Constructing Linear Bicategories

Rose Kudzman-Blais [1]
Rick Blute [1]   Susan Niefield [2]

[1]University of Ottawa

[2]Union College

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Presentation Overview

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$\mathbb{Z}_\infty$-relations

Consider two ordered structures on $\mathbb{Z}_\infty = \mathbb{Z} \cup \{+\infty, -\infty\}$:

Tropical Semiring
$x \times_1 y = x + y$
$-\infty \times_1 \infty = -\infty = \infty \times_1 -\infty$
$x + y = \max\{x, y\}$
Usual order

Artic Semiring
$x \times_2 y = x + y$
$-\infty \times_2 \infty = \infty = \infty \times_2 -\infty$
$x + y = \min\{x, y\}$
Opposite order

Both $\mathbb{Z}_\infty$ and ${\mathbb{Z}_\infty}^{op}$ are not just semirings, but are in fact *quantales*.

They are of great use in the theory of synchronization. For more information about how these structures arise:

- M. Droste, W. Kuich. Semirings and Formal Power Series (2009).
- M. Droste, W. Kuich, H. Vogler. Handbook of Weighted Automata (2009).

$\mathbb{Z}_\infty$-relations (cont'd)

Definition

If X and Y are sets, define a $\mathbb{Z}_\infty$- relation $R\colon X \to\!\!\!\!\!\bullet\;\; Y$ to be a function $R\colon X \times Y \to \mathbb{Z}_\infty$. Then the two semiring constructions above define two distinct relational compositions.

Given $X \xrightarrow{A} Y \xrightarrow{B} Z$, define

$$A \otimes B(x, z) = \max_{y \in Y}(A(x, y) \times_1 B(y, z)),$$

$$A \oplus B(x, z) = \min_{y \in Y}(A(x, y) \times_2 B(y, z)).$$

Remark

These two compositions are related by a linear distribution and determine a locally posetal linear bicategory.

Further, $\mathbb{Z}_\infty$ with the above operations is a Girard quantale and the two structures are related by the Girard duality.

Linear Bicategories

Definition (Cockett, Koslowski, and Seely)

A linear bicategory $\mathcal{B}$ consists of the following data.

- *A class $\mathcal{B}_0$ (of '0-cells')*
- *A category $\mathcal{B}_1$ (with '1-cells' as objects and '2-cells' as morphisms)*
- *Functors $D_0, D_1 : \mathcal{B}_1 \to \mathcal{B}_0$ (domain, codomain).*
- *Functor $\otimes_{X,Y,Z} : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z)$ (tensor)*
- *Functor $\oplus_{X,Y,Z} : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z)$ (par\cotensor)*
- *1-cells $\top_X, \bot_X : 1 \to \mathcal{B}(X, X)$*

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Linear Bicategories (cont'd)

Definition (Cockett, Koslowski, and Seely)

- *Natural isomorphisms (expressing associativity of* $\otimes$ *and* $\oplus$*)*

$$a_\otimes \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \quad a_\oplus \colon A \oplus (B \oplus C) \to (A \oplus B) \oplus C$$

- *Natural isomorphisms (expressing unity of* $\top_X$ *and* $\bot_X$*)*

$$u^L_\otimes \colon A \to \top_X \otimes A \quad u^R_\otimes \colon A \to A \otimes \top_Y$$

$$u^L_\oplus \colon \bot_X \oplus A \to A \quad u^R_\oplus \colon A \oplus \bot_X \to A$$

- *Natural transformations (linear distributivities)*

$$\delta_L \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

$$\delta_R \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

subject to various coherence conditions.

Girard quantales and quantaloids

Definition (Yetter)

- *An element* $\bot \in Q$ *is a* *cyclic dualizing element* *if* $\forall a \in Q$*, we have*

$$\bot \multimapinv a = a \multimap \bot = a^\bot \qquad \text{and} \qquad (a \multimap \bot) \multimap \bot = a$$

- *A* *Girard quantale* *is a quantale* Q *equipped with a chosen cyclic dualizing object* $\bot$.

Definition (Rosenthal)

- *A family of 1-cells* $\mathcal{D} = \{d_a \colon a \to a \mid a \in \mathcal{Q}\}$ *is a* *cyclic dualizing family* *if* $\forall f$,

$$f \multimap d_a = d_b \multimapinv f = f^\bot \qquad \text{and} \qquad d_a \multimapinv (f \multimap d_a) = f$$

- *A* *Girard quantaloid* *is a quantaloid* $\mathcal{Q}$ *together with a cyclic dualizing family* $\mathcal{D}$.

Girard quantales and quantaloids (cont'd)

Noting that a Girard quantale is a $*$-autonomous category:

Theorem

A Girard quantale is a linearly distributive category, where the second monoidal structure is given by $a \oplus b = (b^{\perp} \otimes a^{\perp})^{\perp}$ (Girard\De Morgan duality).

Similarly:

Theorem

A Girard quantaloid is a linear bicategory.

Q-Rel is linear when Q is Girard

Suppose Q is a Girard quantale.
Consider Q-Rel the locally posetal bicategory of sets and Q-relations, then given $X \overset{R}{\multimap} Y \overset{S}{\multimap} Z$, we define

$$R \otimes S(x,z) = \sup_{y \in Y}(R(x,y) \otimes S(y,z))$$

$$R \oplus S(x,z) = \inf_{y \in Y}(R(x,y) \oplus S(y,z))$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases} \qquad \bot_X(x,x') = \begin{cases} \mathbf{1} & \text{if } x \neq x' \\ \bot & \text{if } x = x' \end{cases}$$

Proposition

The above determines Q-Rel as a linear bicategory.

Q-Rel is Girard iff Q is Girard

Result can actually be made more specific:

Theorem

Suppose Q is a unital quantale. Then Q-Rel is a quantaloid.

Q-Rel: *Girard quantaloid, cyclic dualizing family* $D = \{d_X \colon X \to\!\!\!\!\bullet\!\!\to X\}$	$\iff$	Q: *Girard quantale, cyclic dualizing element d corresponding to* $d_1 : 1 \to\!\!\!\!\bullet\!\!\to 1$

$$d_X(x, x') = \begin{cases} \mathbf{1} & x \neq x' \\ d & x = x' \end{cases}$$

LD-quantales

Model theory of linear logic

$*$-autonomous category	Linear distributive (LD) category
Girard quantale	LD-quantale

The analagous structure to LD-categories for quantales:

Definition

An LD-quantale is a complete lattice Q with operations $\otimes$ and $\oplus$ and elements $\top$ and $\bot$ such that

- $(Q, \otimes, \top)$ *and* $(Q^{op}, \oplus, \bot)$ *are quantales.*
- $a \otimes (b \oplus c) \leq (a \otimes b) \oplus c$ *and* $(b \oplus c) \otimes a \leq b \oplus (c \otimes a)$

Clearly a Girard quantale is a LD-quantale and an LD-quantale is a linearly distributive category.

Shift Monoids

Definition (Cockett, Seely)

- *A* *shift monoid* *consists of a 4-tuple* $\mathcal{M} = (M, +, \top, a)$ *where* $(M, +, 0)$ *is a commutative monoid and* a *is an invertible element of* M*.*
- *If* $\mathcal{M}$ *is a shift monoid, define a second multiplication by*

$$x \cdot y = x + y - a$$

Then this is a second monoid structure on M *with unit given by* a*.*

Definition

A commutative monoid M *is* *cancellative* *if for all* $a, b, c \in M$*, one has*

$$a + b = a + c \Rightarrow b = c$$

Shift Monoids (cont'd)

Lemma

- *Let M be a cancellative commutative monoid. We view M as a discrete poset and then add top and bottom elements which we denote* **1** *and* **0**. *Denote this set as M^+, We then extend the addition on M:*

$$\mathbf{1} + b = \mathbf{1} \text{ if } b \in M \text{ or } b = \mathbf{1} \qquad \mathbf{0} + b = \mathbf{0} \text{ for all } b \in M^+$$

  Then $(M^+, +)$ is a commutative quantale.

- *If $\mathcal{M}$ is furthermore a cancellative commutative shift monoid, then extend the second operation in the dual way:*

$$\mathbf{0} \cdot b = \mathbf{0} \text{ if } b \in M \text{ or } b = \mathbf{0} \qquad \mathbf{1} \cdot b = \mathbf{1} \text{ for all } b \in M^+$$

  Then $\mathcal{M}$ is an LD-*quantale.*

Linear quantaloids

Appropriate structure for quantaloids to be linear bicategories:

Definition

A linear quantaloid is a locally small category $\mathcal{Q}$ whose hom-sets are complete lattices with binary operations $\otimes$ and $\oplus$, and families of distinguished morphisms $\{\top_a \mid a \in \mathrm{ob}\mathcal{Q}\}$ and $\{\bot_a \mid a \in \mathrm{ob}\mathcal{Q}\}$ such that

- $(\mathcal{Q}, \otimes, \top_a)$ *and* $(\mathcal{Q}^{co}, \oplus, \bot_a)$ *are quantaloids,*
- $f \otimes (g \oplus h) \leq (f \otimes g) \oplus h$ *and* $(f \oplus g) \otimes h \leq f \oplus (g \otimes h)$ $\forall$ composable $f, g, h \in \mathrm{mor}\mathcal{Q}$

Clearly a Girard quantaloid is a linear quantaloid, which is in turn a linear bicategory.

Q is LD iff Q-Rel is linear

Suppose $(Q, \otimes, \oplus)$ is an LD-quantale, and $X \overset{R}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Y \overset{S}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Z$ are Q-valued relations. Define

$$R \otimes S(x,z) = \sup_{y \in Y}(R(x,y) \otimes S(y,z))$$

$$R \oplus S(x,z) = \inf_{y \in Y}(R(x,y) \oplus S(y,z))$$

The identity 1-cells are given by:

$$\top_X(x,x') = \begin{cases} \mathbf{0} & \text{if } x \neq x' \\ \top & \text{if } x = x' \end{cases} \qquad \bot_X(x,x') = \begin{cases} \mathbf{1} & \text{if } x \neq x' \\ \bot & \text{if } x = x' \end{cases}$$

Theorem

$(Q, \otimes, \oplus)$ is an LD-quantale if and only if (Q-Rel, $\otimes, \oplus$), where $\otimes$ and $\oplus$ are defined above, is a linear quantaloid.

$\mathcal{Q}$-Mod

A structure showed by Rosenthal to be Girard given conditions:

Definition

Let $\mathcal{Q}$ be a quantaloid. A $\mathcal{Q}$-category is a pair $M = (X, \rho)$ where:

- *X is a set.*
- *ρ is a function $\rho : X \to \mathrm{ob}\mathcal{Q}$.*
- *We have a function, called the enrichment assigning to each pair $(x, x') \in X \times X$ a morphism $M(x, x') : \rho(x) \to \rho(x')$ such that*
  1. $\top_{\rho(x)} \leq M(x, x) \ \forall x \in X$
  2. $M(x, x') \otimes M(x', x'') \leq M(x, x'') \ \forall x, x', x'' \in X$

Remark

Alternatively, a $\mathcal{Q}$-category is a lax bifunctor from set X to $\mathcal{Q}$, when X is viewed as an indiscrete\chaotic bicategory (i.e. every hom-cat is trivial).

$\mathcal{Q}$-Mod (cont'd)

Definition

Let $\mathcal{Q}$ be a quantaloid, $M = (X, \rho_M)$ and $N = (Y, \rho_N)$ be $\mathcal{Q}$-categories. A $\mathcal{Q}$-bimodule $\Theta : M \mathrel{\mapstochar\!\!\to} N$ consists of a 1-cell $\Theta(x, y) : \rho_M(x) \to \rho_N(y)$ for every $(x, y) \in X \times Y$ such that

1. $\Theta(x, y) \otimes N(y, y') \leq \Theta(x, y') \ \forall x \in X, y, y' \in Y$
2. $M(x, x') \otimes \Theta(x', y) \leq \Theta(x, y) \ \forall x, x' \in X, y \in Y$

Then, $\mathcal{Q}$-Mod is the locally posetal bicategory of $\mathcal{Q}$-categories and $\mathcal{Q}$-bimodules. Given $M \xrightarrow{\Theta} N \xrightarrow{\Pi} P$, composition $\Theta \otimes \Pi : M \mathrel{\mapstochar\!\!\to} P$ is defined by

$$(\Theta \otimes \Pi)(x, z) = \sup_{y \in Y} \Theta(x, y) \otimes \Pi(y, z)$$

Identity 1-cells are $\iota_M : M \mathrel{\mapstochar\!\!\to} M$ defined by $\iota_M(x, x') = M(x, x')$.

$\mathcal{Q}$ is Girard iff $\mathcal{Q}$-Mod is Girard

If $\mathcal{Q}$ is a Girard quantaloid, $\mathcal{Q}$-Mod has a second bicategorical structure. Given $M \overset{\Theta}{\relbar\!\!\bullet\!\!\rightarrow} N \overset{\Pi}{\relbar\!\!\bullet\!\!\rightarrow} P$, composition $\Theta \oplus \Pi : M \relbar\!\!\bullet\!\!\rightarrow P$ is

$$(\Theta \oplus \Pi)(x, z) = \inf_{y \in Y} \Theta(x, y) \oplus \Pi(y, z)$$

with identity 1-cells $\delta_M : M \relbar\!\!\bullet\!\!\rightarrow M$, $\delta_M(x, x') = M(x', x)^{\perp}$.

Proposition

Suppose $\mathcal{Q}$ is a quantaloid. Then $\mathcal{Q}$-Mod is a quantaloid. Furthermore $\mathcal{Q}$ is a Girard quantaloid if and only if $\mathcal{Q}$-Mod is a Girard quantaloid.

$\mathcal{Q}$-categories and $\mathcal{Q}$-bimodules when $\mathcal{Q}$ is a linear quantaloid

Definition

Let $\mathcal{Q}$ be a linear quantaloid. A linear $\mathcal{Q}$-category is a pair $M = (X, \rho)$ where:

- X is a set,
- ρ is a function $X \to ob\mathcal{Q}$,
- We have a $\otimes$-enrichment and $\oplus$-enrichment, assigning to each pair $(x, x') \in X \times X$:
  - $M_{\otimes}(x, x') : \ \rho(x) \to \rho(x')$
  - $M_{\oplus}(x, x') : \ \rho(x) \to \rho(x')$

  subject to inequalities
- satisfying $M_{\otimes}(x, x'') \leq M_{\oplus}(x, x') \oplus M_{\otimes}(x', x'')$, $\forall x, x', x'' \in X$, and three other inequalities

$\mathcal{Q}$-categories and $\mathcal{Q}$-bimodules when $\mathcal{Q}$ is a linear quantaloid (cont'd)

Remark

A linear $\mathcal{Q}$-category is in fact a linear functor from X to $\mathcal{Q}$, where X is the degenerate linear bicategory (i.e. $\otimes = \oplus$) formed from the indiscrete bicategory.

Definition

Let $\mathcal{Q}$ be a linear quantaloid, $M = (X, \rho_M)$ and $N = (Y, \rho_N)$ be $\mathcal{Q}$-categories. A linear $\mathcal{Q}$-bimodule $\Theta : M \multimap N$ consists of two 1-cells for every $(x, y) \in X \times Y$

- $\Theta_\otimes(x, y) : \rho_M(x) \to \rho_N(y)$
- $\Theta_\oplus(y, x) : \rho_N(y) \to \rho_M(x)$
- satisfying $N_\otimes(y, y') \otimes \Theta_\oplus(y', x) \leq \Theta_\oplus(y, x)$, $\forall x \in X, y, y' \in Y$ and three other inequalities.

Definition

Let $\mathcal{Q}$ be a linear quantaloid.

- $(\mathcal{Q}\text{-Mod}, \otimes, \iota_M)$ *is a locally posetal bicategory whose 0-cells are linear $\mathcal{Q}$-categories, 1-cells are linear $\mathcal{Q}$-bimodules.*

2-cells $\Theta \leq \Pi \Leftrightarrow \forall (x, y) \in X \times Y : \quad \begin{array}{l} \Theta_\otimes(x, y) \leq \Pi_\otimes(x, y) \\ \Pi_\oplus(y, x) \leq \Theta_\oplus(y, x) \end{array}$

composition $\Theta \otimes \Pi : M \multimap P$:

$$(\Theta \otimes \Pi)_\otimes(x, z) = \sup_{y \in Y} \Theta_\otimes(x, y) \otimes \Pi_\otimes(y, z)$$
$$(\Theta \otimes \Pi)_\oplus(z, x) = \inf_{y \in Y} \Pi_\oplus(z, y) \oplus \Theta_\oplus(y, x)$$

$\otimes$-identity 1-cells $\iota_M : M \multimap M$:

$$\iota_{M\otimes}(x, x') = M_\otimes(x, x')$$
$$\iota_{M\oplus}(x, x') = M_\oplus(x, x')$$

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$\mathcal{Q}$-Mod when $\mathcal{Q}$ is a linear quantaloid (cont'd)

Definition

- *($\mathcal{Q}$-Mod^{co}, $\oplus$, δ_M) is a locally posetal bicategory whose 0-cells are linear $\mathcal{Q}$-categories, 1-cells are linear $\mathcal{Q}$-categories, with 2-cells, composition and identities defined similarly.*

Theorem

$(\mathcal{Q}, \otimes, \oplus)$ *is a linear quantaloid if and only if* $(\mathcal{Q}\text{-Mod}, \otimes, \oplus)$ *is a linear quantaloid and therefore a linear bicategory.*

Girard Bicategories

Suppose $\mathcal{D} = \{D_X \colon X \to X \mid X \in \mathcal{B}\}$ is a family of 1-cells in a biclosed bicategory $\mathcal{B}$. Given $A \colon X \to Y$, we get

$$A \xrightarrow{\delta_{A,Y}} (D_Y \multimapinv A) \multimap D_Y \qquad \text{and} \qquad A \xrightarrow{\delta_{X,A}} D_X \multimapinv (A \multimap D_X)$$

Definition

- *A family $\mathcal{D} = \{D_X \colon X \to X \mid X \in \mathcal{B}\}$ is called* ***dualizing*** *if the 2-cell $\delta_{A,Y}$ is invertible, for all $A \colon X \to Y$.*
- *A dualizing family $\mathcal{D}$ is called* ***cyclic*** *if there are invertible 2-cells $\theta_A \colon (D_Y \multimapinv A) \to (A \multimap D_X)$, natural in $A \colon X \to Y$, subject to a commuting diagram. In this case, we let $A^\perp = A \multimap D_X$.*
- *$\mathcal{B}$ is a* ***Girard bicategory*** *if it admits a cyclic dualizing family.*

Examples of Non-locally Posetal Linear Bicategories

Theorem

Every Girard bicategory is a linear bicategory.

Lemma

- *If* $\mathcal{V}$ *is a* $*$*-autonomous category, then* $\mathcal{B}(\mathcal{V})$ *is a linear bicategory. If, in addition,* $\mathcal{V}$ *has equalizers and coequalizers stable under composition, then so is* $\mathcal{B}\mathrm{im}(\mathcal{V})$*. In particular,* $\mathcal{Q}\mathrm{uant}$ *and* $\mathcal{L}\mathrm{oc}$ *are linear bicategories.*
- *If* $\mathcal{V}$ *is a star-autonomous category with set-indexed products and coproducts, then* $\mathcal{V}$-$\mathcal{M}\mathrm{at}$ *is a linear bicategory. If, in addition,* $\mathcal{V}$ *has equalizers and coequalizers, then so is* $\mathcal{V}$-$\mathcal{P}\mathrm{rof}$*. In particular,* $\mathcal{Q}\mathrm{tld}$ *is a linear bicategory.*

Further Work: Formal Theory of Linear Monads

Definition (Street)

Let $\mathcal{B}$ be a 2-cat, then Mnd($\mathcal{B}$) is the 2-cat of monads, monad functors, and monad transformations.

- If $\mathcal{Q}$ is a Girard quantaloid, is Mnd($\mathcal{Q}$) a Girard quantaloid?
- If $\mathcal{Q}$ is a linear quantaloid,
  - what is the appropriate definition for Mnd($\mathcal{Q}$)?
  - is Mnd($\mathcal{Q}$) a linear quantaloid?

- If $\mathcal{B}$ is a Girard bicategory, is Mnd($\mathcal{B}$) a Girard bicategory?
- If $\mathcal{B}$ is a linear bicategory,
  - what is the appropriate definition for Mnd($\mathcal{B}$)?
  - is Mnd($\mathcal{B}$) a linear bicategory?
  - when are linear monads in $\mathcal{B}$ generated by a cyclic linear adjunction?