FMCS 2022

Dagger linear logic and categorical quantum mechanics - Part II

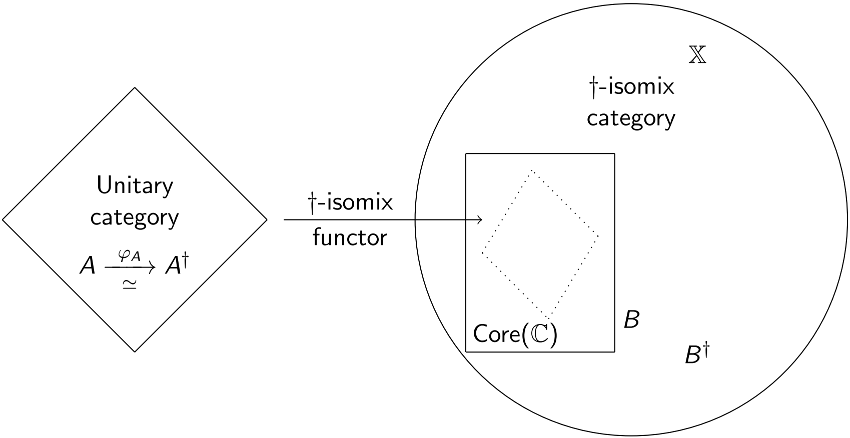
Priyaa Varshinee Srinivasan

Joint work with Robin Cockett

June 25, 2022

Mixed unitary categories (MUCs)

(Take away from Part I)



A mixed unitary category, $M : \mathbb{U} \to \mathbb{C}$, is

†-isomix functor: unitary category $\to$ †-isomix category

An LDC which is not $*$-autonomous: Shift Monoids

Let M be any set. A **shift monoid** is a commutative monoid, $(M, +, 0)$ with a designated element s such that there exists an inverse to s i.e, $s - s = 0$.

A second multiplication can be defined on the set M as follows: for all $x, y \in M$,

$$x \circ y = (x + y) - s$$

The unit of the second multiplication is s.

A shift monoid considered as a discrete category (the elements of the monoid are the objects and the maps are identity maps).

Note that the distributors are identity maps but $\otimes$ and $\oplus$ are distinct.

†-shift monoid: A discrete category of pairs of shift monoids with the dagger interchanging the monoids?

Complementary quantum observables

A quantum observable refers to a measurable property of quantum system.

A pair of quantum observables are complementary if measuring one observable increases uncertanity regarding the value of the other.

Example: position and momentum of an electron

Complementary quantum observables

A quantum observable refers to a measurable property of quantum system.

A pair of quantum observables are complementary if measuring one observable increases uncertainty regarding the value of the other.

Example: position and momentum of an electron

Complementary quantum observables

A quantum observable refers to a measurable property of quantum system.

A pair of quantum observables are complementary if measuring one observable increases uncertanity regarding the value of the other.

Example: position and momentum of an electron

Complementary quantum observables

A quantum observable refers to a measurable property of quantum system.

A pair of quantum observables are complementary if measuring one observable increases uncertanity regarding the value of the other.

Example: position and momentum of an electron

Our goal: Fill this table

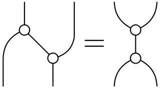
	†Monoidal Cats	MUCs
Observable		
Complementary observables		
Measurement		

Quantum observables and Frobenius algebras

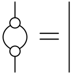
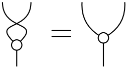
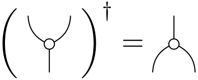
A quantum observable is described by an orthonormal basis for a Hilbert space.

In category of Hilbert spaces and linear maps, an orthonormal basis is precisely a special **commutative dagger Frobenius algebras**[1].

In a monoidal category, a Frobenius algebra (A, , , , ) consists of a monoid (A, , ), and a comonoid (A, , ) satisfying:

[Frob.] 

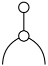
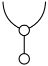
Special commutative dagger Frobenius algebra in symmetric †-monoidal category:

[Spl.]  [Comm.]  [Dagger.] 

[1] Coecke, B., Pavlovic, D. and Vicary, J. "A new description of orthogonal bases"

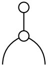
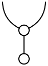
Properties of Frobenius algebras

If $(A,$  $,$  $,$  $,$  $)$ is a Frobenius algebra, then A is self-dual:

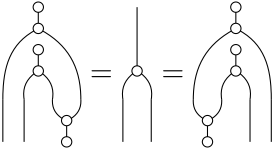
$\eta : I \longrightarrow A \otimes A :=$  $\epsilon : A \otimes A \longrightarrow I :=$ 

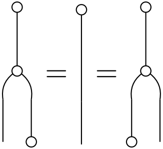
Properties of Frobenius algebras

If $(A,$  $,$  $,$  $,$  $)$ is a Frobenius algebra, then A is self-dual:

$\eta : I \longrightarrow A \otimes A :=$  $\epsilon : A \otimes A \longrightarrow I :=$ 

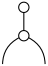
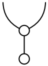
The monoid and comonoid are dual to one another by the above cap and cup:



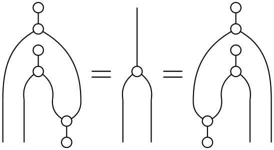


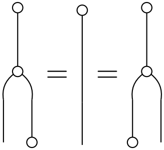
Properties of Frobenius algebras

If $(A,$  $)$ is a Frobenius algebra, then A is self-dual:

$\eta : I \longrightarrow A \otimes A :=$  $\qquad \epsilon : A \otimes A \longrightarrow I :=$ 

The monoid and comonoid are dual to one another by the above cap and cup:





Linear monoids generalize Frobenius Algebras to LDCs.

Linear monoids

In a symmetric LDC, a (symmetric) **linear monoid**, $A \overset{\circ}{\dashv\!\vdash} B$, contains a:

- a monoid $(A, \curlyvee : A \otimes A \to A, \; \circ\!\!\mid : \top \to A)$

- a dual for A, $(\eta, \epsilon) : A \dashv\!\vdash B$

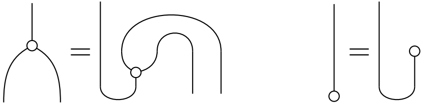
Linear monoids

In a symmetric LDC, a (symmetric) **linear monoid**, $A \overset{\circ}{\dashv\!\mid} B$, contains a:

- a monoid $(A, \; : A \otimes A \to A, \; : \top \to A)$

- a dual for A, $(\eta, \epsilon) : A \dashv\!\mid B$

together producing a comonoid $(B, \; : B \to B \oplus B, \; : B \to \bot)$



A **self-linear monoid** is a linear monoid, $A \overset{\circ}{\dashv\!\mid} B$, with $A \simeq B$.
For all practical purposes, a self-linear monoid is $A \overset{\circ}{\dashv\!\mid} A$

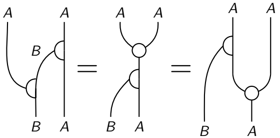
Alternate characterization of linear monoids

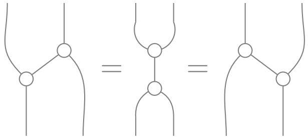
A linear monoid[2], A ⟶ B, consists of a $\otimes$-monoid, $(A, \; , \;)$, and a $\oplus$-comonoid, $(B, \; , \;)$ and:

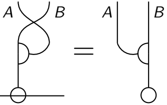
- monoid actions: $\; : A \otimes B \longrightarrow B$; $\; : B \otimes A \longrightarrow A$

- comonoid coactions: $\; : B \longrightarrow A \oplus B$; $\; : B \longrightarrow A \oplus B$

satisfying certain equations. The Frobenius equation is given as follows:





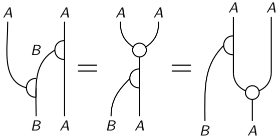
Symmetric linear monoid: 

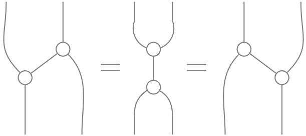
[2] Cockett, Koslowski and Seely (1999) "Introduction to linear bicategories"

Are self-linear monoids same as Frobenius algebras?

In a monoidal category, are self-linear monoids $A \overset{\circ}{\dashv\!\vdash} A$ same as Frobenius algebras?

Clue:

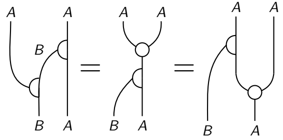


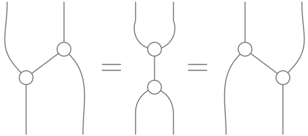


Are self-linear monoids same as Frobenius algebras?

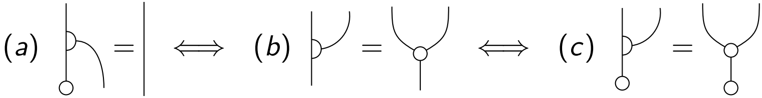
In a monoidal category, are self-linear monoids $A \overset{\circ}{\dashv} A$ same as Frobenius algebras?

Clue:



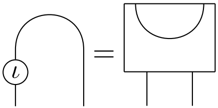


Yes! but..
Lemma A self-linear monoid $A \overset{\circ}{\dashv} A$ is Frobenius if and only if:

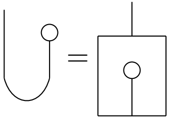


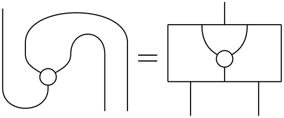
Dagger linear monoids

In a $\dagger$-LDC, $A \dashv\!\!\dashv A^\dagger$ is a $\dagger$-**dual** iff:



A $\dagger$-**linear monoid**, $(A, \curlyvee, \circ) \overset{\dagger\circ}{\dashv\!\!\dashv} (A^\dagger, \curlywedge, \circ)$, in a $\dagger$-LDC, is a linear monoid such that $(\eta, \epsilon) : A \dashv\!\!\dashv A^\dagger$ are $\dagger$-duals and:





Our goal: Fill this table

	†Monoidal Cats	MUCs
Observable	†-Frobenius algebra	†-linear monoid
Complementary observables		
Measurement		

Complementary observables

A pair of quantum obsevables are **complementary** if measuring one observable increases uncertainty regarding the value of the other.

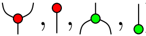
In a symmetric †-**monoidal category**, two special commutative †-Frobenius algebras

are strongly **complementary**[3] if

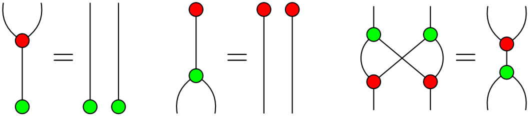
are bialgebras which are also Hopf.

[3]Bob Coecke and Ross Duncan (2008). "Interacting quantum observables"

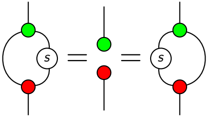
Complementary observables (cont...)

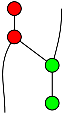
$(A,$  $)$ is a Hopf algebra:

Bialgebra rules:



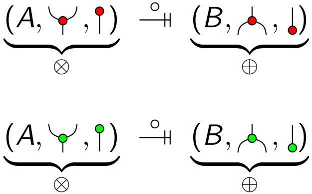
Hopf rule:



where, $s :=$ 

Complementary observables in isomix categories

In an LDC, can a linear monoid interact to produce two bialgebras?



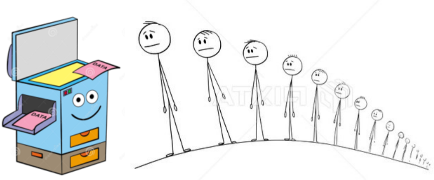
A $\otimes$-monoid and a $\oplus$-comonoid interacting to produce a bialgebra is simply **NOT POSSIBLE** due to the direction of the linear distributor

So what is solution?

Then came the spring.. In FMCS 2019, here at the Field Station, we got our first hint during my conversation with J.S. – **exponential modalities in linear logic**

Exponential modalities

Linear logic accommodates non-linear resources using the exponential modalities !,(of course / bang) and ? (why not / whimper):



!**A** refers to an infinite supply of the resource A

In this sense, the type $!A$ is an infinitely-copyable and freely disposable storage for resource A.

?**A** represents the notion of infinite demand.

Dually, the type $?A$ is a spontaneously-appearing infinite consumption for resource A.

! for monoidal categories

A colagebra modality for a symmetric monoidal category consists of a comonad $(!, \delta, \epsilon)$ and natural transformations $\Delta_A : !A \longrightarrow !A \otimes !A$ and $e_A : !A \longrightarrow I$ such that $(!A, \Delta_A, e_A)$ is a cocommutative comonoid and $\delta_A : !A \longrightarrow !!A$ preserves the comultiplication

An bialgebra modality[4] on an additive symmetric monoidal category is a monoidal coalgebra modality $(!, \delta, \epsilon, \Delta, e)$, a natural transformation $\nabla : !A \otimes !A \longrightarrow !A$ and a natural transformation $u : I \longrightarrow !A$ such that $(!A, \nabla, u)$ is a commutative monoid, and $(!A, \nabla, u, \Delta, e)$ is a bialgebra.

[4] Richard Blute, Robin Cockett, Robert Seely (2006) "Differential Categories"

! and ? for LDCs

In a $(!, ?)$-LDC[5]

- ! is a monoidal coalgebra comodality
  - $(!, \delta :! \Rightarrow !!, \epsilon :! \Rightarrow \mathbb{I})$ is a monoidal comonad
  - For each A, $(!A, \Delta_A, e_A)$ is a cocommutative comonoid

- ! is a comonoidal algebra modality
  - $(?, \mu :?? \Rightarrow ?, \eta : \mathbb{I} \Rightarrow ?)$ is a comonoidal monad
  - For each A, $(?A, \nabla_A, u_A)$ is a commutative monoid

- $(!, ?)$ is a linear functor

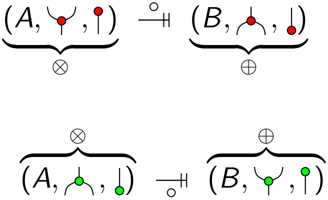
- The pairs (δ, μ), (ϵ, η), (Δ, ∇) are linear transformations

Examples: Category of finiteness relations, category of finiteness matrices over a commutative rig

[5] Richard Blute, Robin Cockett, and Robert Seely (1996). "! and ? - Storage as tensorial strength."

Solution

Hint: In a (!-?) LDC, $!A$ is a $\otimes$-comonoid and $?A$ is a $\oplus$-monoid.



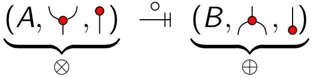
$\longleftarrow$ linear monoid

$\longleftarrow$ hypothetical!

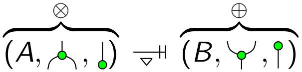
Now, obtaining two bialgebras – a $\otimes$ one and a $\oplus$ one – is straightforward!

The solution was quite simple

Hint: In a (!-?) LDC, $!A$ is a $\otimes$-comonoid and $?A$ is a $\oplus$-monoid.



$\longleftarrow$ linear monoid



$\longleftarrow$ Linear comonoid!

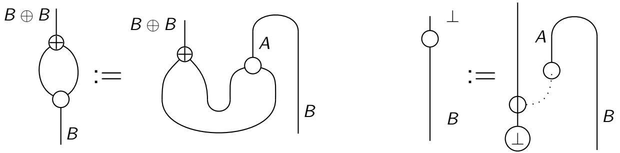
Now, obtaining two bialgebras – a $\otimes$ one and a $\oplus$ one – is straightforward!

Linear comonoids

In a symmetric LDC, a (symmetric) linear comonoid, $A \multimap B$, contains a:

- a $\otimes$-comonoid $(A, \Delta : A \to A \otimes A, \ e : A \to \top)$
- a dual for A, $(\eta, \epsilon) : A \dashv\!\dashv B$

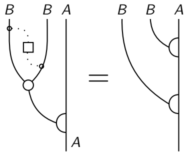
together producing a $\oplus$-monoid $(B, \nabla : B \oplus B \to B, u : \bot \to B)$

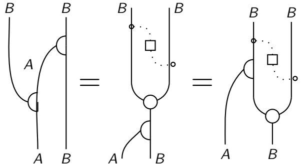


Alternate characterization of linear comonoids

A **linear comonoid**, $A \underset{\triangledown}{\dashv} B$, in an isomix category is equivalent to the following data:

- a $\oplus$-monoid $\quad (B, \;: B \oplus B \to B, \;: \bot \to B)$
- a $\otimes$-comonoid $\quad (A, \;: A \to A \otimes A, \;: A \to \top)$
- actions $\quad : B \otimes A \to A; \qquad : A \otimes B \to A,$
- coactions $\quad : B \to A \oplus B; \qquad : B \to B \oplus A,$





Linear bialgebra

In a symmetric LDC, a linear bialgebra

$$\frac{(a,b)}{(c,d)} : A \xrightarrow[\triangledown]{\circ} B$$

is given by a linear monoid $(a,b) : A \xrightarrow{\circ} B$ and a linear comonoid $(c,d) : A \xrightarrow[\triangledown]{} B$ interacting bialgebraically.

Thereby, $(A, \ , \ , \ , \ )$ is a $\otimes$-bialgebra and $(B, \ , \ , \ , \ )$ is a $\oplus$-bialgebra.

Exponential modalities linear bialgebras

In a (!-?)-LDC, for every object A, $(!A, \Delta_A, \multimap\!\!\!\!\perp_A)$ is a cocommutative $\otimes$-comonoid and A, $(?A, \nabla_A, \top_A)$ is a commutative $\oplus$-monoid

Exponential modalities linear bialgebras

In a (!-?)-LDC, for every object A, $(!A, \Delta_A, \overset{|}{\triangle}_A)$ is a cocommutative $\otimes$-comonoid and A, $(?A, \nabla_A, \underset{|}{\triangledown}_A)$ is a commutative $\oplus$-monoid

Lemma: In a (!-?)-LDC, for any dual gives a linear comonoid using the natual cocommutative comonoid $(!A, \Delta_A, \overset{|}{\triangle}_A)$.

$$(a, b) : A \dashv\!\mid B \Longrightarrow (a, b) : !A \underset{\triangledown}{\dashv\!\mid} ?B$$

Exponential modalities linear bialgebras

In a (!-?)-LDC, for every object A, $(!A, \Delta_A, \overset{|}{\vartriangle}_A)$ is a cocommutative $\otimes$-comonoid and A, $(?A, \nabla_A, \underset{|}{\triangledown}_A)$ is a commutative $\oplus$-monoid

Lemma: In a (!-?)-LDC, for any dual gives a linear comonoid using the natual cocommutative comonoid $(!A, \Delta_A, \overset{|}{\vartriangle}_A)$.

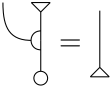
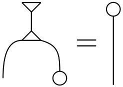
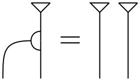
$$(a, b) : A \dashv\!\!\mid B \Longrightarrow (a, b) :!A \underset{\triangledown}{\dashv\!\!\mid} ?B$$

Lemma: In a (!-?)-LDC, any linear monoid and an arbitary dual gives a linear bialgebra using the natural cocommutative comonoid on !.

$$(a, b) : A \overset{\circ}{\dashv\!\!\mid} B \text{ and } (c, d) : A \dashv\!\!\mid B \Longrightarrow \frac{(a, b)}{(c, d)} :!A \underset{\triangledown}{\overset{\circ}{\dashv\!\!\mid}} ?B$$

Complementary systems in isomix cats

A complementary system $A \overset{\circ}{\underset{\triangledown}{-}}\!\!\dashv A$ in an isomix category a self-linear bialgebra, A (not necessarily in the core), such that:

[comp.1]  **[comp.2]**  **[comp.3]** 

Lemma: If A is a complementary system, then A is a $\otimes$-Hopf and $\oplus$-Hopf.

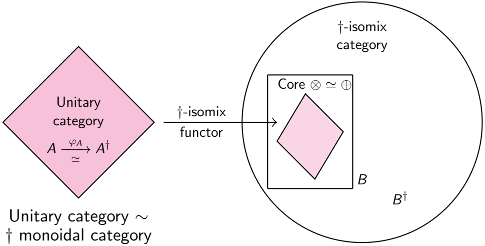
Our goal: Fill this table

	†Monoidal Cats	MUCs
Observable	†-Frobenius algebra	†-linear monoid †-linear comonoid
Complementary observables	Bialgebra	Linear bialgebra
Measurement		

Measurement in †-monoidal cats

A **demolition measurement**[6] on an object A is **retract** from A to a spl. comm. †-Frobenius algebra (an abstract quantum observable), E.

$$A \underset{r^\dagger}{\overset{r}{\rightleftarrows}} E \;\; \text{such that } r^\dagger r = 1_E$$



[6] Coecke and Pavlovic (2006). "Quantum measurements without sums"

Measurement in †-monoidal cats

A **demolition measurement**[7] on an object A is **retract** from A to a spl. comm. †-Frobenius algebra (an abstract quantum observable), E.

$$A \underset{r^\dagger}{\overset{r}{\rightleftarrows}} E \quad \text{such that } r^\dagger r = 1_E$$

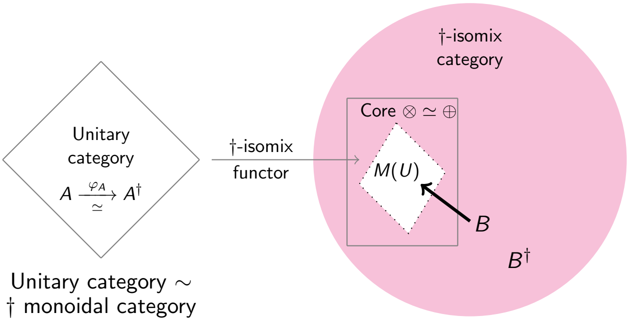
Classical types obtained by †-**splitting** †- **idempotents**[8] in a †-category

An idempotent $e : A \longrightarrow A$ is a †-idempotent if $e = e^\dagger$

e †-splits if $e = rr^\dagger$ and $r^\dagger r = 1$

[7] Coecke and Pavlovic (2006). "Quantum measurements without sums"
[8] Selinger (2008) "Idempotents in Dagger Categories: (Extended Abstract)"

MUC measurement = Compaction followed by demolition



A compaction in a MUC, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a **retraction** to an object in the unitary core $r : B \longrightarrow M(U)$.

Binary idempotents

Binary idempotent **(any category)**: $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

$$
\begin{tikzcd}
A \arrow[dr, "u"] & & & & \\
& B \arrow[dl, "v"'] & = & A \arrow[r, "u"] & B \\
A \arrow[dr, "u"'] & & & & \\
& B & & &
\end{tikzcd}
\qquad \vdots \qquad
\begin{tikzcd}
A \arrow[dr, "v"] & & & & \\
& B \arrow[dl, "u"'] & = & B \arrow[r, "v"] & A \\
A \arrow[dr, "v"'] & & & & \\
& B & & &
\end{tikzcd}
$$

Binary idempotents

Binary idempotent **(any category)**: $A \underset{v}{\overset{u}{\rightleftarrows}} B$ such that:

$$A \xrightarrow{u} B \xrightarrow{v} A \xrightarrow{u} B \;=\; A \xrightarrow{u} B$$

$$A \xrightarrow{v} B \xrightarrow{u} A \xrightarrow{v} B \;=\; B \xrightarrow{v} A$$

Suppose,
$e_A := uv$ splits through E, and

$e_B := vu$ splits through F

then $E \simeq F$

In a mix category, suppose $E \simeq F$ is in the core, then uv is **coring**

Dagger Binary idempotent

†-binary idempotent: **(†-LDC)** $A \underset{v}{\overset{u}{\rightleftarrows}} A^\dagger$

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\mathsf{u}"'] & A^{\dagger\dagger} \arrow[d, "\mathsf{u}^\dagger"] \\ & A^\dagger \end{tikzcd} \qquad \begin{tikzcd} A^\dagger \arrow[r, "\mathsf{v}"] \arrow[dr, "\mathsf{v}^\dagger"'] & A \arrow[d, "\iota"] \\ & A^{\dagger\dagger} \end{tikzcd}$$

Dagger Binary idempotent

†-binary idempotent: **(†-LDC)** $A \underset{v}{\overset{u}{\rightleftarrows}} A^\dagger$

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\mathrm{u}"'] & A^{\dagger\dagger} \arrow[d, "\mathrm{u}^\dagger"] \\ & A^\dagger \end{tikzcd} \qquad \begin{tikzcd} A^\dagger \arrow[r, "\mathrm{v}"] \arrow[dr, "\mathrm{v}^\dagger"'] & A \arrow[d, "\iota"] \\ & A^{\dagger\dagger} \end{tikzcd}$$

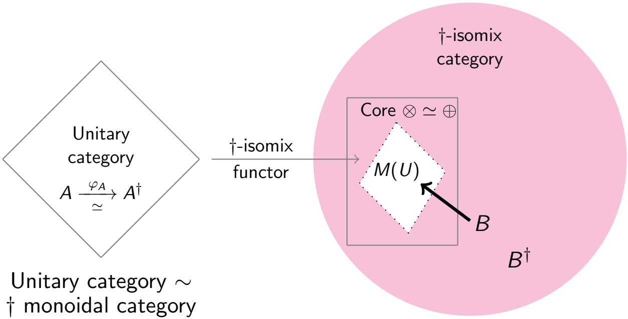
Observation:

$$(e_A)^\dagger = (\mathrm{uv})^\dagger = v^\dagger u^\dagger = v\iota u^\dagger = vu = e_{A^\dagger}$$

In a †-monoidal category where $A = A^\dagger$, †-binary idempotents are precisely †-idempotents.

In a †-monoidal category when a †-binary idempotent splits, it always †-splits.

Characterizing compaction using binary idempotents



Unitary category ∼
† monoidal category

A compaction in a MUC, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is a **retraction** to an object in the unitary core $r : B \longrightarrow M(U)$.

Theorem: In a †-isomix category, $r : A = B \longrightarrow M(U)$ is a compaction if and only if $M(U)$ is given by splitting a coring †-binary idempotent.

Goal accomplished

	†Monoidal Cats	MUCs
Observable	†-Frobenius algebra	†-linear monoid †-linear comonoid
Complementary observables	Bialgebra	Linear bialgebra
Measurement	†-idempotents	†-binary idempotent

Main Result **New**

A connection between exponential modalities and complementary observables

Theorem: In a $(!, ?)$-isomix category with **free** exponential modalities, every complementary system is given by splitting a binary idempotent on the linear bialgebra induced on the free exponentials.

(The proof uses a series of results)

The structures and results discussed extend directly to †-linear bilagebras in †-isomix categories with free exponential modalities due to the †-linearity of $(!, ?)$, (η, ϵ), (Δ, ∇), and $(\perp, \top)$.

Theorem sketch

$$\frac{(a,b)}{(c,d)} : A \xrightarrow[\nabla]{\circ}\!\!\shortmid\!\shortmid\; A$$

Linear Monoid

$$(a,b) : A \xrightarrow{\circ}\!\!\shortmid\!\shortmid\; A$$

$$(a_!,b_?) : !A \xrightarrow{\boxed{\circ}}\!\!\shortmid\!\shortmid\; ?A$$

Linear Comonoid

$$(c,d) : A \xrightarrow[\nabla]{}\!\!\shortmid\!\shortmid\; A$$

$$(c_!,d_?) : (!A, \Delta_A, \delta_A) \xrightarrow[\nabla]{}\!\!\shortmid\!\shortmid\; (?A, \nabla_A, \curlyvee_A)$$

$$\frac{(a_!,b_?)}{(c_!,d_?)} : !A \xrightarrow[\nabla]{\boxed{\circ}}\!\!\shortmid\!\shortmid\; ?A$$

Linear Bialgebra

$$!A \underset{I^\sharp}{\overset{\varrho}{\rightleftarrows}} A \underset{1^\flat}{\overset{\varepsilon}{\rightleftarrows}} ?A$$

Binary idempotents

Why is this result interesting?

Quantum harmonic oscillator(QHO) is one of the most important system model used in quantum mechanics.

A harmonic oscillator is a system which has been displaced from its equilibrium and has a restoring force acting on it.

Quantum harmonic oscillator is modelled as Bosonic Fock spaces which are in turn modelled using ! **exponential modality**[9],[10].

Total energy of QHO is given by the equation: $\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$

Alternately, ladder operators allow calculation of energy eigen value without solving a differential equation.

Position and momentum operators, which are complementary, can be recovered from the ladder operators for harmonic oscillators.

[9] Blute, Panandagen, Seely (1994) "Fock Space: A Model of Linear Exponential Types"

[10] Vicary (2007) "A categorical framework for the quantum harmonic oscillator"

Linear comonoids and linear bialgebras

(Take away from Part II)

Linear comonoid:

a $\otimes$-comonoid $(A,\ \curlywedge : A \to A \otimes A,\ \dot{\bot} : A \to \bot)$

and a dual $(\eta, \epsilon) : A \dashv\!\!\vdash B$

Linear bialgebras in symmetric LDCs: bialgebraic interaction between a linear monoid and a linear comonoid giving a $\otimes$-bialgebra, and a $\oplus$-bialgebra.

References (Linearly distributive categories)

- Linearly distributive categories:
  Robin Cockett, Robert Seely (1997) "Weakly distributive categories"
- Graphical calculus for LDCs:
  Robin Cockett and Robert Seely (1997) "Proof theory for full intuitionistic linear logic, bilinear logic, and mix categories".
- Linear functors and transformations:
  Robin Cockett, Robert Seely (1999) "Linearly distributive functors"
- Linear duals, monoids:
  Robin Cockett, Jürgen Koslowski, and Robert Seely (2000) "Introduction to linear bicategories"
- Exponential modalities:
  Richard Blute, Robin Cockett, and Robert Seely (1996) "! and ? - Storage as tensorial strength".
  Richard Blute et. al. (2019) "Differential categories revisited"

References (Categorical quantum mechanics)

- Categorical quantum mechanics:
  - Bob Coecke, Aleks Kissinger (2017) “Picturing Quantum Processes”
  - Chris Heunen, Vicary Jamie (2019) “Categories for Quantum Theory: An Introduction”
- Dagger compact closed categories, completely positive maps:
  - Peter Selinger (2007) “Dagger compact closed categories and completely positive maps”
  - Bob Coecke, Chris Heunen (2016) “Pictures of complete positivity in arbitrary dimension”
- Measurement, complementarity:
  - Bob Coecke and Dusko Pavlovic (2007) “Quantum measurements without sums”
  - Peter Selinger (2008) “Idempotents in dagger Categories”
  - Bob Coecke and Ross Duncan (2011) “Interacting quantum observables: categorical algebraand diagrammatics”
  - Bob Coecke, Dusko Pavlovic, and Jamie Vicary (2012) “A new description of orthogonal bases”

Other References

- Jamie Vicary (2008) "A categorical framework for the quantum harmonic oscillat"
- Jeff Egger (2011) "On involutive monoidal categories
- Jeff Egger (2012) "The Frobenius relations meet linear distributivity"

Publications and preprints

- Robin Cockett, Priyaa Srinivasan, "Complementarity and exponential modalities" ArXiv: 2103. 05191 [math.CT], Mar. 2021 (To appear in the proceedings of ACT 2021)
- Robin Cockett, Priyaa Srinivasan,"Completely positive maps for mixed unitary categories" ArXiv:1905.08877 [math.CT], May. 2019. (Presented at QPL 2019)
- Robin Cockett, Cole Comfort, and Priyaa Srinivasan "Dagger linear logic for categorical quantum mechanics" ArXiv:1809.00275 [math.CT], Sep. 2018. (under review in Logical Methods in Computer Science)