$\mathcal{O}$-categories

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What we're doing

Short version: We define a notion of $\mathcal{O}$-category which unifies:

1. monads and lax monoidal categories,
2. algebras over a monad and monoids.

Long version: For every (**Set**-based) operad $\mathcal{O}$, we define a type of categorical structure, that we call $\mathcal{O}$-categories.

They are a weakened form of $\mathcal{O}$-algebras in the cartesian monoidal category **Cat**: equalities are replaced by directed natural transformations.

Both monads and lax (unbiased) monoidal categories are the $\mathcal{O}$-categories for two different $\mathcal{O}$s.

Every $\mathcal{O}$-category $\mathcal{C}$ has a notion of algebras. Both algebras over a monad and monoids in a lax monoidal category are the algebras over the $\mathcal{O}$-categories $\mathcal{C}$ for two different $\mathcal{O}$s.

Operads

Definition

An operad $\mathcal{O}$ is given by a family of sets $(\mathcal{O}(n))_{n\in\mathbb{N}}$ together with:

- functions

$$\begin{aligned} \mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) &\longrightarrow \mathcal{O}(k_1 + \ldots + k_n) \\ (t, s_1, \ldots, s_n) &\longmapsto t \circ (s_1, \ldots, s_n) \end{aligned}$$

- an element $\mathrm{id} \in \mathcal{O}(1)$

such that:

- Associativity:

$$\begin{aligned} & t \circ (s_1 \circ (u_1^1, \ldots, u_{n(1)}^1), \ldots, s_p \circ (u_1^p, \ldots, u_{n(p)}^p)) \\ = {} & (t \circ (s_1, \ldots, s_p)) \circ (u_1^1, \ldots, u_{n(p)}^p) \end{aligned}$$

- Unitality:

$$\mathrm{id} \circ t = t = t \circ (\mathrm{id}, \ldots, \mathrm{id})$$

Operads from a set of arities

Definition

We call set of arities any subset $\mathbb{I}$ of $\mathbb{N}$ such that:

- $1 \in \mathbb{I}$,
- if $n, k_1, ..., k_n \in \mathbb{I}$, then $k_1 + ... + k_n \in \mathbb{I}$.

Example (of set of arities)

1. If X is a subsemigroup of $(\mathbb{N}, +)$, then $X \cup \{1\}$ is a set of arities.
2. If $A \subseteq \mathbb{N}$, we have the set of arities $\langle A \rangle$ generated by A. If $A \not\subseteq \{1\}$ and $A \subseteq 2\mathbb{N} + 1$, then $\langle A \rangle$ is not of the first form. Such an example is $\langle 2\mathbb{N} + 1 \rangle = 2\mathbb{N} + 1$.

Example (of operad)

Let $\mathbb{I}$ be a set of arities. We define an operad $\mathcal{O}_{\mathbb{I}}$ by setting:

- $\mathcal{O}_{\mathbb{I}}(n) = \begin{cases} \{\text{n}\} & \text{if n} \in \mathbb{I} \\ \emptyset & \text{if n} \notin \mathbb{I} \end{cases}$,
- $\mathrm{id} = 1$,
- $n \circ (k_1, ..., k_n) = k_1 + ... + k_n$.

We will note $\mathbf{I} = \mathcal{O}_{\{1\}}$ and $\mathbf{AS} = \mathcal{O}_{\mathbb{N}}$. They will be our main examples of operads.

$\mathcal{O}$-algebras 1: endomorphism operads

We recall what is an $\mathcal{O}$-algebra in order to explain how $\mathcal{O}$-categories are a weakened form of $\mathcal{O}$-algebras.

We first need to know what is an endomorphism operad.

Example (of operad)

Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let $A \in \mathcal{C}$. The endomorphism operad End_A of A is obtained by setting:

- $\mathrm{End}_A(n) = \mathcal{C}[A^{\otimes n}, A]$,
- $\mathrm{id} = \mathrm{id}_A \in \mathrm{End}_A(1)$,
- $u \circ (v_1, ..., v_n) = u \circ (v_1 \otimes ... \otimes v_n)$.

$\mathcal{O}$-algebras 2: homomorphism of operads and the definition of $\mathcal{O}$-algebras

Definition

Let $\mathcal{O}_1, \mathcal{O}_2$ be two operads. An homomorphism of operads $f : \mathcal{O}_1 \to \mathcal{O}_2$ is given by a function $f_n : \mathcal{O}_1(n) \to \mathcal{O}_2(n)$ for every $n \in \mathbb{N}$ such that:

- $f_{k_1+\ldots+k_n}(u \circ (v_1, \ldots, v_n)) = f_{k_1+\ldots+k_n}(u) \circ (f_{k_1}(v_1), \ldots, f_{k_n}(v_n))$,
- $f_1(\mathrm{id}) = \mathrm{id}$.

Definition

Let $\mathcal{O}$ be an operad and $(\mathcal{C}, \otimes, I)$ a monoidal category. An $\mathcal{O}$-algebra in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with an homomorphism of operads $\mathcal{O} \to \mathrm{End}_A$.

$\mathcal{O}$-categories

Notations:

- we will write $\mathcal{O} = \coprod_{n \geq 0} \mathcal{O}(n)$,
- if $t \in \mathcal{O}$, we write $t \in \mathcal{O}(n(t))$.

Definition

Let $\mathcal{C}$ be a category. A structure of $\mathcal{O}$-category on $\mathcal{C}$ is given by a functor

$$T^t : \mathcal{C}^{n(t)} \to \mathcal{C}$$

for every $t \in \mathcal{O}$, together with:

- natural transformations

$$m_{t,s_1,\dots,s_{n(t)}} : T^t \circ (T^{s_1} \times \dots \times T^{s_n}) \Rightarrow T^{t \circ (s_1,\dots,s_n)}$$

- a natural transformation

$$u : 1_{\mathcal{C}} \Rightarrow T^{\mathrm{id}}$$

such that some associativity and unitality diagrams commute.

Warning: In the definition of an $\mathcal{O}$-algebra in the cartesian monoidal category **Cat**, the natural transformations are degenerated ones = equalities.

Monads and lax monoidal categories as $\mathcal{O}$-categories

Example (of $\mathcal{O}$-category)

Recall that we can define an operad $\mathbf{I}$ by setting $\mathbf{I}(1) = \{\mathrm{id}\}$ and $\mathbf{I}(n) = \emptyset$ if $n \neq 1$. An $\mathbf{I}$-category is a couple $(\mathcal{C}, S)$ where S is a monad on $\mathcal{C}$.

Example (of $\mathcal{O}$-category)

Recall that we can define an operad $\mathbf{AS}$ by $\mathbf{AS}(n) = \{n\}$ and:

$$n \circ (k_1, ..., k_n) = k_1 + ... + k_n.$$

An $\mathbf{AS}$-category is a lax (unbiased) monoidal category i.e. a category with functors

$$\bigotimes_n : \mathcal{C}^n \to \mathcal{C}$$

and natural transformations

$$\bigotimes_n \Big(\bigotimes_{k_1} (A^1_1, ..., A^1_{k_1}), ..., \bigotimes_{k_n} (A^n_1, ..., A^n_{k_n})\Big) \to \bigotimes_{k_1 + ... + k_n} (A^1_1, ..., A^n_{k_n})$$

$$A \to \bigotimes_1 (A)$$

such that some associativity and unitality diagrams commute.

Examples of lax monoidal categories

Example

Every monoidal category is a lax monoidal category.

Example

Let $\mathcal{C}$ be a monoidal category with finite products, together with a monad S on $\mathcal{C}$. Then, $T_n(A_1, ..., A_n) = S(A_1) \times ... \times S(A_n)$ equips $\mathcal{C}$ with a structure of a lax monoidal category.

Algebras over an $\mathcal{O}$-category

Definition

Let $\mathcal{C}$ be an $\mathcal{O}$-category. An algebra over $\mathcal{C}$ is an object A together with a morphisms $a_t : T^t(A, ..., A) \to A$ for every $t \in \mathcal{O}$, such that this diagram commutes:

$$\begin{array}{ccc}
T^t(T^{s_1}(A, ..., A), ..., T^{s_{n(t)}}(A, ..., A)) & \xrightarrow{m_{t,s_1,\ldots,s_{n(t)}}} & T^{t\circ(s_1,\ldots,s_{n(t)})}(A, ..., A) \\
\downarrow{\scriptstyle T^t(a_{s_1},\ldots,a_{s_{n(t)}})} & & \downarrow{\scriptstyle a_{t\circ(s_1,\ldots,s_{n(t)})}} \\
T^t(A, ..., A) & \xrightarrow[a_t]{} & A
\end{array}$$

Definition

Let $(A, (a_t))$ and $(B, (b_t))$ be algebras over some $\mathcal{O}$-category $\mathcal{C}$. A homomorphism of algebras from A to B is a morphism $f \in \mathcal{C}[A, B]$ such that this diagram commutes:

$$\begin{array}{ccc}
T^t(A, ..., A) & \xrightarrow{T^t(f,\ldots,f)} & T^t(B, ..., B) \\
\downarrow{\scriptstyle a_t} & & \downarrow{\scriptstyle b_t} \\
A & \xrightarrow[f]{} & B
\end{array}$$

Algebras over a monad and monoids in a lax monoidal category as algebras over an $\mathcal{O}$-category

Example

We have seen that an **I**-category is a couple $(\mathcal{C}, S)$ where S is a monad on $\mathcal{C}$. An algebra over the **I**-category $(\mathcal{C}, S)$ is just an algebra over the monad S.

Example

We have seen that an **AS**-category is a lax monoidal category. An algebra over the **AS**-category $\mathcal{C}$ is just a monoid in the lax monoidal category $\mathcal{C}$.

The homomorphisms of algebras also correspond to the usual notions.

An exercise

Proposition

Let $(A, (a_t))$ be an algebra over an $\mathcal{O}$-category $(\mathcal{C}, T)$. Let $B \in \mathcal{C}$. Suppose we have such a section-retraction pair in $\mathcal{C}$:

$$A \underset{s}{\overset{r}{\rightleftarrows}} B.$$

Then, there exists a structure of T-algebra on B such that r is a homomorphism of T-algebras iff this diagram commutes for every $t \in \mathcal{O}$:

$$\begin{array}{ccc}
T^t(A, \ldots, A) & \xrightarrow{T^t(r,\ldots,r)} & T^t(B, \ldots, B) \\
\downarrow{\scriptstyle T^t(r,\ldots,r)} & & \\
T^t(B, \ldots, B) & & \downarrow{\scriptstyle b_t} \\
\downarrow{\scriptstyle T^t(s,\ldots,s)} & & \\
T^t(A, \ldots, A) & & \\
\downarrow{\scriptstyle a_t} & & \\
A & \xrightarrow[r]{} & B
\end{array}$$

In this case, this structure of algebra is unique.

Fun fact: The notion of $\mathcal{O}$-category has been invented in order to make this exercise possible.

But now the concept of $\mathcal{O}$-category seems more important than the exercise!

Thank you!