gaussian quantum circuits

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based on: arXiv:2401.07914 and arXiv:2403.10479

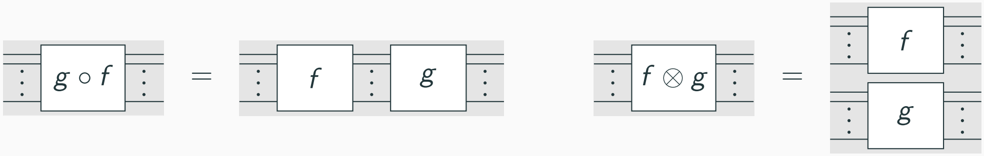
July 12, 2024

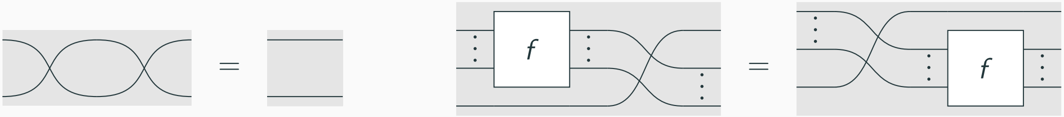
## Motivation

We seek to formulate aspects of mathematics and physics using string diagrams.

- **For mathematics**, this means giving a syntactic presentation, and exhibiting and equivalence to the semantics;
- **For physics**, often we must also construct the *semantics*.

A **prop** is a strict symmetric monoidal category generated by a single object...




A **compact prop** also allows for wires to be bent/unbent:

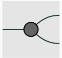
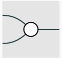
# Graphical linear algebra

# Affine matrices: generators

Given a field $\mathbb{K}$, finite dimensional affine transformations can be represented their **homogeneous coordinates matrices** ($T, S$ are matrices, $\vec{a}, \vec{b}$ are vectors):

$$\left[\begin{array}{c|c} T & \vec{a} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} S & \vec{b} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} TS & T\vec{b}+\vec{a} \\ \hline 0 & 1 \end{array}\right]$$

The prop of affine transformations between finite dimensional vector spaces is generated by the homogeneous coordinate matrices:

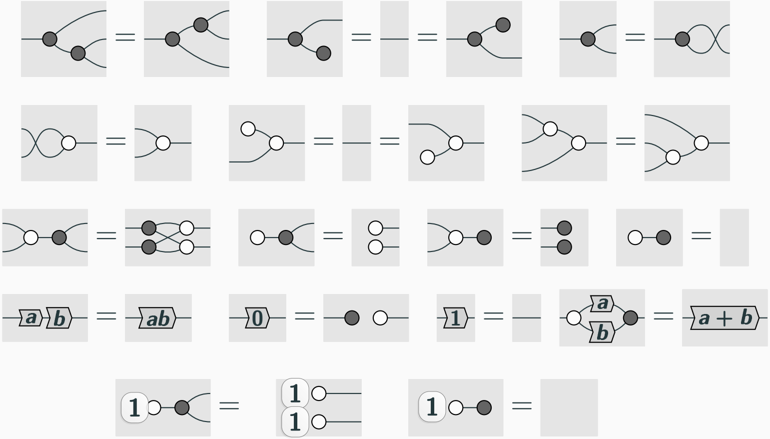
$[\![$  $]\!]$ $= \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ \hline & 1 \end{array}\right]$   $[\![$  $]\!]$ $= \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline & & 1 \end{array}\right]$   $[\![$  $]\!]$ $= \left[\begin{array}{c|c} a & 0 \\ \hline & 1 \end{array}\right]$



$[\![$  $]\!]$ $= \left[\begin{array}{c|c} * & * \\ \hline & 0 \end{array}\right]$   $[\![$  $]\!]$ $= \left[\begin{array}{c|c} * & 0 \\ \hline & 0 \end{array}\right]$   $[\![$  $]\!]$ $= \left[\begin{array}{c|c} * & 0 \\ \hline & 1 \end{array}\right]$

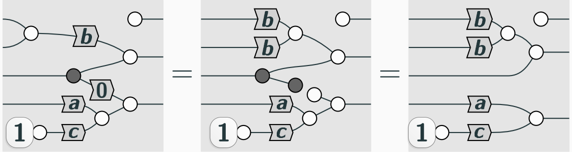
# Affine matrices: axioms

Modulo the equations:

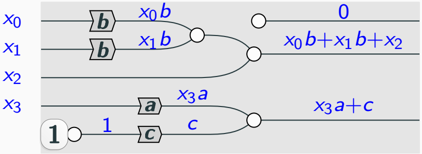


# Example of matrix multiplication

The following diagram can be simplified to a normal form:



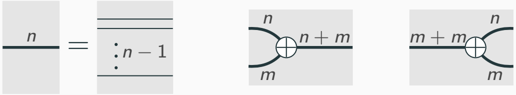

Following the paths from left to right gives us the homogeneous coordinate matrix:

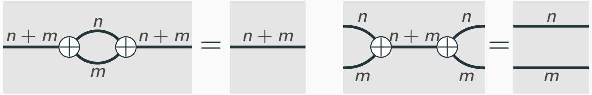



$$\rightsquigarrow \left[\begin{array}{cccc|c} 0&0&0&0&0\\ b&b&1&0&0\\ 0&0&0&a&c\\ \hline &&&&1\end{array}\right] \left[\begin{array}{c} x_0\\ x_1\\ x_2\\ x_3\\ \hline 1\end{array}\right] = \left[\begin{array}{c} 0\\ x_0 b + x_1 b + x_2\\ x_3 a + c\\ \hline 1\end{array}\right]$$

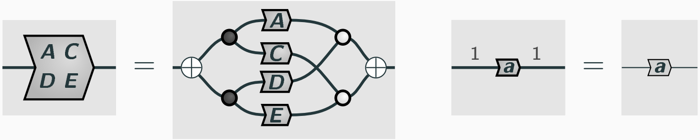
# Strictification and block matrices

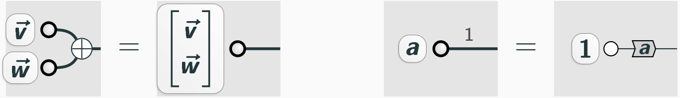
Every prop can be strictified to an $\mathbb{N}$-coloured prop:




This allows us to define block matrices/vectors diagrammatically:



# Affine relations (Bonchi et al. [Bon+19],Bonchi et al. [BSZ17])

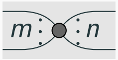
Given a field $\mathbb{K}$, the compact prop of $\mathbb{K}$-affine relations, $\mathsf{AffRel}_{\mathbb{K}}$, has:

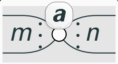
- **Morphisms** $n \to m$ are affine subspaces $S \subseteq \mathbb{K}^n \oplus \mathbb{K}^m$.
- **Composition** relational, for $S : n \to m$, $T : m \to k$

$$R \circ S := \{(\vec{x}, \vec{z}) \in \mathbb{K}^n \oplus \mathbb{K}^k \mid \exists \vec{y} \in \mathbb{K}^m : (\vec{x}, \vec{y}) \in S \quad \text{and} \quad (\vec{y}, \vec{z}) \in R\}$$

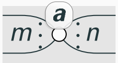
- **Symmetric monoidal structure** given by direct sum;
- **Compact structure** same as Rel.

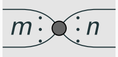
$\mathsf{AffRel}_{\mathbb{K}}$ is generated by the following relations, for all $a \in \mathbb{K}$:

$[\![$   $]\!] := \left\{ \left( \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}, \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \right) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; a \in \mathbb{K} \right\}$

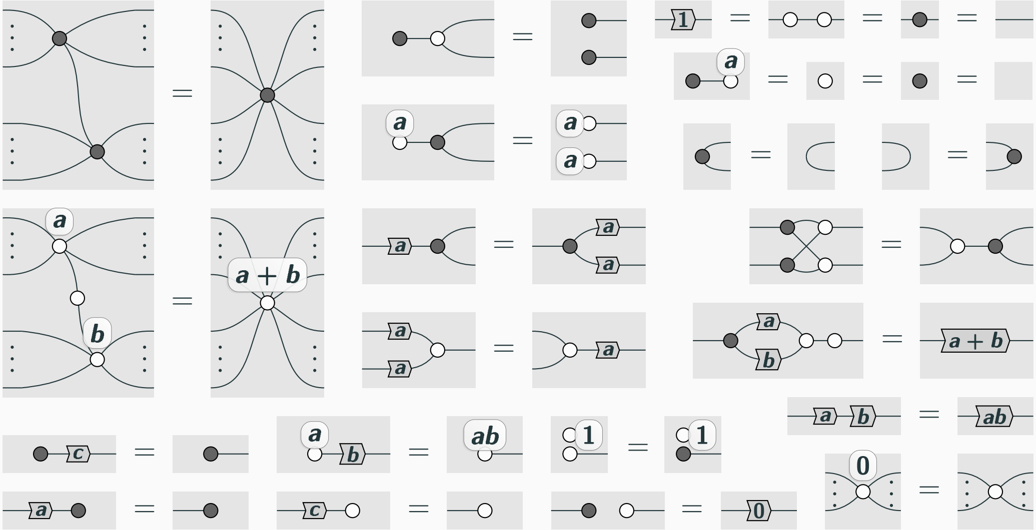
$[\![$   $]\!] := \left\{ (\vec{b}, \vec{c}) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; \sum_{j=0}^{n-1} b_j + \sum_{k=0}^{m-1} c_k = a \right\}$

$[\![$   $]\!] := \{(b, ab) \mid b \in \mathbb{K}\}$

Modulo, the “spiders” 

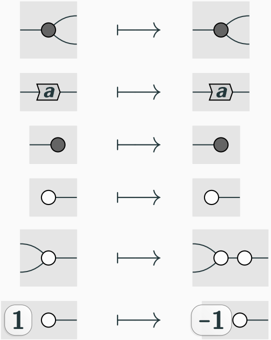
and 

being commutative, undirected and,




for all $a, b \in \mathbb{K}$, $c \in \mathbb{K}^{\times}$.

The embedding $\mathsf{AffMat}_{\mathbb{K}} \hookrightarrow \mathsf{AffRel}_{\mathbb{K}}$ taking an affine transformation $T : n \to m$ to it's graph $\{(\vec{x}, T\vec{x}) \mid \vec{x} \in \mathbb{K}^m\}$ sends:



# Classical mechanics and symplectic geometry

# Electrical circuits: current and voltage

The extensional behaviour of an electrical circuits is characterised by how it transforms current and voltage;

- **Ohm's law:** The voltage around the node in a circuit is equal to the current multiplied by the resistance.
- **Kirchhoff's current law:** The sum of currents flowing into a node is equal to the sum of currents flowing out of the node.

**Example**

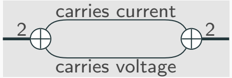
Given a linear resistor with resistance $r \in \mathbb{R}^{>0}$ on a wire with incoming current/votage $(z_0, x_0)$ and outgoing current/voltage $(z_1, x_1)$:

- by KCL, currents equalize: $z_0 = z_1$;
- by OL, the outgoing current becomes: $x_1 = x_0 + z_0 r$.

# String diagrams for electrical circuits, take I

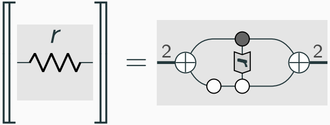
Following Baez et al. [BCR18] and Baez and Fong [BF18], we can represent electrical circuit components as real affine relations.

Using the string diagrams from Bonchi et al. [Bon+19], decompose a wire into a current and voltage

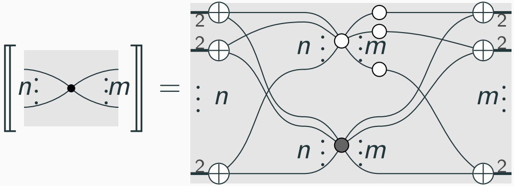



...the resistor is represented as follows:

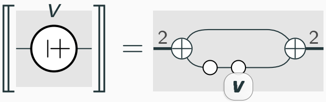
**Example**



**Example**
Ideal wire junctions *sum currents*, and *equalize voltages*:




**Example**
Constant voltage source *does nothing to current* and *adds to the voltage*:




What is the more conceptual picture?

# Symplectic geometry

Classical mechanical systems can be represented by the configurations of abstract **positions** $Z$ and **momenta** $X$:

| *Classical mechanics* | $Z$ | $dZ/dt$ | $X$ | $dX/dt$ |
|---|---|---|---|---|
| **Translation** | position | velocity | momentum | force |
| **Electronic** | charge | current | flux linkage | voltage |
| **Hydraulic** | volume | flow | pressure momentum | pressure |
| **Thermal** | entropy | entropy flow | temperature momentum | temperature |

For $n$-particles in Euclidean space, *the space of possible configurations of positions/momenta* $\mathbb{R}^{2n} \cong \mathbb{R}^n_Z \oplus \mathbb{R}^n_X$ is the **phase space**.

Table adapted from Smith [Smi93, page 23, table 2.1] and Baez and Fong [BF18]

# Affine Lagrangian subspaces

**Definition**
Two configurations $(\vec{z}, \vec{x}), (\vec{q}, \vec{p}) \in \mathbb{K}^{2n}$ of phase-space are **compatible** when:

$$\vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q} = 0$$

The bilinear map

$$\omega_n : \mathbb{K}^{2n} \oplus \mathbb{K}^{2n} \to \mathbb{K} \quad ((\vec{z}, \vec{x}), (\vec{q}, \vec{p})) \mapsto \vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q}$$

is a **symplectic form**, and the phase space $(\mathbb{K}^{2n}, \omega_n)$ is a **symplectic vector space**.

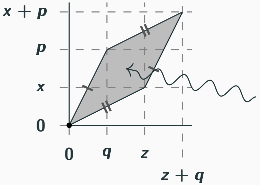
An **affine Lagrangian subspace** is a *maximally compatible* affine subspace of a symplectic vector space.

**Remark (Baez and Fong [BF18], Baez et al. [BCR18])**
*Resistors, voltages sources and junctions of wires are affine Lagrangian subspaces.*

# Geometric interpretation of compatibility

**Example**

In the phase-space of a single particle, $(\mathbb{K}^2, \omega_1)$, the symplectic form measures area:




$$\omega_1\left(\begin{bmatrix} z \\ x \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}\right)$$

Compatible points are colinear, so affine Lagrangian subspaces are lines.

An affine Lagrangian subspaces don't represent single particle; but an ensemble of particles *flowing along a trajectory*.

# Affine Lagrangian relations

**Definition (Guillemin and Sternberg [GS79],Weinstein [Wei82])**
The **compact prop of affine Lagrangian relations** $\mathsf{AffLagRel}_{\mathbb{K}}$ has:

- **Morphisms** $n \to m$, given by (possibly empty) affine Lagrangian subspaces of $(\mathbb{K}^{2n} \oplus \mathbb{K}^{2m}, \omega_n - \omega_m : \mathbb{K}^{2(n+m)} \oplus \mathbb{K}^{2(n+m)} \to \mathbb{K})$.
- **Composition** is given by relational composition.
- **Symmetric monoidal structure** is given by the direct sum.

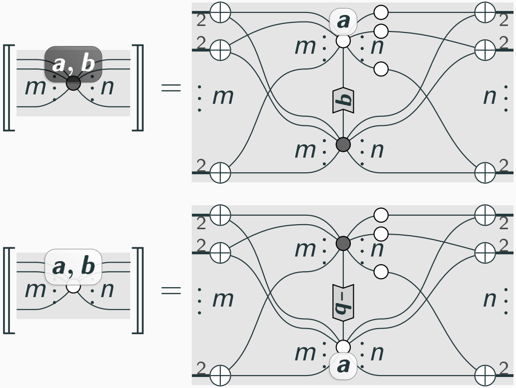
**Lemma**
*There is an embedding* $\mathsf{AffRel}_{\mathbb{K}} \to \mathsf{AffLagRel}_{\mathbb{K}}$ *given*

- **on objects by:** $n \mapsto 2n$;
- **on morphisms by:** $(S + \vec{a}) \mapsto S^{\perp} \oplus (S + \vec{a})$.

*For the geometrically inclined, this is induced by the embedding of a vector space* $\mathbb{R}^n \hookrightarrow T^*(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \oplus \mathbb{R}^n \cong \mathbb{R}^{2n}$ *into its cotangent bundle.*

# Generators of affine Lagrangian relations (Comfort and Kissinger [CK22])

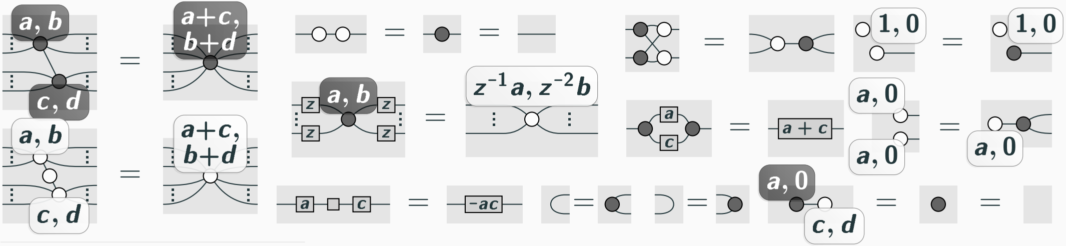
$\mathsf{AffLagRel}_{\mathbb{K}}$ is generated by two spiders decorated by $\mathbb{K}^2$; interpreted in $\mathsf{AffRel}_{\mathbb{K}}$ as:



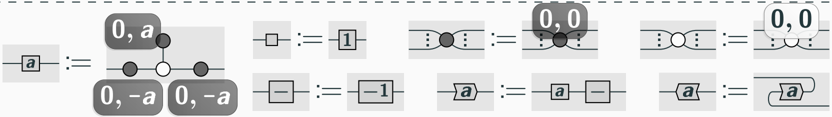
# Equations of affine Lagrangian relations (Booth et al. [BCC24b])

Modulo both spiders, being commutative, undirected nodes,

as well as for all $a, b, c, d \in \mathbb{K}$ and $z \in \mathbb{K}^{\times}$:

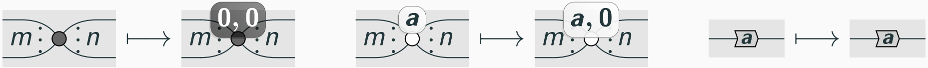



With derived generators:

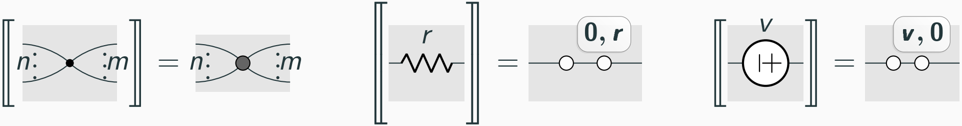


# Interpreting electirical circuits

The embedding $\mathsf{AffRel}_{\mathbb{K}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{K}}$ takes:




Now that the position/momentum wires are bundled together, we have a more concise description of electrical circuit components:

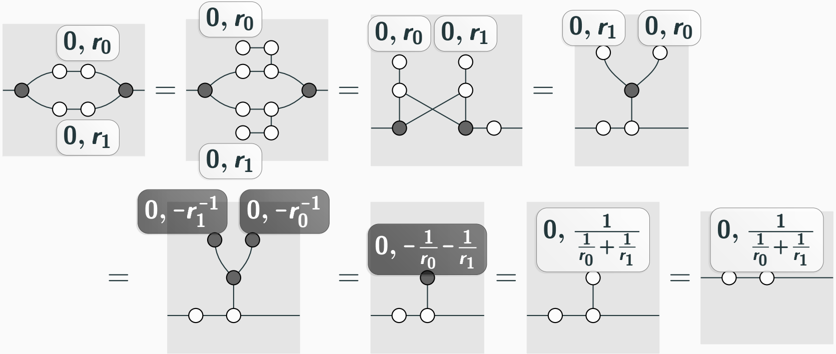


# Example: composing resistors in parallel

$\mathsf{AffLagRel}_{\mathbb{R}}$ allows us to cleanly compose electrical circuits:

**Example**

Consider two resistors with resistances $r_0, r_1 \in \mathbb{R}^{>0}$ composed in parallel.



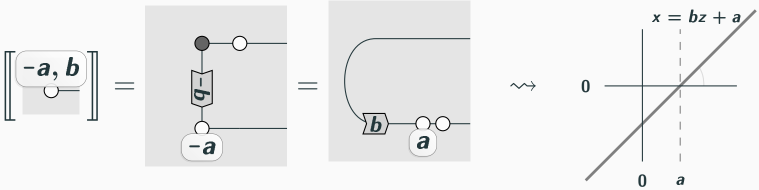

Electrons nondeterministically flow through both resistors, where they are impeded.

They extensionally behave like a resistor with resistance $1/(1/r_0 + 1/r_1)$.

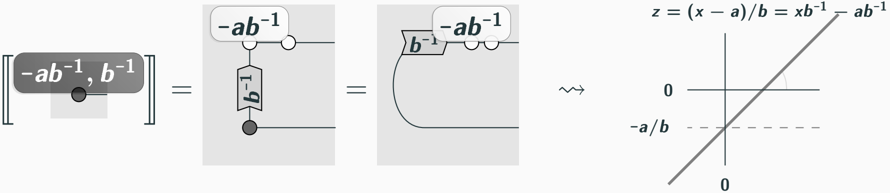
# Interpreting the two spiders:

This colour-swap rule corresponds to a change of refrence frame.

Where configurations of phase space can be represented as functions of position:



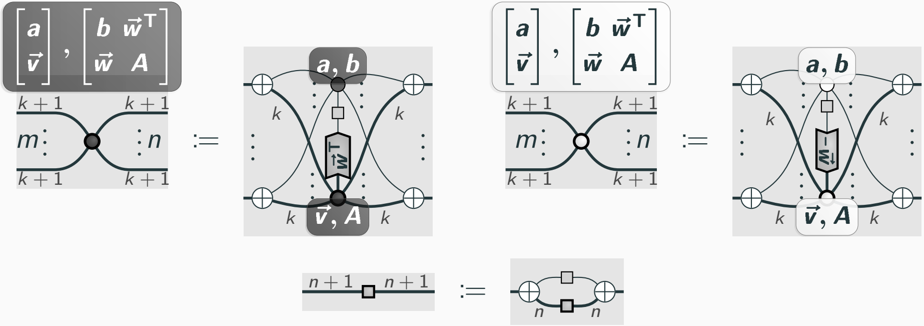

...or of momentum:



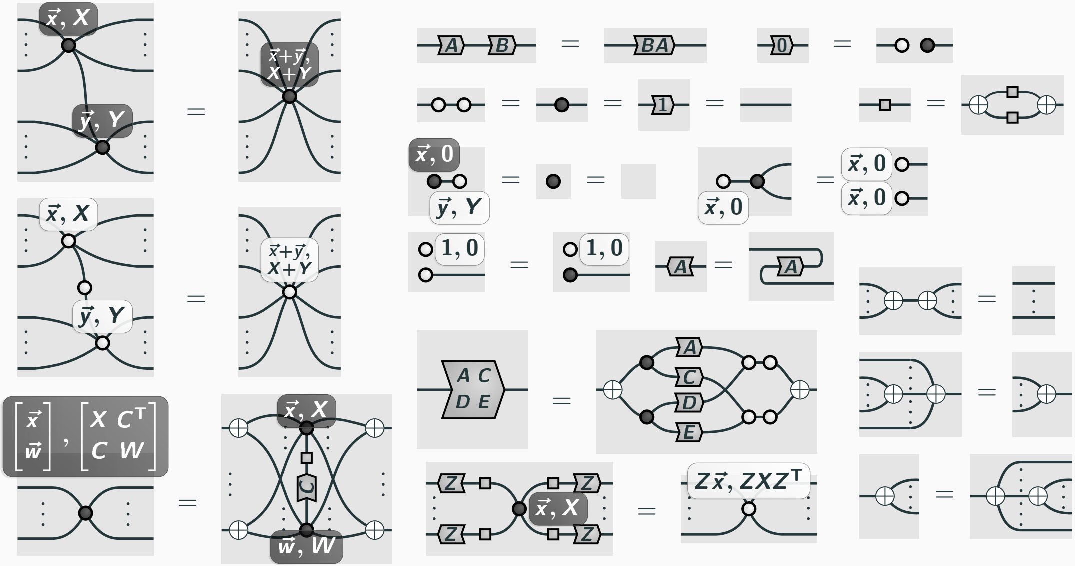
# Scalable spiders

We can define higher-dimensional spiders by induction on the number of wires $k \in \mathbb{N}$.

Take $n, m \in \mathbb{N}$, $a, b \in \mathbb{K}$, $\vec{v}, \vec{w} \in \mathbb{K}^k$ and $A \in \text{Sym}_k(\mathbb{K})$.

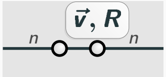


# Scalable identities



Consider a network of resistors/voltage sources acting on $n$ wires.

The extensional behaviour can be represented by a positive-definite $0 \prec R \in \mathrm{Sym}_n(\mathbb{R})$ called the **impedance matrix**, and a voltage $\vec{v} \in (\mathbb{R}^{>0})^n$



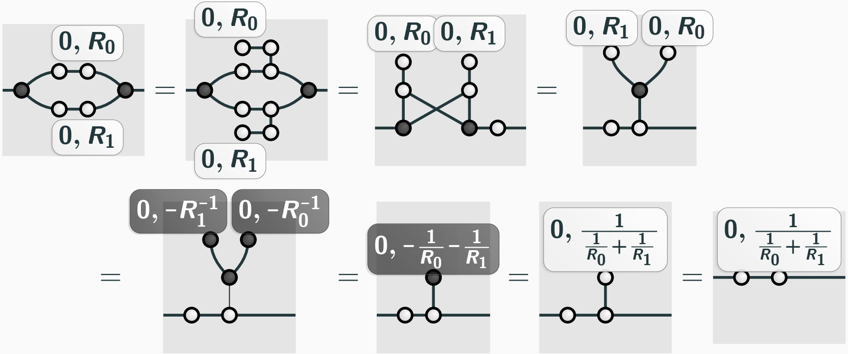

$$\left[\!\left[ \;\cdot\; \right]\!\right] = \left\{ \left( \begin{bmatrix} \vec{z} \\ \vec{x} \end{bmatrix}, \begin{bmatrix} \vec{z} \\ , \vec{x} + R\vec{z} + \vec{v} \end{bmatrix} \right) \mid \forall \vec{z}, \vec{x} \in \mathbb{R}^n \right\}$$

The resistance between the $j$th and $k$th wire is $r_{j,k} = r_{k,j} \in \mathbb{R}$.

The change in voltage on wire $j$ is $v_j \in \mathbb{R}$.

# Composing networks of resistors in parallel

Black-boxed networks of resistors compose in parallel in the same way as single resistors composed in parallel:




We don't know the internal structure of the two networks, but we still can compute their extensional behaviour in parallel.

# Quantized phase-space

# Recap: classical phase space

Recall that the phase-space on $n$ particles in Euclidean space is the symplectic vector spaces $(\mathbb{R}^{2n} \cong (\mathbb{R}^n)_Z \oplus (\mathbb{R}^n)_X, \omega_n)$:

| *Classical mechanics* | $Z$ | $dZ/dt$ | $X$ | $dX/dt$ |
|---|---|---|---|---|
| **Translation** | position | velocity | momentum | force |
| **Electronic** | charge | current | flux linkage | voltage |
| **Hydraulic** | volume | flow | pressure mom'um | pressure |
| **Thermal** | entropy | entropy flow | temperature mom'um | temperature |

Where idealized flows are morphisms in $\mathsf{AffLagRel}_{\mathbb{R}}$.

# Quantized fragments of quantum mechanics

"Quantized fragments" of quantum mechanics admit similar phase-space semantics:

- **Stabiliser quantum mechanics**
    - discrete, finite
    - nondeterministic
- **Unconstrained Gaussian quantum mechanics**
    - continuous, infinite
    - nondeterministic+probabilistic

# Stabiliser quantum mechanics

# The Pauli group

Finite dimensional quantum mechanics "lives in" $(\mathrm{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$...

**Definition**
Fix some odd prime $p$. The state space of a **quopit** is the $p$-dimensional vector space:

$$\mathcal{H}_d := \ell^2(\mathbb{Z}/d\mathbb{Z}) \cong \mathbb{C}[\mathbb{Z}/d\mathbb{Z}] = \mathrm{span}_{\mathbb{C}}\{|0\rangle, \cdots, |d-1\rangle\}$$

**Definition**
The $n$-quopit **Pauli group** $\mathcal{P}_p^{\otimes n} \subset \mathrm{U}(p^n)$ is generated under tensor product and composition by:

$$\mathcal{X}|k\rangle := |k+1\rangle \quad \text{and} \quad \mathcal{Z}|k\rangle := e^{i\frac{2\pi}{p}k}|k\rangle$$

**Lemma**
*Because* $\mathcal{X}\mathcal{Z} = e^{-i\frac{2\pi}{p}}\mathcal{Z}\mathcal{X}$ *every element of* $\mathcal{P}_p^{\otimes n}$ *has the following form,*

$$\chi(a)\mathcal{W}(\vec{z},\vec{x}) := e^{i\frac{2\pi}{p}a}\bigotimes_{j=0}^{n-1}\mathcal{Z}^{z_j}\mathcal{X}^{x_j}$$

*for some* $a \in \mathbb{F}_p$, $\vec{z},\vec{x} \in \mathbb{F}_p^n$.

# Stabiliser states

**Lemma**
*Up to scalars, a maximal Abelian subgroups* $S \subseteq \mathcal{P}_p^{\otimes n}$ *uniquely determines a normalised state* $|S\rangle : \mathcal{H}_p^{\otimes n}$ *such that for all* $P \in S$, $P|S\rangle = |S\rangle$.

*Such states are called* **stabiliser states**.

**Remark**
*Two n-quopit Pauli operators* $\chi(a)\mathcal{W}(\vec{z}, \vec{x})$ *and* $\chi(b)\mathcal{W}(\vec{q}, \vec{p})$ *commute if and only if* $\omega_n((\vec{z}, \vec{x}),\ (\vec{q}, \vec{p})) = 0$.

**Corollary (Gross [Gro06])**
*There is a bijection:*

$$
\begin{aligned}
\{\textit{Maximal Abelian subgroups } S \subseteq \mathcal{P}_p^{\otimes n}\} &\cong \{\textit{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{F}_p^{2n}, \omega_n)\} \\
&\cong \{\textit{stabiliser states } |S\rangle : \mathcal{H}_p^{\otimes n}\}
\end{aligned}
$$

*Given a Pauli* $\chi(a)\mathcal{W}(\vec{z}, \vec{x}) \in S$:

- $\vec{z}$ *are the positions;*
- $\vec{x}$ *are the momenta;*
- $a$ *is determined by the affine shift.*

# Phase-space representation of stabiliser states

Using the compact-closed structure of $(\mathsf{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$:

**Definition**
The compact prop of quopit **stabiliser circuits** is generated under tensor and composition of the linear operators:

- All quopit stabiliser states $0 \to n$;
- Caps $|j\rangle \otimes |k\rangle \mapsto \delta_{i,j}$ of type $2 \to 0$;
- The cup $\sum_{j=0}^{p-1} |j\rangle \otimes |j\rangle$ is already a stabiliser state of type $0 \to 2$.

The composition of $\mathsf{AffLagRel}_{\mathbb{F}_p}$ agrees with that of in $\mathsf{FVect}_{\mathbb{C}}$:

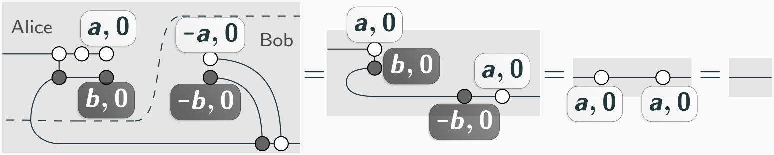
**Theorem (Comfort and Kissinger [CK22])**
$\mathsf{AffLagRel}_{\mathbb{R}_p}$ *isomorphic to quopit stabiliser circuits, modulo scalars.*

**Remark**

*The presentation of* $\mathsf{AffLagRel}_{\mathbb{F}_p}$ *is the stabiliser ZX-calculus of Poór et al. [Poó+23], modulo scalars.*

# Picturing quantum teleportation

This is powerful enough to do quantum teleportation à la Abramsky and Coecke [AC04] and Coecke and Kissinger [CK18]:



# Gaussian quantum mechanics

## Continuous-variable quantum mechanics

**Definition**
The continuous-variable 1-D quantum state space is the Hilbert space:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \, \mathrm{d}x < \infty \right\}$$

The morphisms are bounded linear maps $(L^2(\mathbb{R}))^{\otimes n} \to (L^2(\mathbb{R}))^{\otimes m}$.

**Definition**
The **displacement** operators $\hat{Z}, \hat{X} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the CV-version of Paulis:

$$\hat{Z}(s) \circ \varphi(r) := e^{i2\pi rs}\varphi(r) \quad \text{and} \quad \hat{X}(s) \circ \varphi(r) := \varphi(r-s) \quad \text{for all} \quad r, s \in \mathbb{R},\ \varphi \in L^2(\mathbb{R})$$

The $n$-qumode **Heisenberg-Weyl group** $\mathcal{HW}^{\otimes n}$ is generated by displacement operators by tensor product and composition, where every Heisenberg-Weyl operator has the form:

$$\chi(a)\mathcal{W}(\vec{z}, \vec{x}) := e^{i2\pi a} \bigotimes_{j=0}^{n-1} \hat{Z}(z_j)\hat{X}(x_j)$$

## The failure of CV stabilizer states

**Lemma**
*Affine Lagrangian subspaces of* $(\mathbb{R}^{2n}, \omega_n)$ *are in bijection with maximally Abelian subgroups of* $\mathcal{HW}^{\otimes n}$*, modulo scalars.*

**Problem:** *Given an affine Lagrangian subspace* $S \subseteq (\mathbb{R}, \omega_n)$*, there is no non-zero state* $|S\rangle : (L^2(\mathbb{R}))^{\otimes n}$ *such that* $\mathcal{W}(\vec{z}, \vec{x}) \, |S\rangle$ *for all* $(\vec{z}, \vec{x}) \in \mathbb{R}^n$*!*

None of the states in $\mathrm{AffLagRel}_{\mathbb{R}}$ can be represented in Hilbert spaces!!!

$$\{\text{Maximal Abelian subgroups } S \subseteq \mathcal{HW}^{\otimes n}\} \cong \{\text{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{R}^{2n}, \omega_n)\}$$
$$\ncong \{\text{stabiliser states } |S\rangle : (L^2(\mathbb{R}))^{\otimes n}\}$$

**Definition**
An $n$-variate **Gaussian distribution** $\mathcal{N}(\Sigma, \vec{\mu})$ consists of a positive semidefinite covariance matrix $\Sigma \in \mathrm{Sym}_n(\mathbb{R})$ and a **mean** vector $\vec{\mu} \in \mathbb{R}^n$.

When $\Sigma$ is positive-definite, $\mathcal{N}(\Sigma, \vec{\mu})$ admits a probability density function.

**Proposition**
*A $2n$-variate Gaussian probability distribution $\mathcal{N}(\Sigma, \vec{\mu})$ on phase-space $(\mathbb{R}^{2n}, \omega_n)$ corresponds to a bounded state on $(L^2(\mathbb{R}))^{\otimes n}$ if and only if:*

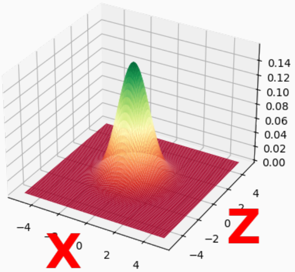
- *$\Sigma$ is positive definite;* } *so that $\mathcal{N}(\Sigma, \vec{\mu})$ has a density function*
- *$\det(\Sigma) = 1$;*
- *$\Sigma + i \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is positive semidefinite.*

} *respects Heisenberg's uncertainty principle for pure states*

Call this a **quantum Gaussian distribution**.

**Example**

The **quantum vacuum state** $|0\rangle : L^2(\mathbb{R})$ is represented by the Gaussian distribution $\Phi_1$ on $(\mathbb{R}^2, \omega_1)$:




$$\rightsquigarrow \mathcal{N}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0\\ 0\end{bmatrix}\right) =: \Phi_1$$

$\Phi_1$ is the unique quantum Gaussian distribution on $(\mathbb{R}^2, \omega_1)$ invariant under rotation.

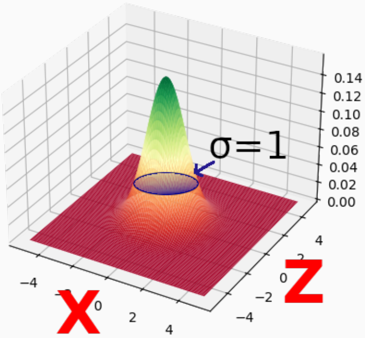
The Quantum Gaussian distribution for $|0\rangle^{\otimes n}$ has the universal property of being invariant under symplectic rotations: $\mathrm{SO}(\mathbb{R}, 2n) \cap \mathrm{Sp}(\mathbb{R}, 2n)$.

Phase-space diagrams generated by Strawberry Fields/matplotlib

**Lemma**
*Quantum Gaussian states are vacuum states acted on by affine symplectomorphisms.*

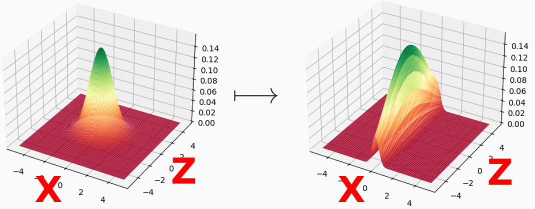
**Example**
For $n = 1$, recall that $\omega_1 : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ measures area in $\mathbb{R}^2$.

Symplectomorphisms preserve the area of the unital covariance elipse:

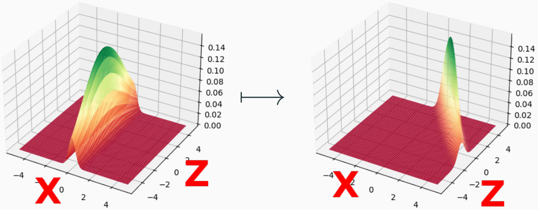


# Picturing area-preservation

For example, we can squeeze the Gaussian distribution for the vacuum state state:

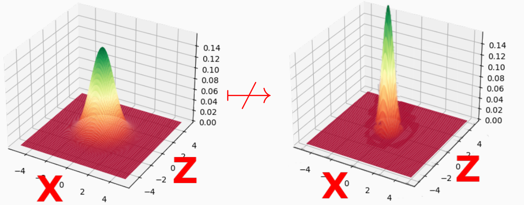



Changing the mean and rotating still is allowed.



# Picturing area-preservation

*But we can not make $\Phi_1$ more concentrated:*




This violates Heisenberg's uncertainty principle.

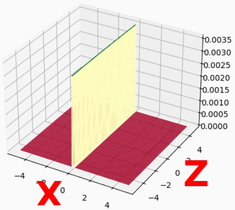
# Approximating stabiliser states with Gaussian convolution

In phase-space CV stabiliser states do not have strictly positive definite covariance.

So they are not quantum Gaussian states.

However, they can be approximated with quantum Gaussian states:

Dirac delta distribution

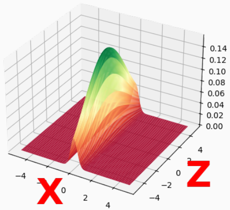



convolution by

$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$$

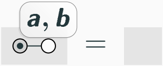
$$\longmapsto$$

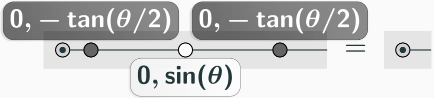
Gaussian density function



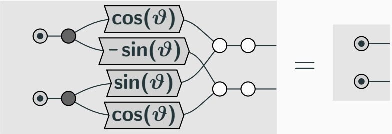
Because the vacuum state is the unique permissible Gaussian distribution in phase-space distribution invariant under rotation:

**Theorem (Booth et al. [BCC24a])**
*The Gaussian state can be freely added to* $\mathsf{AffLagRel}_{\mathbb{R}}$ *as a generator* ⊙—, *such that for all* $\vartheta \in [0, 2\pi)$ *and* $\theta \in (-\pi, \pi)$*:*




*rotation of* $\mathbb{R}^2$ *by* $\theta$
*decomposed into three shears*




*rotation of* $\mathbb{R}^2$ *by* $\vartheta$*, embedded along* $\mathsf{AffRel}_{\mathbb{R}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{R}}$

This contains both quantum Gaussian states and formal CV stabilisers.

# Quantum Gaussian states and complexification

There is an equivalent formulation using the complex numbers

**Proposition**

*Quantum Gaussian states/CV stabilisers can be represented by affine Lagrangian subspaces* $S + \vec{a} \subseteq (\mathbb{C}^{2n}, \omega_n)$*, where:*

- $\vec{a}$ *is real;*
- *for all* $\vec{x} \in S$*,* $i\omega_n(\overline{\vec{x}}, \vec{x}) \geqslant 0$*.*

In other, words, we can represent the vacuum state as follows:

**Theorem (Booth et al. [BCC24a])**

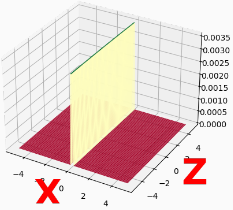
*The Gaussian ZX-calculus is equivalent to adding the state* 

*to the image of the embedding* $\mathrm{AffLagRel}_{\mathbb{R}} \hookrightarrow \mathrm{AffLagRel}_{\mathbb{C}}$*.*

# Picturing Gaussian convolution

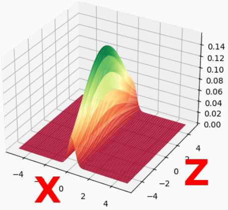
Dirac delta distribution

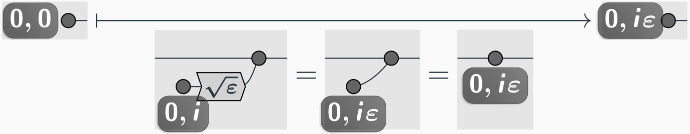
Gaussian density function




convolution by

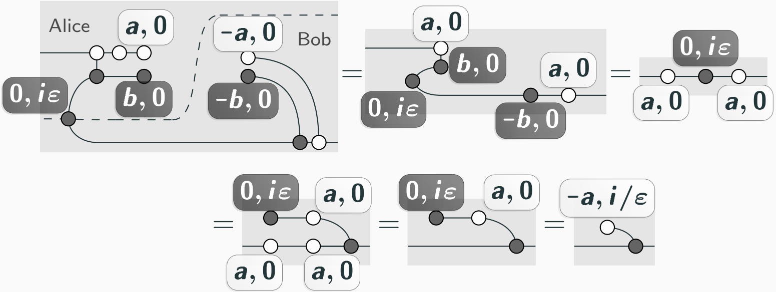
$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$$



# Picturing continuous-variable quantum teleportation

We can interpret the continuous-variable quantum teleportation algorithm of Braunstein and Kimble [BK98]:



# Fin

# References

[AC04] S. Abramsky and B. Coecke. “A categorical semantics of quantum protocols”. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004.* 2004, pp. 415–425.

[BCR18] John C. Baez, Brandon Coya, and Franciscus Rebro. “Props in network theory”. In: *Theory and Applications of Categories* 33.25 (2018), pp. 727–783.

[BF18] John C. Baez and Brendan Fong. “A compositional framework for passive linear networks”. In: *Theory and Applications of Categories* 33.38 (2018), pp 1158–1222.

[Bon+19] Filippo Bonchi, Robin Piedeleu, Paweł Sobociński, and Fabio Zanasi. “Graphical affine algebra”. In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2019, pp. 1–12.

[BSZ17] Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. “Interacting Hopf algebras”. In: *Journal of Pure and Applied Algebra* 221.1 (2017), pp. 144–184.

[BCC24a] Robert I. Booth, Titouan Carette, and Cole Comfort. *Complete equational theories for classical and quantum Gaussian relations*. 2024.

[BCC24b] Robert I. Booth, Titouan Carette, and Cole Comfort. *Graphical Symplectic Algebra*. 2024.

[BK98] Samuel L Braunstein and H J Kimble. "Teleportation of Continuous Quantum Variables". In: *Physical Review Letters* 80.4 (1998), p. 4.

[CK18] Bob Coecke and Aleks Kissinger. "Picturing Quantum Processes: A First Course on Quantum Theory and Diagrammatic Reasoning". In: *Lecture Notes in Computer Science*. Springer International Publishing, 2018, pp. 28–31.

[CK22] Cole Comfort and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations". In: *Electronic Proceedings in Theoretical Computer Science* 372 (Nov. 2022), pp. 338–351.

[Gro06] D. Gross. "Hudson's theorem for finite-dimensional quantum systems". In: *Journal of Mathematical Physics* 47.12 (Dec. 2006).

[GS79] Victor Guillemin and Shlomo Sternberg. “Some Problems in Integral Geometry and Some Related Problems in Micro-Local Analysis”. In: *American Journal of Mathematics* 101.4 (Aug. 1979), p. 915.

[Poó+23] Boldizsár Poór, Robert I. Booth, Titouan Carette, John Van De Wetering, and Lia Yeh. “The Qupit Stabiliser ZX-travaganza: Simplified Axioms, Normal Forms and Graph-Theoretic Simplification”. In: *Electronic Proceedings in Theoretical Computer Science*. Twentieth International Conference on Quantum Physics and Logic. Vol. 384. Paris, France, Aug. 29, 2023, pp. 220–264.

[Smi93] Lorcan Stuart Peter Stillwell Smith. “Bond Graph Modelling of Physical Systems”. PhD thesis. University of Glasgow, 1993.

[Wei82] Alan Weinstein. “The symplectic “category””. In: ed. by Heinz-Dietrich Doebner, Stig I. Andersson, and Herbert Rainer Petry. Red. by A. Dold and B. Eckmann. Vol. 905. Book Title: Differential Geometric Methods in Mathematical Physics Series Title: Lecture Notes in

Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1982, pp. 45–51.