Hurewicz for Symmetric Spectra

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Outline

Nice Spaces

Our category of “nice” topolicial spaces is going to be the category of compactly generated, weak Hausdorff spaces, **CGwH**, a full subcategory of **Top**. Call the objects of **CGwH** *spaces*.

Briefly, a space X is compactly generated if, for every compact Hausdorff space K, every map $f\colon K \to X$, and every subset $Y \subseteq X$ we have that Y is closed if and only if $f^{-1}(Y)$ is.

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A space X is weakly Hausdorff if and only if the image of any compact Hausdorff space K in X is closed.

Smash Product

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$$X \wedge Y := X \times Y / X \amalg Y$$

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Further, one can topologize **CGwH**$_*(X, Y)$ to make $\wedge$ into a closed symmetric monoidal product.

Homotopy Relation

Call parallel maps $f, g\colon X \to Y$ *homotopic* if there exist a map $H\colon X \wedge I_+ \to Y$ such that $f = H(- \wedge 0)$ and $g = H(- \wedge 1)$.

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Intuitively f is homotopic to g if "f can be deformed into g"

Spheres

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Happily: $S^n \cong \{v \in \mathbb{R}^{n+1} | |v| = 1\}$ the usual n-sphere.

Homotopy Groups

There are functors $\pi_n \colon \mathbf{CGwH}_* \to \mathbf{Grp}$ (pron: '*n*th homotopy group') for $n \geq 1$ which takes a pointed space to the group obtained by taking the homotopy classes of pointed maps $[S^n, -]$ with multiplication given by composition of loops.

Example

X	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$	$\pi_4(X)$	$\pi_5(X)$	$\pi_6(X)$	$\pi_7(X)$
S^1	$\mathbb{Z}$	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$

Homology Groups

There are also functors $H_n \colon \mathbf{CGwH} \to \mathbf{Grp}$ (pron: 'nth homology groups') for $n \geq 0$ which takes a (non-pointed) space first to the chain complex whose n-th level is $\hom(I^n, X)$ with differential map ∂ taking the alternating sum of the faces of I^n, then takes returns $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

Example

$$H_n(S^k) = \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

Hurewicz Map

There is a natural map $h_*\colon \pi_* \Rightarrow H_*$ called the *hurewicz map* defined as follows:

Given $f\colon S^n \to X$ we get a map $H_n(f)\colon H_n(S^n) \to H_n(X)$

Since $H_n(S^n) \cong \mathbb{Z}$ we may set $h_n(f) = H_n(f)(1)$ by abuse of notation.

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Proposition: If the set of homotopy classes of maps from $S^0 \to X$ is a singleton and if $\pi_1(X,*) = \cdots = \pi_k(X,*) = 0$ then $h_{k+1}\colon \pi_{k+1}(X,x_0) \to H_{k+1}(X)$ is an isomorphism and $H_1(X) = H_2(X) = \cdots = H_k(X) = 0$

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Theorem (Freudenthal): For any space X the sequence

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This allows us to define the *nth stable homotopy group* functor $\pi_n^S \colon \mathbf{sSet}_* \to \mathbf{Grp}$ by $\pi_n^S(X,*) = \operatorname{colim} \pi_{n+k}(S^k \wedge X)$.

It would be nice to get a setting in which this particular functor was more akin to the original.

The Original Solution

Originally the appropriate category, *spectra*, in which to study stable homotopy had, as objects, $\mathbb{N}$-indexed sequences

$$\begin{array}{ccccccccc} \cdots & & S^1 \wedge X_n & & S^1 \wedge X_{n+1} & & S^1 \wedge X_{n+2} & & \cdots \\ & \searrow^{\sigma_{n-1}} & & \searrow^{\sigma_n} & & \searrow^{\sigma_{n+1}} & & \searrow^{\sigma_{n+2}} & \\ \cdots & & X_n & & X_{n+1} & & X_{n+2} & & \cdots \end{array}$$

This worked but...

The original category of spectra was plagued by the unfortunate truth that **CGwH** (and the other categories of "nice" spaces) are symmetric monoidal, the homotopy categories of these categories are symmetric monoidal, the homotopy category of the category of spectra is symmetric monoidal, but the category of spectra itself is not...

Symmetric Spectra

A *symmetric spectrum*, X, is a sequence $\{X_n | n \in \mathbb{N}\}$ of spaces such that each X_n is equipped with a basepoint preserving left Σ_n-action together with $\Sigma_1 \times \Sigma_n$ equivariant *structure maps* $S^1 \wedge X_n \to X_{n+1}$ such that the composite

$$S^p \wedge X_q \xrightarrow[S^{p-1} \wedge \sigma_{1,q}]{} S^{p-1} \wedge X_{q+1} \longrightarrow \cdots \xrightarrow[\sigma_{1,q+p-1}]{} X_{p+q}$$

is $\Sigma_p \times \Sigma_q$ equivariant.

Example

Let X be any space X, then define the spectrum $\underline{X}$ to be:

$\underline{X}_n := S^n \wedge X$

The structure map $\sigma_n \colon S^1 \wedge \underline{X}_n \to \underline{X}_{n+1}$ is the isomorphism $S^1 \wedge S^n \wedge X \to S^{n+1} \wedge X$

The groupoid Σ

Let Σ denote the groupoid whose objects are the natural numbers, and where $\Sigma(n, m) = \begin{cases} \Sigma^n, & n = m \\ \emptyset & \text{otherwise} \end{cases}$

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Notice that this category is symmetric monoidal via the addition functor: $(p, q) \mapsto p + q, (\tau, \gamma) \in \Sigma_p \times \Sigma_q \mapsto \tau \times \gamma \in \Sigma_{p+q}$

$\mathbf{CGwH}^{\Sigma}$

So we have two categories enriched over **CGwH**. Consider the functor category $\mathbf{CGwH}^{\Sigma}$. The day convolution provides a symmetric monoidal product:

$$X(-) \otimes Y(-) := \int^{p,q} X(p) \wedge Y(q) \wedge \Sigma(p+q,-)$$

Which in this case resolves to:

$$(X \otimes Y)(n) = \coprod_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} X(p) \wedge Y(q)$$

A monoid for $\otimes$ is: S^- which takes n to S^n and Σ_n acts on S^n via

$$s_1 \wedge \ldots \wedge s_n \mapsto s_{\tau 1} \wedge \ldots \wedge s_{\tau n}$$

So now we have a monoidal category $\mathbf{CGwH}^{\Sigma}$ and a monoid S^- thus we may consider modules over that monoid:

$$\begin{array}{ccc} S^- \otimes S^- \otimes X & \longrightarrow & S^- \otimes X \\ \downarrow & & \downarrow \\ S^- \otimes X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} U \otimes X & \longrightarrow & S^- \otimes X \\ & \searrow & \downarrow \\ & & X \end{array}$$

What are these things?

Let X be one of these S^- modules, then X is described by the following data:

- A sequence of spaces X_n for each $n \in \mathbb{N}$ such that each space is equipped with an action of Σ_n
- A collection of $\Sigma_p \times \Sigma_q$ maps $\sigma_{p,q} \colon S^p \wedge X_q \to X_{p+q}$ such that all reasonable diagrams commute

Note that it is enough to specify the $\sigma_{1,q} \colon S^1 \wedge X_q \to X_{q+1}$ and verify that the composites

$$S^p \wedge X_q \xrightarrow[S^{p-1} \wedge \sigma_{1,q}]{} S^{p-1} \wedge X_{q+1} \longrightarrow \cdots \xrightarrow[\sigma_{1,q+p-1}]{} X_{p+q}$$

are $\Sigma_p \times \Sigma_q$-equivariant.

Homotopy

Let X be a symmetric spectrum with spaces X_n and structure maps σ_n.

$$\pi_k(X_n, *) \xrightarrow{S^1 \wedge -} \pi_{k+1}(S^1 \wedge X_n, *) \xrightarrow{\pi_{k+1}(\sigma_{n+1}, *)} \pi_{k+1}(X_{n+1}, *)$$

From this we may for the colimit:

$$\pi_k^S(X) = \operatorname*{colim}_k \pi_{n+k}(X_n)$$

A Fun Fact

Given any symmetric spectrum E one can define an *E-homology*, $E_k(-)$, on **CGwH** by snetting $E_k(X) = \pi_k(X \wedge K)$ where $X \wedge K$ is the result of smashing each level of E with the space X and ignoring X when considering the action.

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Warning!

Homology

This suggests that, given a spectrum E we can define the E-homology for spectra the same way! If X is a symmetric spectrum then define the kth E-homology, $E_k(X)$ to be $\pi_k(E \wedge X)$.

Hurewicz

There is a map $S^0 \to \mathrm{H}\,\mathbb{Z}$, now, given any pointed symmetric spectrum X we have that $X \cong X \wedge S^0$ so we get a hurewicz map for a 'nice ' subcategory of $\mathbf{Sp}^{\Sigma}$

$$\pi_*(X) \cong \pi_*(X \wedge S^0) \to \pi_*(X \wedge \mathrm{H}\,\mathbb{Z}) := H_*(X)$$