FMCS 2022

Dagger linear logic and categorical quantum mechanics

Part I

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Joint work with Robin Cockett and Cole Comfort

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Linear logic

In 1987, Girard introduced linear logic as a logic for resources manipulation.

Classical logic treats statements as truth values; linear logic treats statements as resources which cannot be duplicated or destroyed.

p : to spend a dollar

q : to buy an apple

"$p \Rightarrow q$" has the meaning that if a dollar is spent then an apple can be bought.

A person can either have a dollar or an apple at a given time but not both.

The word "linear" refers to this resource sensitivity of the logic.

Linear logic (cont..)

A proof of a statement in linear logic may be regarded as a series of resource transformations

Linear logic is concerned with how a statement can be proved, rather than, merely if the statement is provable or not.

Thus, linear logic, emphasizes the structure of proofs rather than provability

Girard introduced proof-nets drawn as circuits to represent proofs in linear logic

To know when a circuit is a valid proof-net paved way to the study of categorical proof theories of linear logic.

Categorical semantics for multiplicative linear Logic (MLL)

Linear logic fragment	**Connectives**
Negation	$A^{\perp}$
Multiplicative	$(\otimes, 1)$ and $(\parr, \perp)$
Additive	$(\&, \top)$ and $(\oplus, 0)$
Exponentials	! and ?

Linear logic fragment	**Categorical proof theory**
MLL $(\otimes, 1)$ and $(\parr, \perp)$	Linearly distributive categories[1] $(\mathbb{X}, \otimes, \top, \oplus, \perp)$
MLL with negation	$*$-autonomous categories[2]
Compact MLL $(\otimes = \parr, 1 = \perp)$	Monoidal categories $(\mathbb{X}, \otimes, I)$
Compact closed categories[3]	Compact MLL with negation

[1] Cockett and Seely (1997) "Weakly Distributive Categories"

[2] Barr (1991) "$*$-autonomous categories and linear logic"

[3] Kelly and Laplaza (1980) "Coherences for compact closed categories"

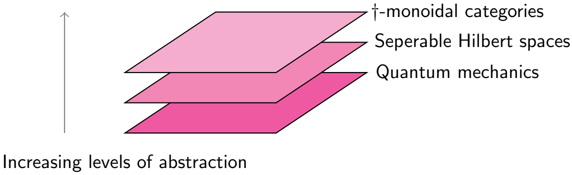
Linear logic for quantum mechanics

Linear logic captures the essence of quantum mechanics of quantum mechanics owing to its resource-sensitive character.

(In linear logic) *Thou shall not duplicate or discard an arbitrary resource* $\approx$ (By no-cloning theorem) *Thou shall not duplicate an arbitrary quantum state*

Categorical Quantum Mechanics (CQM) uses this connection to develop a diagrammatic framework based on the graphical calculus of monoidal categories for describing quantum mechanics

CQM introduced a dagger functor for monoidal and compact closed which abstracts unitary evolution of quantum systems



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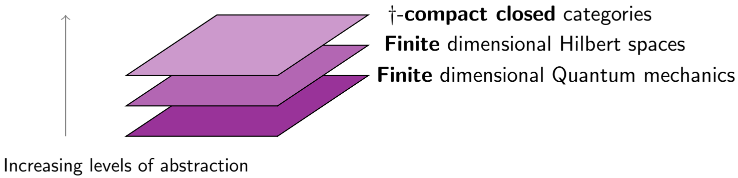

On dimensionality

Dagger compact closed categories $\Rightarrow$ Finite-dimensionality on Hilbert Spaces

Practical applications of CQM has been focused on quantum computing, and quantum information theory (finite dimensional branches of quantum mechanics).

However, **infinite-dimensional systems** occur in many quantum settings - quantum chemistry, quantum dynamics, to name a few.

How can one model infinite dimensional systems using dagger linear logic?

Our approach towards infinite-dimensional systems

Monoidal categories are the categorical semantics of **compact** multiplicative linear logic.

Compact closed categories are monoidal categories with negation.

Linearly distributive categories (LDCs) are the categorical semantics of **non-compact** multiplicative linear logic.

$*$-autonomous categories are LDCs with negation.

$*$-autonomous categories have already been used to model systems in infinite dimensions.

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Move beyond Hilbert spaces to non-compact †-linear logic . . .

Categorical semantics of multiplicative linear logic

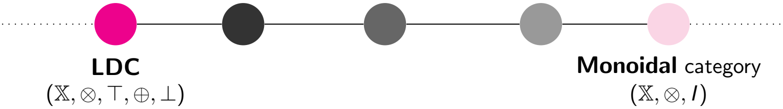
Linearly distributive categories (LDCs):

$$(\mathbb{X}, \otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes) \qquad (\mathbb{X}, \oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$$

linked by linear distributors:

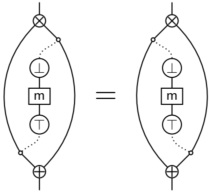
$$\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

Monoidal categories: LDCs in which $\otimes = \oplus; \top = \bot$



Mix category

A **mix category** is a LDC with a mix map $\mathsf{m} : \bot \to \top$ in $\mathbb{X}$ such that

$\mathsf{mx}_{A,B} : A \otimes B \to A \oplus B :=$ 

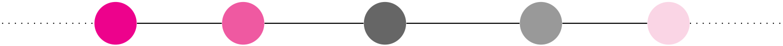
$$(1 \oplus (u_\oplus^L)^{-1})(1 \otimes (\mathsf{m} \oplus 1))\delta^L (u_\otimes^R \oplus 1) = ((u_\oplus^R)^{-1} \oplus 1)((1 \oplus \mathsf{m}) \otimes 1)\delta^R (1 \oplus u_\otimes^R)$$

mx is called a **mixor**. The mixor is a natural transformation.

Mix category

$\mathsf{m} : \bot \to \top$

$\mathsf{mx} : A \otimes B \to A \oplus B$



LDC

$(\mathbb{X}, \otimes, \top, \oplus, \bot)$

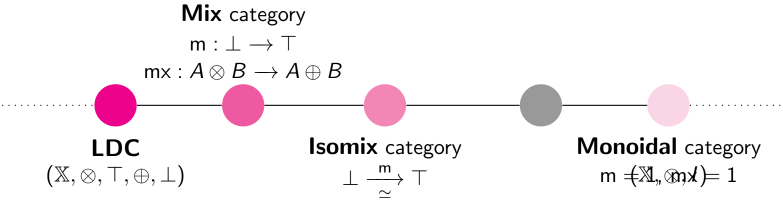
Monoidal category

$(\mathbb{X}, \otimes, I)$

Isomix categories

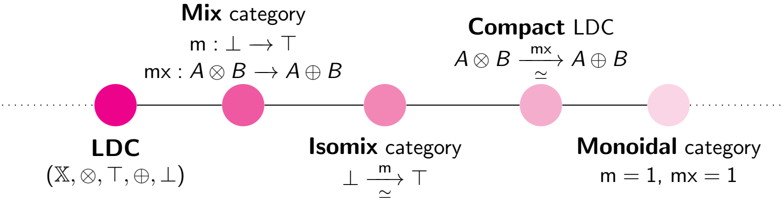
It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the mixor an isomorphism.



Compact LDCs

A **compact LDC** is an LDC in which every mixor is an isomorphism i.e., in a compact LDC $\otimes \simeq \oplus$.



Compact LDCs $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ are linearly equivalent to underlying monoidal categories $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$.

The core of a mix category

The **core** of a mix category, $\text{Core}(\mathbb{X}) \subseteq \mathbb{X}$, is the full subcategory determined by objects $U \in \mathbb{X}$ for which the natural transformation is also an isomorphism:

$$U \otimes A \xrightarrow{\mathsf{mx}_{U,A}} U \oplus A$$

The core of a **mix** category is closed to $\otimes$ and $\oplus$.

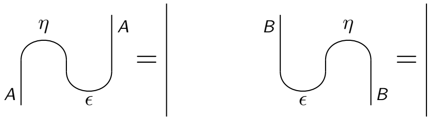
The core of an isomix category contains the monoidal units $\top$ and $\bot$.

$*$-autonomous categories

In a symmetric LDC, an object A has a dual B if there exist:

$$\eta : \top \longrightarrow A \oplus B \qquad \epsilon : B \otimes A \longrightarrow \bot$$

such that:



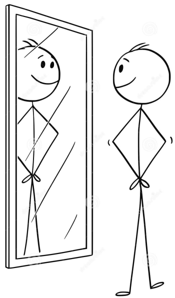
A symmetric $*$-**autonomous category** is an LDC in which every object has a chosen dual object.

Dagger monoidal categories

In a †-**monoidal category**, dagger is a contravariant functor:

- Stationary on objects $A = A^\dagger$
- Involution on maps $f^{\dagger\dagger} = f$
- Coherent with the tensor $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- The basic natural isomorphisms are unitary:

$$a_\otimes^{-1} = a_\otimes^\dagger; \quad u_\otimes^{-1} = u_\otimes^\dagger; \quad c_\otimes^{-1} = c_\otimes^\dagger$$



Dagger for LDCs

The definition of $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ cannot be directly imported to LDCs because the dagger minimally has to **flip the tensor products**: $(A \otimes B)^\dagger = A^\dagger \oplus B^\dagger$.

Why? If dagger is identity-on-objects, then the linear distributor degenerates to associator:

$$\frac{\delta^R : (A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)}{(\delta_R)^\dagger : A \oplus (B \otimes C) \longrightarrow (A \oplus B) \otimes C}$$

Dagger for LDCs

The dagger for an LDC is a contravariant Frobenius functor which is a linear involutive equivalence.

A †-LDC is a LDC $\mathbb{X}$ with a dagger functor $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ and the natural isomorphisms:

$$
\begin{aligned}
\text{tensor laxtors: } & \lambda_\oplus : A^\dagger \oplus B^\dagger \longrightarrow (A \otimes B)^\dagger \\
& \lambda_\otimes : A^\dagger \otimes B^\dagger \longrightarrow (A \oplus B)^\dagger \\
\text{unit laxtors: } & \lambda_\top : \top \longrightarrow \bot^\dagger \\
& \lambda_\bot : \bot \longrightarrow \top^\dagger \\
\text{involutor: } & \iota : A \longrightarrow A^{\dagger\dagger}
\end{aligned}
$$

such that certain coherence conditions hold.

†-isomix categories

A †-mix category is a †-LDC with $m : \bot \longrightarrow \top$ such that:

$$\begin{array}{ccc} \bot & \xrightarrow{\mathsf{m}} & \top \\ {\scriptstyle \lambda_\bot}\downarrow & & \downarrow{\scriptstyle \lambda_\top} \\ \top^\dagger & \xrightarrow[\mathsf{m}^\dagger]{} & \bot^\dagger \end{array}$$

Lemma 1: The following diagram commutes in a mix †-LDC:

$$\begin{array}{ccc} A^\dagger \otimes B^\dagger & \xrightarrow{\mathsf{mx}} & A^\dagger \oplus B^\dagger \\ {\scriptstyle \lambda_\otimes}\downarrow & & \downarrow{\scriptstyle \lambda_\oplus} \\ (A \oplus B)^\dagger & \xrightarrow[\mathsf{mx}^\dagger]{} & (A \otimes B)^\dagger \end{array}$$

For a †-mix category, if m is an isomorphism, then $\mathbb{X}$ is a †-isomix category.

Finiteness spaces and relations

A finiteness space, $(X, \mathcal{A}, \mathcal{A}^{\perp})$, consists of a set X and a subset $\mathcal{A} \subseteq P(X)$ where

$$\mathcal{A}^{\perp} = \{b \mid b \in P(X) \text{ with for all } a \in A, |a \cap b| < \infty\},$$

and $\mathcal{A} = \mathcal{A}^{\perp\perp}$.

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Suppose X is a finite set, $(X, P(X), P(X))$ is a finiteness space.

A finiteness relation, $(X, \mathcal{A}, \mathcal{A}^{\perp}) \xrightarrow{R} (Y, \mathcal{B}, \mathcal{B}^{\perp})$ is relation $X \xrightarrow{R} Y$ such that

$$\forall A \in \mathcal{A},\ AR \in \mathcal{B} \ \text{ and } \ \forall B \in \mathcal{B}^{\perp},\ RB \in \mathcal{A}^{\perp}$$

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Finiteness spaces with finiteness relation is a †-**isomix** $*$-**autonomous** category:

$$(X, \mathcal{A}, \mathcal{A}^\perp)^\dagger := (X, \mathcal{A}^\perp, \mathcal{A}) \text{ and } R^\dagger \text{ is the converse relation.}$$

Finiteness matrices over complex numbers

FMat($\mathbb{C}$) is defined as follows:

Objects: Finiteness spaces $(X, \mathcal{A}, \mathcal{A}^{\perp})$

Maps: **Finiteness matrices** $(X, \mathcal{A}, \mathcal{A}^{\perp}) \xrightarrow{M} (Y, \mathcal{B}, \mathcal{B}^{\perp})$ is a matrix $XxY \xrightarrow{M} \mathbb{C}$ such that

$$supp(M) := \{(x, y) | x \in X, y \in Y \text{ and } M(x, y) \neq 0\}$$

is a finiteness relation $(X, \mathcal{A}, \mathcal{A}^{\perp}) \rightarrow (Y, \mathcal{A}, \mathcal{B}^{\perp})$.

Composition: Matrix multiplication

Dagger: $(X, \mathcal{A}, \mathcal{B})^{\dagger} := (X, \mathcal{B}, \mathcal{A})$
$M^{\dagger}$ is the conjugate transpose of M.

Mat($\mathbb{C}$) is a full subcategory of FMat($\mathbb{C}$) which is determined by the objects, $(X, P(X), P(X))$, where X is a finite set and is the core of FMat($\mathbb{C}$).

An intuition for finiteness Matrices

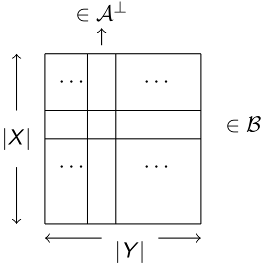

Figure: Finiteness matrix $(X, \mathcal{A}, \mathcal{A}^\perp) \xrightarrow{M} (Y, \mathcal{B}, \mathcal{B}^\perp)$

Composition of finiteness matrices

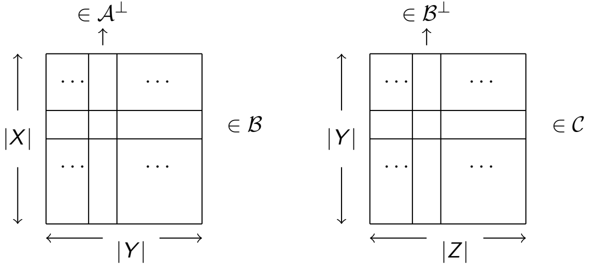

Figure: Finiteness matrix $(X, \mathcal{A}, \mathcal{A}^\perp) \xrightarrow{M} (Y, \mathcal{B}, \mathcal{B}^\perp) \xrightarrow{N} (Z, \mathcal{C}, \mathcal{C}^\perp)$

Unitary structure for dagger isomix cats (1)

In †-monoidal categories, unitary evolution of quantum systems is axiomatized as **unitary isomorphisms**.

$f : A \longrightarrow B$ is a unitary isomorphism in a †-monoidal category iff:

$$f^\dagger : B \longrightarrow A = f^{-1}$$

What are unitary isomorphisms in †-LDCs?

In a †-LDC, suppose $f : A \xrightarrow[\simeq]{} B$:

$$A \xrightarrow{f} B$$

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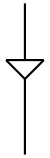
Pre-unitary and unitary objects

An object, A, in a $\dagger$-isomix category is pre-unitary if A is in the core and has a unitary structure map:

$$A \xrightarrow[\simeq]{\varphi_A} A^\dagger =$$



such that:

$$\varphi_{A^\dagger} = {\varphi_A^{-1}}^\dagger : A^\dagger \to A^{\dagger\dagger} \qquad A \xrightarrow{\varphi_A} A^\dagger \xrightarrow{{\varphi_A^{-1}}^\dagger} A^{\dagger\dagger} = A \xrightarrow{\iota} A^{\dagger\dagger}$$

In a $\dagger$-isomix category, unitary objects are an essentially small class of pre-unitary objects, such that the unitary structure of each object preserves the $\dagger$-isomix structure.

Unitary isomorphisms

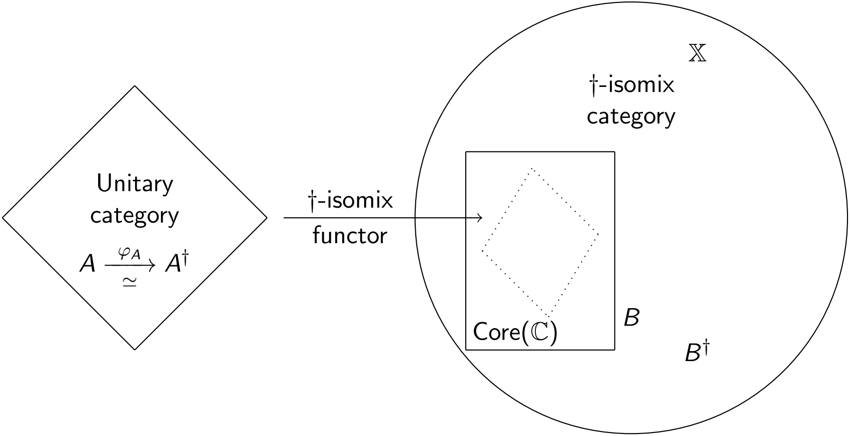
Unitary isomorphism: In a †-isomix cat, A, B are unitary objects,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \varphi_A}\downarrow & & \downarrow{\scriptstyle \varphi_B} \\ A^\dagger & \xleftarrow[f^\dagger]{} & B^\dagger \end{array}$$

Unitary category: compact †-LDC in which every object is unitary.

Every unitary category is †-linearly equivalent to a †-monoidal category.

Mixed unitary categories (MUCs)



A mixed unitary category, $M : \mathbb{U} \longrightarrow \mathbb{C}$, is

†-isomix functor: unitary category $\longrightarrow$ †-isomix category

Examples of MUCs

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- FHilb $\hookrightarrow$ Chus_I(Vec($\mathbb{C}$)): Finite-dimensional Hilbert spaces embedded into Chu spaces over complex vector spaces
- Unitary construction: Given any †-isomix category $\mathbb{C}$ one can construct a canonical MUC, Unitary($\mathbb{C}$) $\hookrightarrow \mathbb{C}$, by choosing its pre-unitary objects.

Unitary($\mathbb{C}$):

Objects: Pre-unitary objects (U, α),

Maps: $(U, \alpha) \xrightarrow{f} (V, \beta)$ where $U \xrightarrow{f} V$ is any map of $\mathbb{X}$.

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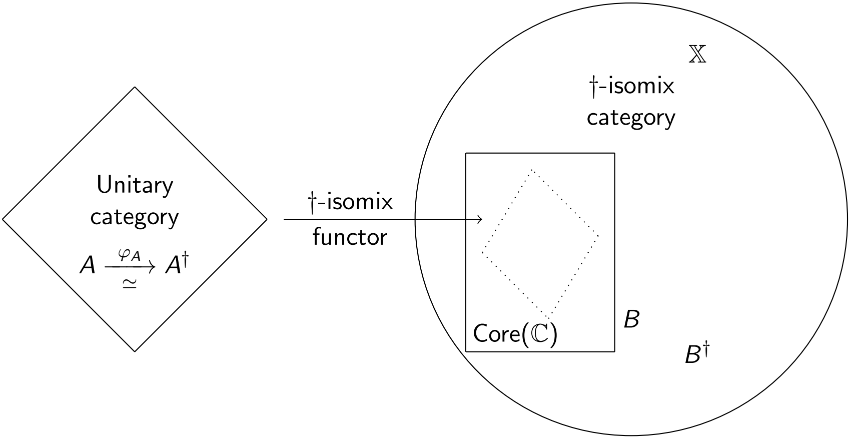

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