The free cornering as a functor

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Talk plan

1. The idea
2. Single-object double categories and their graphical calculus
3. The free cornering: construction and properties
4. Functoriality
5. Connection to higher-order quantum transformations?

The free cornering: the idea

Let $\mathbb{X}$ be a strict monoidal category. We construct a strict double category ${}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner}$ called **the free cornering** of $\mathbb{X}$.

${}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner}$ has “the same vertical cells as $\mathbb{X}$,” but with freely added “corner” cells.

These corner cells provide ${}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner}$ with companions and conjoints, making it a proarrow equipment.

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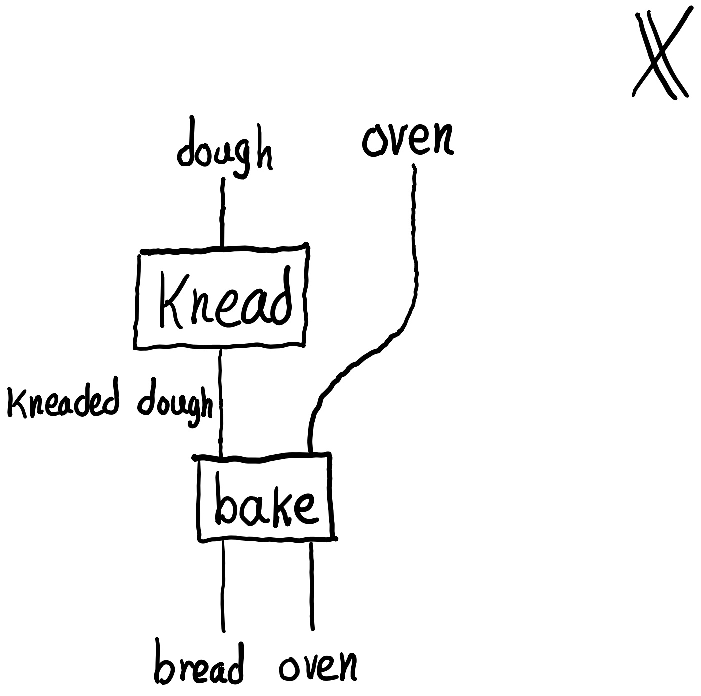
Graphically:



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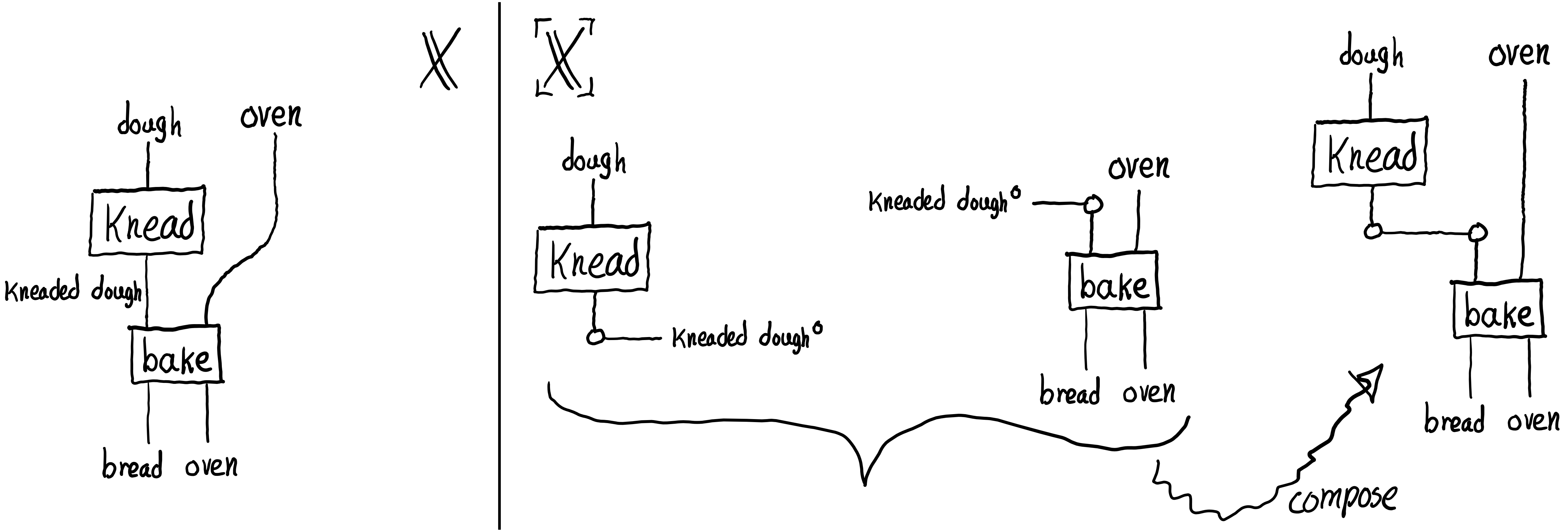
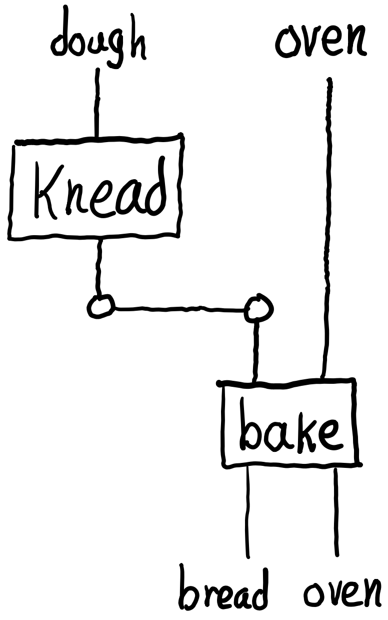


Motivation

The free cornering has been proposed by Chad Nester as a theory of concurrent computations.

Nester suggests that a diagram in $\ulcorner\mathbb{X}\urcorner$ may be understood as a *material history* of a concurrent computation — an account of the exchanges and computations performed by two interacting processes.

$\ulcorner\mathbb{X}\urcorner$



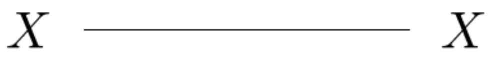
Single-object double categories

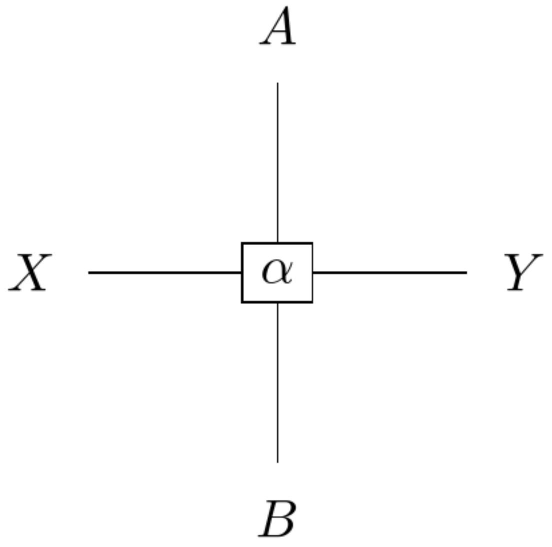
Definition. A <u>strict double category</u> $\mathbb{D}$ is an internal category in the category of categories.

A single-object double category $\mathbb{D}$ consists of:

1. Vertical wires $A, B, \ldots$
2. Horizontal wires $X, Y, \ldots$
3. 2-cells $\alpha, \beta, \ldots$

We can visualize this data using our graphical calculus for double categories (Myers 2016):





Notation

$\alpha \in \mathbb{D}\begin{pmatrix} & A & \\ X & & Y \\ & B & \end{pmatrix}$

"cell-set"

Single-object double categories

$\mathbb{D}$ has composition operations for wires and 2-cells, so that...

Vertical wires form a monoid $(\mathbb{D}, \star, I_V)$, and horizontal wires form a monoid $(\mathbb{D}, \heartsuit, I_H)$ ...

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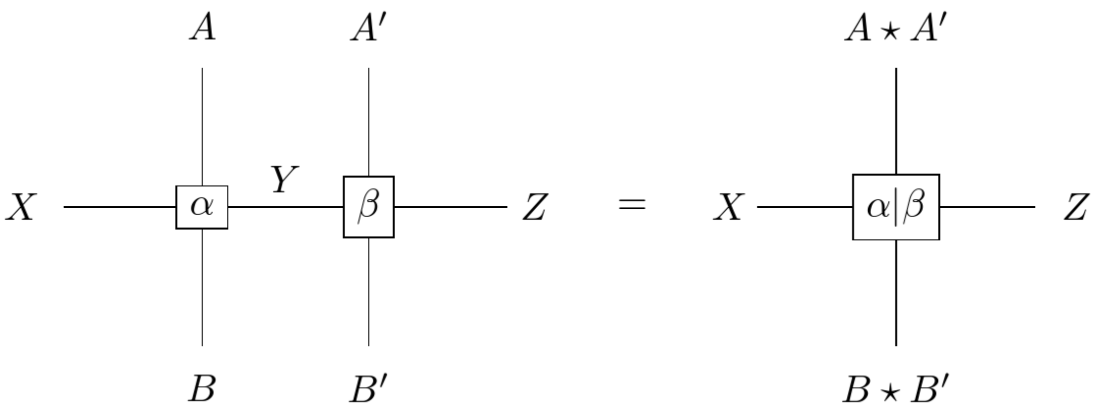
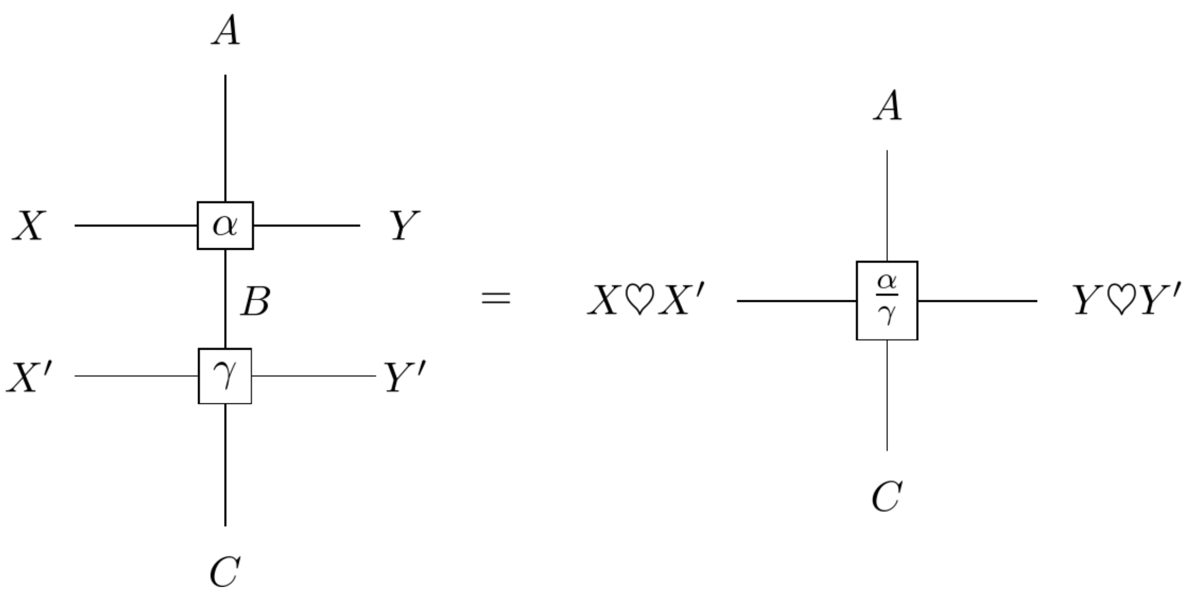
Vertical wires form a monoid $(\mathbb{D}, \star, I_V)$, and horizontal wires form a monoid $(\mathbb{D}, \heartsuit, I_H)$ ...

And 2-cells can be composed horizontally and vertically along matching boundaries:

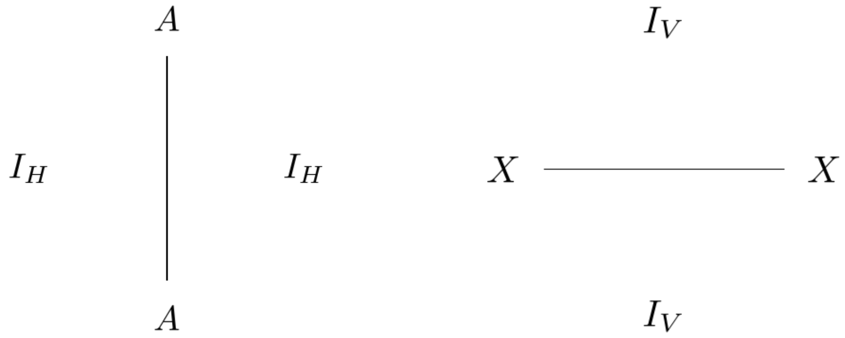




Single-object double categories

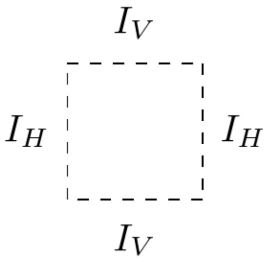
Horizontal and vertical composition are associative and unital.

Identity 2-cells are drawn as bare wires:



We omit unit wires I_V, I_H.

The shared identity 2-cell of I_V, I_H is "depicted" by empty space:



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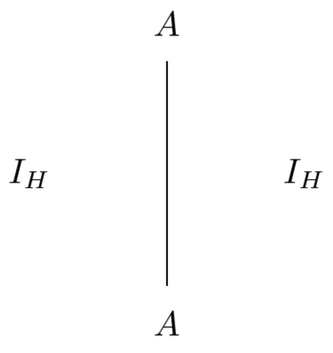

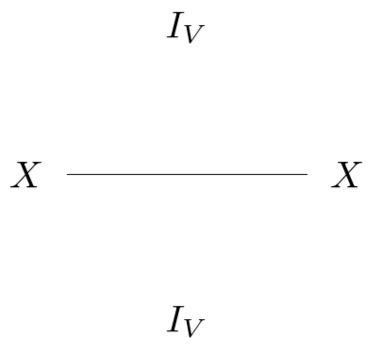

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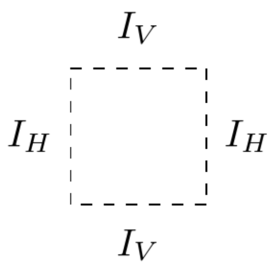
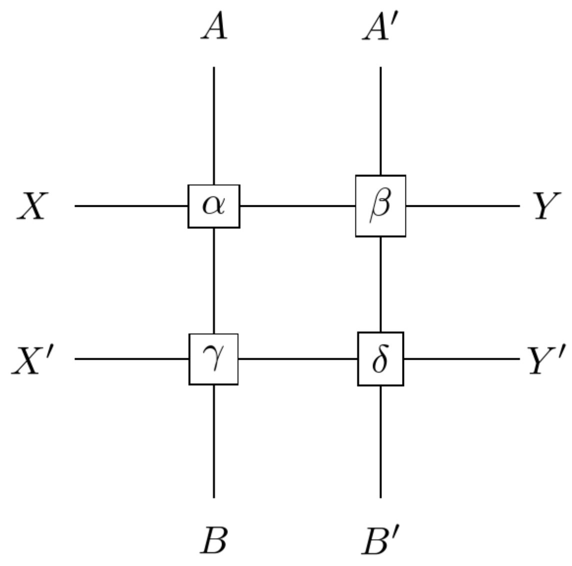


Horizontal and vertical composition satisfy the interchange law:

$$\frac{\alpha}{\gamma}\bigg|\frac{\beta}{\delta} = \frac{\alpha|\beta}{\gamma|\delta}$$

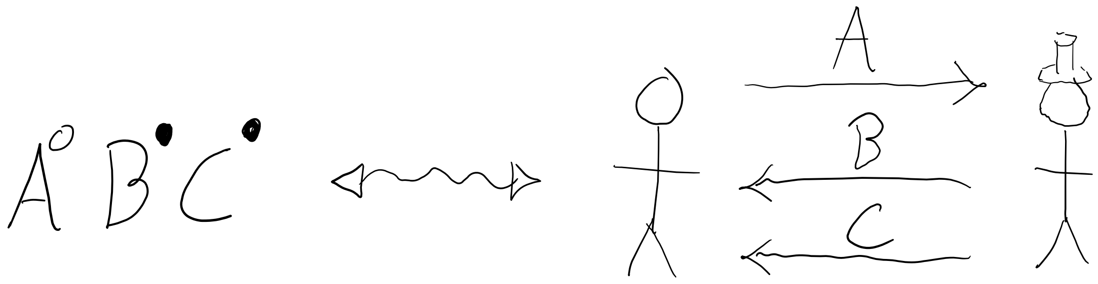
This means the following diagram can be interpreted unambiguously:



Construction of the free cornering

We refer to the free monoid $(\mathrm{Obj}\ \mathbb{X} \times \{\circ, \bullet\})^*$ as the monoid of $\mathbb{X}$-valued exchanges.

An exchange such as $A^\circ B^\bullet C^\bullet$ is to be understood as a sequence according to which two parties may exchange resources.

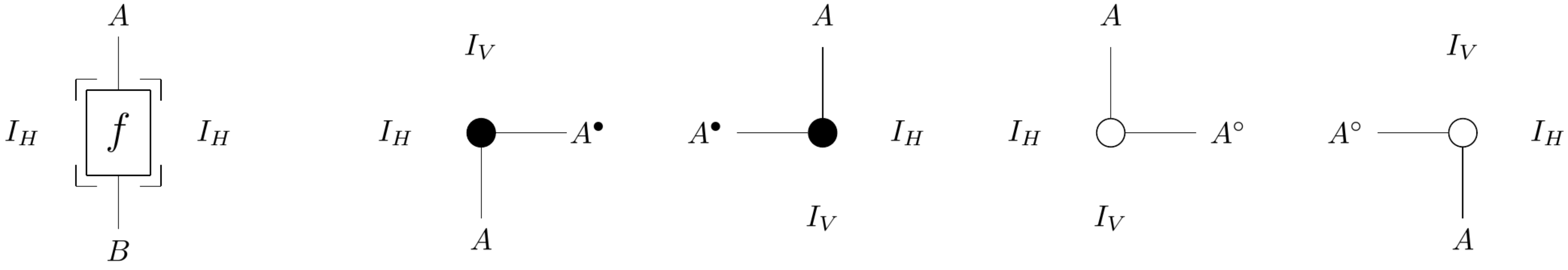


$(\mathrm{Obj}\ \mathbb{X} \times \{\circ, \bullet\})^*$ will be taken as the monoid of horizontal wires of the free cornering.

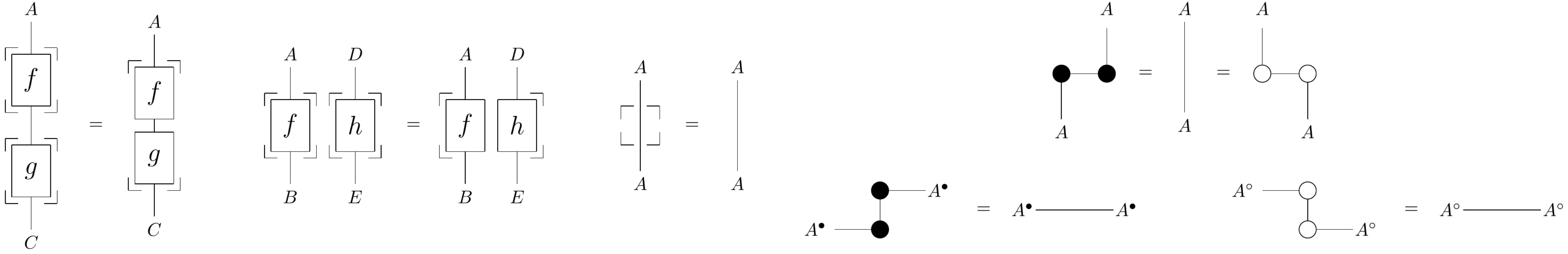
Construction of the free cornering

Definition (Nester 2021). $\ulcorner\mathbb{X}\lrcorner$ is the free single-object double category having:

1. Vertical wires the monoid $\mathrm{Obj}\ \mathbb{X}$ (with the monoidal product as multiplication).
2. Horizontal wires the free monoid $(\mathrm{Obj}\ \mathbb{X} \times \{\circ, \bullet\})^*$.
3. Generating 2-cells



for each object $A \in \mathrm{Obj}\ \mathbb{X}$ and map $f : A \to B \in \mathrm{Arr}\ \mathbb{X}$, subject to the equations



Example

${}^{\ulcorner}_{\llcorner}\mathbb{1}^{\urcorner}_{\lrcorner}$: the free cornering of the terminal strict monoidal category is a single-object strict double category with:

- One vertical wire $\star$
- Horizontal wire monoid the free monoid $\{\star^\circ, \star^\bullet\}^*$
- Exactly one 2-cell of each kind (all horizontal wires are isomorphic!)

Consequently, ${}^{\ulcorner}_{\llcorner}\mathbb{1}^{\urcorner}_{\lrcorner}$ is equivalent to the terminal double category.

Some results

Proposition[1]. The vertical underlying monoidal category of ${}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner}$ is isomorphic to $\mathbb{X}$:

$$\mathbf{V}{}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner} \cong \mathbb{X}$$

This is essentially because, for a cell in $\mathbf{V}{}^{\ulcorner}_{\llcorner}\mathbb{X}^{\urcorner}_{\lrcorner}$, any corner in it must be canceled by a matching corner, and all that can remain is a promoted $\mathbb{X}$-map.

1: Nester 2021
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Theorem[2]. The horizontal underlying monoidal category $\mathbf{H}\ulcorner\mathbb{X}\urcorner$ of $\ulcorner\mathbb{X}\urcorner$ contains $\mathrm{Optic}_{\mathbb{X}}$ as a full subcategory: for each $A, B, C, D \in \mathrm{Obj}\ \mathbb{X}$

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Proposition[3]. $\mathbf{H}\ulcorner\llcorner\mathbb{X}\lrcorner\urcorner$ is a linear actegory.

(though a somewhat degenerate one)

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How *exactly* do we construct $\llcorner\!\ulcorner \mathbb{X} \urcorner\!\lrcorner$?

Is $\mathbb{X} \mapsto \llcorner\!\ulcorner \mathbb{X} \urcorner\!\lrcorner$ functorial?

Free double categories

Definition[1]**.** A **double derivation scheme** consists of all the same data and properties of a double category, but only a *set* of 2-cells without a composition operation.

A one-object double derivation scheme consists of:

- A vertical wire monoid
- A horizontal wire monoid
- A set of 2-cells

A morphism of one-object double derivation schemes consists of monoid homomorphisms of vertical wires and horizontal wires, and a function on 2-cells preserving their boundary wires.

Fiore, Paoli, and Pronk (2008) construct a free-forgetful adjunction

$$R \dashv U : \mathbf{DblDerSch} \to \mathbf{DblCat}$$

1: Restated from Fiore, Paoli, & Pronk 2008

Free double categories

Definition (Fiore, Paoli & Pronk 2008). A **congruence** on a double category $\mathbb{D}$ consists of an equivalence relation $\sim \,=\, \sim_{A,B,X,Y}$ on each cell-set $\mathbb{D}\begin{pmatrix} & A & \\ X & & Y \\ & B & \end{pmatrix}$, such that if $\alpha \sim \alpha'$, $\beta \sim \beta'$, and $\gamma \sim \gamma'$ then

$$\alpha|\beta \;\sim\; \alpha'|\beta' \qquad\qquad \frac{\beta}{\gamma} \;\sim\; \frac{\beta'}{\gamma'}$$

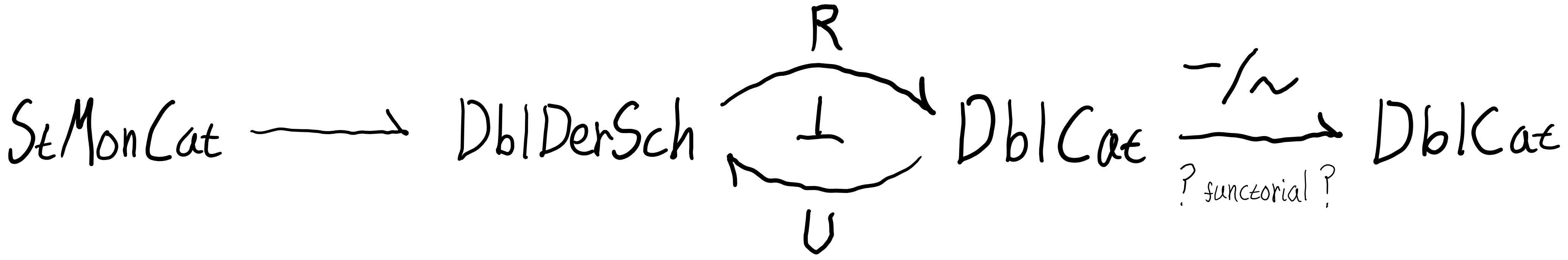
whenever these composites exist.

With such a congruence, one may take the quotient double category $\mathbb{D}/\sim$, satisfying the usual universal property.

Functoriality

Lemma. $\ulcorner - \urcorner : \mathrm{StMonCat} \to \mathrm{StDblCat}$ is functorial.

Given $\mathbb{X} \in \mathrm{StMonCat}$, we assemble a double derivation scheme following the recipe presented earlier, form the free double category, and then quotient:



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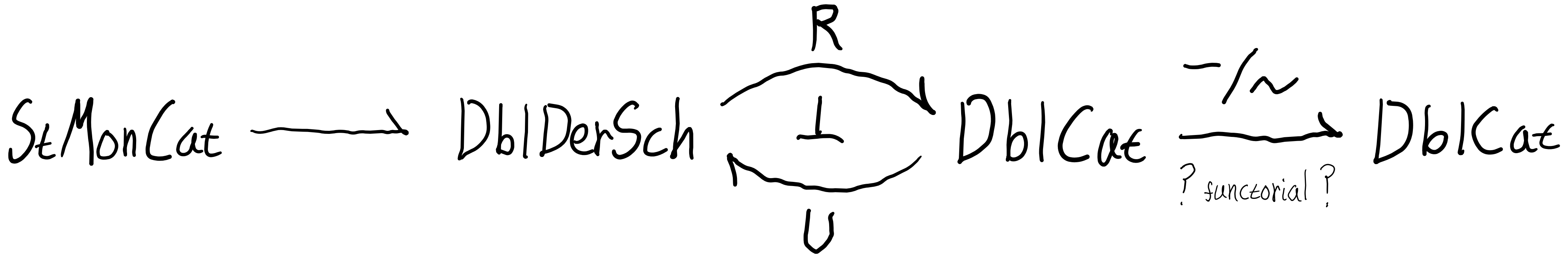
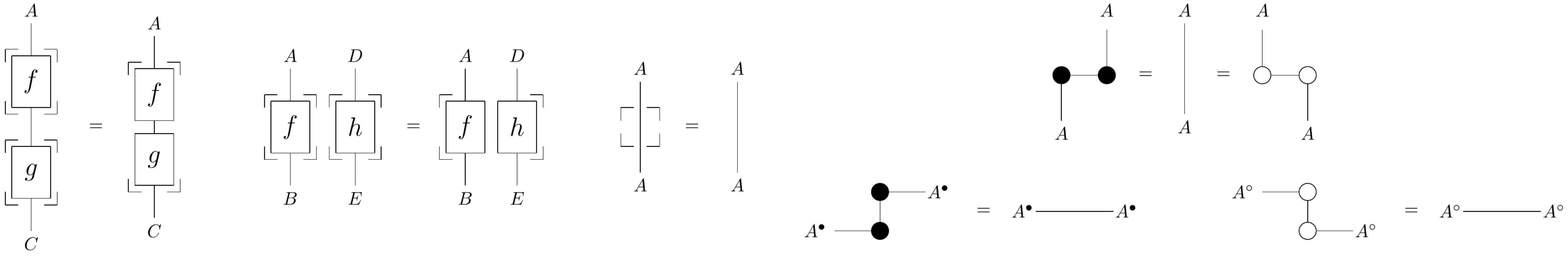


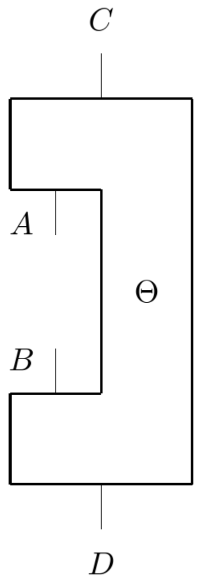
Recall the generating equations of the quotient:



Comb diagrams

$\in \mathrm{CPTP}(A, B)$

(function sending states to states)



$\in \mathrm{CCPPTPP}(A, B; C, D)$

(function sending channels to channels)

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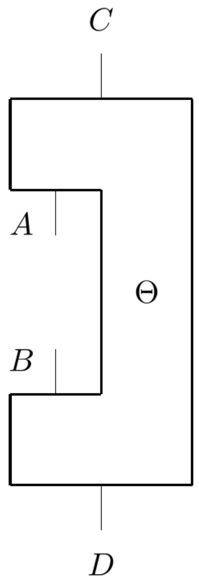

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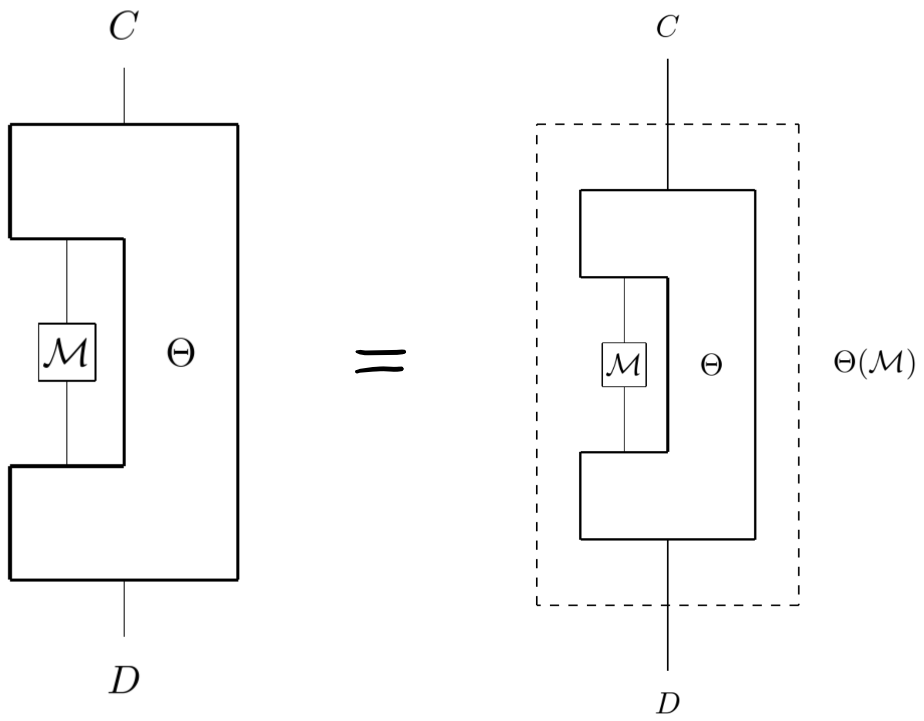

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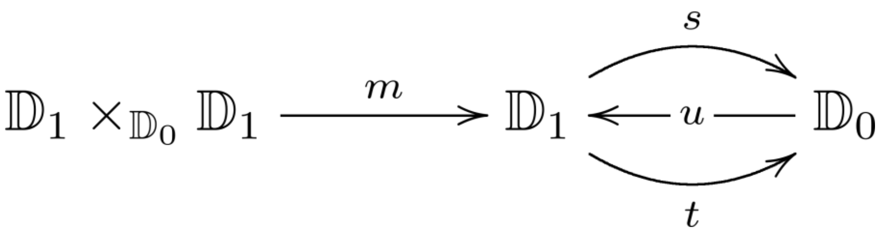
Double categories

Defn. A (strict) <u>double category</u> $\mathbb{D}$ is an internal category of the category of small categories.

This means $\mathbb{D}$ consists of categories and functors:



satisfying the usual axioms of a category.

Unpacking the definition, $\mathbb{D}$ consists of sets of

Objects: $\mathsf{Obj}\,\mathbb{D} \coloneqq \mathsf{Obj}\,\mathbb{D}_0$

Horizontal arrows: $\mathsf{Hor}\,\mathbb{D} \coloneqq \mathsf{Arr}\,\mathbb{D}_0$

Vertical arrows: $\mathsf{Ver}\,\mathbb{D} \coloneqq \mathsf{Obj}\,\mathbb{D}_1$

Squares: $\mathsf{Sq}\,\mathbb{D} \coloneqq \mathsf{Arr}\,\mathbb{D}_1$

which can be composed along matching boundaries.