Shameless Promotion!

The talks by Robin & Robert from last year's FMCS have (finally!) appeared "in print" (*i.e.* on-line):

Cartesian differential storage categories
(Blute-Cockett-Seely)

http://arxiv.org/abs/1405.6973 and

http://www.math.mcgill.ca/rags/

Revisiting the term calculus for proof-nets

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June 2014

http://www.math.mcgill.ca/rags/

Typed Circuits

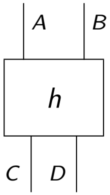
A Typed Circuit is built from

- types
- components (with type signatures giving types of inputs and outputs)

which are then “juxtaposed” together (appropriate wires joined).
(Details . . .)

Circuit Expressions

A typical component looks like this:



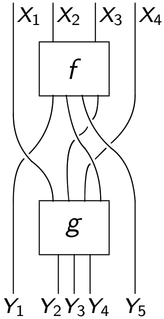
Naming the "wires": $x_1 \colon A, x_2 \colon B, y_1 \colon C, y_2 \colon D$, we'd form the circuit expression

$$[x_1, x_2]h[y_1, y_2]$$

Juxtaposition

$$([x_2, x_3]f[y_1, z_1, y_5, z_2]; [x_1, z_2, x_4, z_1]g[y_2, y_3, y_4])$$

represents



(Think "composition")

Abstraction

Given a circuit expression C, we can abstract it:

$$T_1, \ldots, T_n \colon \langle x_1, \ldots, x_n \mid C \mid y_1, \ldots, y_m \rangle : T'_1, \ldots, T'_m$$

(the types are "optional"; the variables are abstracted in just the way λ-abstraction, $\lambda x f$, abstracts variables).
An abstraction can be dissipated by reinserting the variables

$$[x_1, \ldots, x_n] \langle x_1, \ldots, x_n \mid C \mid y_1, \ldots, y_m \rangle [y_1, \ldots, y_m]$$

(think "β-reduction" $(\lambda x f[x])(x') = f[x']$)
and coalesced

$$\langle x_1, \ldots, x_n \mid [x_1, \ldots, x_n] h [y_1, \ldots, y_m] \mid y_1, \ldots, y_m \rangle$$

(think "η-reduction" $\lambda x f(x) = f$, x not free in f)
(as long as there are no variable "clashes"—rename if necessary!)

Operations on circuit expressions

- (Legal) juxtaposition
- Abstraction

Circuit expression equivalences

- Renaming of bound variables (often needed for the following)
- Reassociation: $c_1; (c_2; c_3) = (c_1; c_2); c_3$
- Elimination of empty circuits: $c; \emptyset = c = \emptyset; c$
- Non-interacting subcircuits exchange: $c_1; c_2 = c_2; c_1$
- Abstraction coalescing and dissipating
- "Surgery rules": other equivalences a theory might impose.

(Symmetric) Monoidal Categories

Given a monoidal category $\langle \mathbf{C}, \otimes, \top \rangle$, we have natural isos:

$$
\begin{array}{rcl}
u_{\otimes}^{R} & : & A \otimes \top \longrightarrow A \\
u_{\otimes}^{L} & : & \top \otimes A \longrightarrow A \\
a_{\otimes} & : & (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \\
(\, c_{\otimes} & : & A \otimes B \longrightarrow B \otimes A \,)
\end{array}
$$

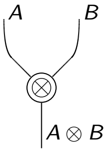
satisfying

$$
\begin{array}{rcl}
a_{\otimes}; 1 \otimes u_{\otimes}^{L} & = & u_{\otimes}^{R} \otimes 1 \\
a_{\otimes}; a_{\otimes} & = & a_{\otimes} \otimes 1; a_{\otimes}; 1 \otimes a_{\otimes} \\
(\, a_{\otimes}; c_{\otimes}; a_{\otimes} & = & c_{\otimes} \otimes 1; a_{\otimes}; 1 \otimes c_{\otimes} \,)
\end{array}
$$

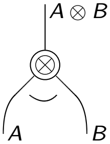
Circuits for Monoidal Cats

These correspond to the following circuits

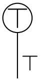
$(\otimes I)$



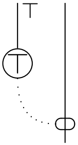
$(\otimes E)$



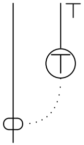
$(\top I)$



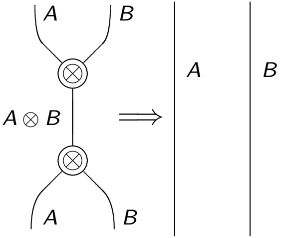
$(\top E)^L$

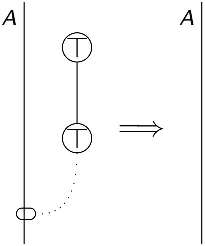


$(\top E)^R$



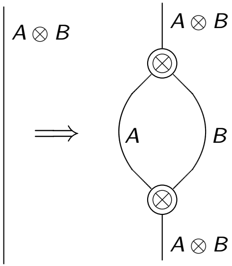
Circuit equivalences: Reductions

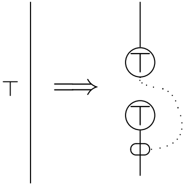




(There is a mirror image rewrite for the unit, with the unit edge and nodes on the other side of the A edge.)

Expansions

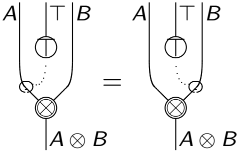


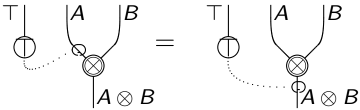


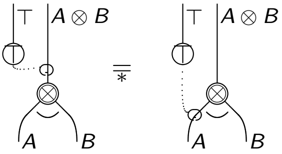
(Again, there is a mirror image rewrite for the unit, with the thinning edge on the other side of the unit edge and node.)

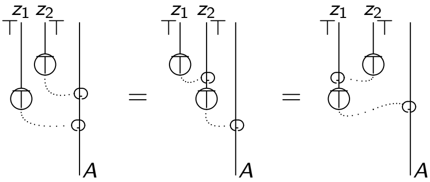
Unit rewirings

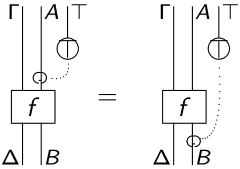
In addition to these rewrites, there are also a number of equivalences we must directly impose, to account for the unit isomorphisms.

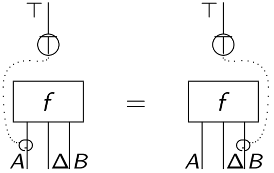




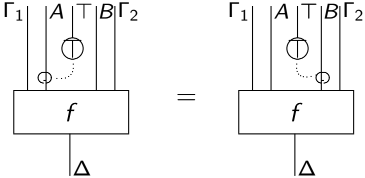


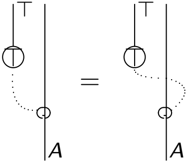


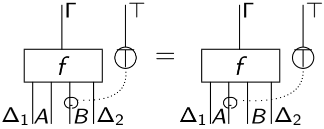




*(One must check that the "net condition" remains satisfied after such a move.)


Γ1 A ⊤ B Γ2
⊤
f
Δ
=
Γ1 A ⊤ B Γ2
⊤
f
Δ


⊤
⊤
A
=
⊤
⊤
A


Γ ⊤
f
⊤
Δ1 A B Δ2
=
Γ ⊤
f
⊤
Δ1 A B Δ2

As circuit expressions . . .

All of the above can be expressed in terms of circuit expressions, of course(!):

Basic components

$[A, B] \otimes I[A \otimes B]$	$\otimes$–introduction
$[A \otimes B] \otimes E[A, B]$	$\otimes$–elimination
$[] \top I[\top]$	unit introduction
$[A, \top] \top E^R[A]$	unit right elimination (thinning)
$[\top, A] \top E^L[A]$	unit left elimination (thinning)

Reductions

$$A, B\colon \langle x_1, x_2 \mid [x_1, x_2] \otimes I[z]; [z] \otimes E[y_1, y_2] \mid y_1, y_2 \rangle\colon A, B$$

$$\Rightarrow \quad A, B\colon \langle x_1, x_2 \mid\mid x_1, x_2 \rangle : A, B$$

$$A\colon \left\langle x \mid [] \top I[z]; [z, x] \top E^L[x] \mid x \right\rangle : A \quad \Rightarrow \quad A\colon \langle x \mid\mid x \rangle : A$$

$$A\colon \left\langle x \mid [] \top I[z]; [x, z] \top E^R[x] \mid x \right\rangle : A \quad \Rightarrow \quad A\colon \langle x \mid\mid x \rangle : A$$

Expansions

$$
\begin{array}{rcl}
A \otimes B \colon \langle z \mid\mid z \rangle : A \otimes B & \Rightarrow & A \otimes B \colon \langle z \mid [z] \otimes E[z_1, z_2]; [z_1, z_2] \otimes I[z] \mid z \rangle \\
\top \colon \langle x \mid\mid x \rangle : \top & \Rightarrow & \top \colon \left\langle x \mid [x] \top E^L[]; [] \top I[x] \mid x \right\rangle : \top \\
\top \colon \langle x \mid\mid x \rangle : \top & \Rightarrow & \top \colon \left\langle x \mid [x] \top E^R[]; [] \top I[x] \mid x \right\rangle : \top
\end{array}
$$

Unit rewirings

$$
\begin{aligned}
&A, \top, B\colon \big\langle x, z, y \mid [x, z]\top E^R[x]; [x, y] \otimes I[w] \mid w \big\rangle : A \otimes B \\
&\qquad = \quad A, \top, B\colon \big\langle x, z, y \mid [z, y]\top E^L[y]; [x, y] \otimes I[w] \mid w \big\rangle : A \otimes B \\
&\top, A \otimes B\colon \big\langle z, x \mid [z, x]\top E^L[x]; [x] \otimes E[x_1, x_2] \mid x_1, x_2 \big\rangle : A, B \\
&\qquad = \quad \top, A \otimes B\colon \big\langle z, x \mid [x] \otimes E[x_1, x_2]; [z, x_1]\top E^L[x_1] \mid x_1, x_2 \big\rangle : A, B \\
&A \otimes B, \top\colon \big\langle x, z \mid [x, z]\top E^R[x]; [x] \otimes E[x_1, x_2] \mid x_1, x_2 \big\rangle : A, B \\
&\qquad = \quad A \otimes B, \top\colon \big\langle x, z \mid [x] \otimes E[x_1, x_2]; [x_2, z]\top E^R[x_2] \mid x_1, x_2 \big\rangle : A, B \\
&A, B, \top\colon \big\langle x_1, x_2, z \mid [x_2, z]\top E^R[x_2]; [x_1, x_2] \otimes I[x] \mid x \big\rangle : A \otimes B \\
&\qquad = \quad A, B, \top\colon \big\langle x_1, x_2, z \mid [x_1, x_2] \otimes I[x]; [x, z]\top E^R[x] \mid x \big\rangle : A \otimes B
\end{aligned}
$$

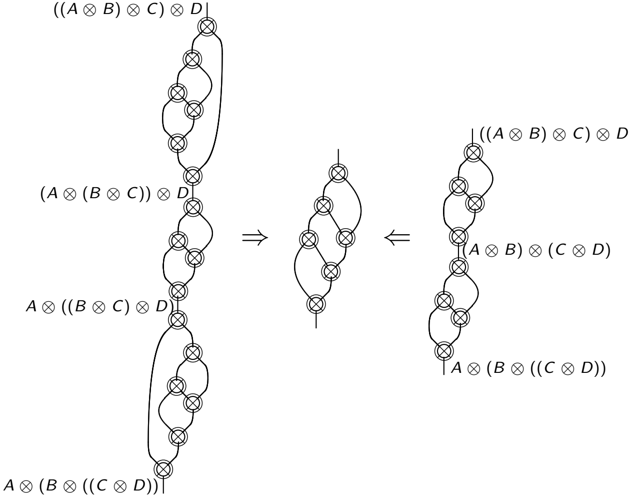
$$\top, A, B \colon \left\langle z, x_1, x_2 \mid [z, x_1] \top E^L[x_1]; [x_1, x_2] \otimes I[x] \mid x \right\rangle : A \otimes B$$
$$= \quad A, B, \top \colon \left\langle z, x_1, x_2 \mid [x_1, x_2] \otimes I[x]; [z, x] \top E^L[x] \mid x \right\rangle : A \otimes B$$
$$\top, \top, A \colon \langle z_1, z_2, x \mid\mid x \rangle : A$$
$$= \quad \top, \top, A \colon \left\langle z_1, z_2, x \mid [z_1, z_2] \top E^L[z_2]; [z_2, x] \top E^L[x] \mid x \right\rangle : A$$
$$= \quad \top, \top, A \colon \left\langle z_1, z_2, x \mid [z_1, z_2] \top E^R[z_1]; [z_1, x] \top E^L[x] \mid x \right\rangle : A$$
$$A, \top, \top \colon \left\langle x, z_1, z_2 \mid [x, z_1] \top E^R[x]; [x, z_2] \top E^R[x] \mid y \right\rangle : A$$
$$= \quad A, \top, \top \colon \left\langle x, z_1, z_2 \mid [z_1, z_2] \top E^R[z_1]; [x, z_1] \top E^R[x] \mid x \right\rangle : A$$
$$= \quad A, \top, \top \colon \left\langle x, z_1, z_2 \mid x, z_1, z_2 \mid [z_1, z_2] \top E^L[z_2]; [x, z_2] \top E^R[x] \mid x \right\rangle : A$$
$$\top, A, \top \colon \left\langle z_1, x, z_2 \mid [z_1, x] \top E^L[x]; [x, z_2] \top E^R[x] \mid y \right\rangle : A$$
$$= \quad \top, A, \top \colon \left\langle z_1, x, z_2 \mid [x, z_2] \top E^R[x]; [z_1, x] \top E^L[x] \mid x \right\rangle : A$$

$$\Gamma_1, A, \top, B, \Gamma_2 \colon \left\langle .., x_1, z, x_2, .. \mid [x_1, z] \top E^R[x_1]; [.., x_1, x_2, ..] f[..] \mid .. \right\rangle : \Delta$$
$$= \Gamma_1, A, \top, B, \Gamma_2 \colon \left\langle .., x_1, z, x_2, .. \mid [z, x_2] \top E^L[x_2]; [.., x_1, x_2, ..] f[..] \mid .. \right\rangle : \Delta$$
$$\top, A, \Gamma \colon \left\langle z, x_1, .. \mid [z, x_1] \top E^L[x_1]; [x_1, ..] f[x_2, ..] \mid x_2, .. \right\rangle : B, \Delta$$
$$= \top, A, \Gamma \colon \left\langle z, x_1, .. \mid [x_1, ..] f[x_2, ..]; [z, x_2] \top E^L[x_2] \mid x_2, .. \right\rangle : B, \Delta$$
$$\Gamma, A, \top \colon \left\langle .., x_1, z \mid [x_1, z] \top E^R[x_1]; [.., x_1] f[.., x_2] \mid .., x_2 \right\rangle : \Delta, B$$
$$= \Gamma, A, \top \colon \left\langle .., x_1, z \mid [.., x_1] f[.., x_2]; [x_2, z] \top E^R[x_2] \mid .., x_2 \right\rangle : \Delta, B$$
$$\top \colon \left\langle z \mid [] f[x_1, .., x_2]; [z, x_1] \top E^L[x_1] \mid x_1, .., x_2 \right\rangle : A, \Delta, B$$
$$= \top \colon \left\langle z \mid [] f[x_1, .., x_2]; [x_2, z] \top E^R[x_2] \mid x_1, .., x_2 \right\rangle : A, \Delta, B$$

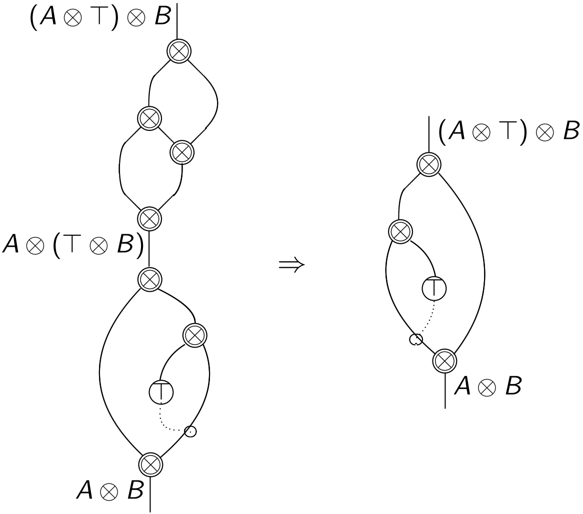
(And two more unit rewirings for symmetry . . .)

Examples: The Pentagon

To prove the standard coherences are a consequence of the equivalences is a simple matter of using the circuit rewrites above.



Unit Coherence



Using the circuit expressions

We shall now express these (simple!) graph rewrites using the circuit term notation—but with some simplifying shortcuts which should make them less (?intimidating?) cumbersome. We'll use numerals as variable names, numbering the wires as we come upon them (reading top down, left to right), with the understanding that variable renaming is an equality.

So for example, an (abstracted) expression such as

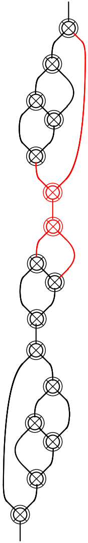
$$((A \otimes B) \otimes C) \otimes D \colon \langle x_1 \mid [x_1] \otimes E[x_2, x_3] \mid x_2, x_3 \rangle \colon A, (B \otimes C) \otimes D$$

would simply become $1 \otimes E\, {}^{2}_{3}$.

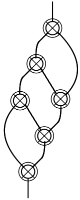
(Reading this on its side, and doing a mirror-image, one can almost "see" the circuit this represents.)

With this simplified (though underspecified!) notation, we can look at the individual steps in showing the pentagon commutes. We'll leave the end target in the picture, so you can see where we're going at each stage. I'll highlight the step where an equivalence is used, in the circuit expression (and initially in the circuit itself).

The Pentagon, step by step



$\Rightarrow$



$1 \otimes E \genfrac{}{}{0pt}{}{2}{3}; 2 \otimes E \genfrac{}{}{0pt}{}{4}{5}; 4 \otimes E \genfrac{}{}{0pt}{}{6}{7}; \genfrac{}{}{0pt}{}{7}{5} \otimes I\, 8;$

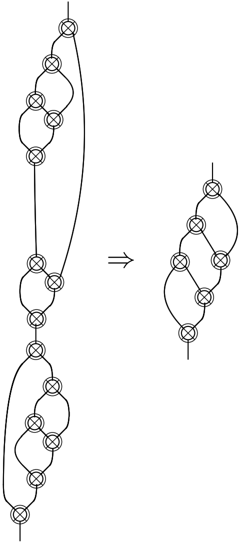
$\genfrac{}{}{0pt}{}{6}{8} \otimes I\, 9; \genfrac{}{}{0pt}{}{9}{3} \otimes I\, 10; 10 \otimes E \genfrac{}{}{0pt}{}{11}{12}; 11 \otimes E \genfrac{}{}{0pt}{}{13}{14}$

$\genfrac{}{}{0pt}{}{14}{12} \otimes I\, 15; \genfrac{}{}{0pt}{}{13}{15} \otimes I\, 16; 16 \otimes E \genfrac{}{}{0pt}{}{17}{18};$

$18 \otimes E \genfrac{}{}{0pt}{}{19}{20}; 19 \otimes E \genfrac{}{}{0pt}{}{21}{22}; \genfrac{}{}{0pt}{}{22}{20} \otimes I\, 23;$

$\genfrac{}{}{0pt}{}{21}{23} \otimes I\, 24; \genfrac{}{}{0pt}{}{17}{24} \otimes I\, 25 \qquad \Rightarrow \ldots$

The Pentagon, step by step 2



$${}^{}1 \otimes E\,{}^{2}_{3};\ 2 \otimes E\,{}^{4}_{5};\ 4 \otimes E\,{}^{6}_{7};\ {}^{7}_{5} \otimes I\,8;$$

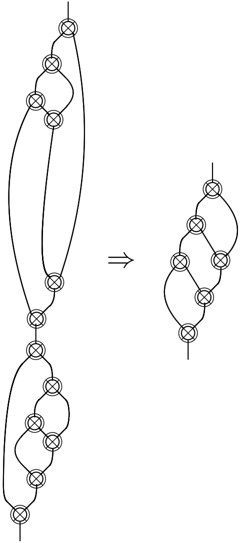
$${}^{6}_{8} \otimes I\,9;\ \emptyset^{9=11}_{3=12};\ 9 \otimes E\,{}^{13}_{14};$$

$${}^{14}_{3} \otimes I\,15;\ {}^{13}_{15} \otimes I\,16;\ 16 \otimes E\,{}^{17}_{18};$$

$$18 \otimes E\,{}^{19}_{20};\ 19 \otimes E\,{}^{21}_{22};\ {}^{22}_{20} \otimes I\,23;$$

$${}^{21}_{23} \otimes I\,24;\ {}^{17}_{24} \otimes I\,25 \qquad \Rightarrow \ldots$$

The Pentagon, step by step 2



$1 \otimes E\, {}^{2}_{3};\ 2 \otimes E\, {}^{4}_{5};\ 4 \otimes E\, {}^{6}_{7};\ {}^{7}_{5} \otimes I\ 8;$

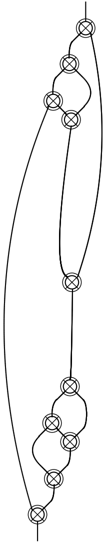
$\emptyset^{9=11,6=13}_{3=12,8=14}\ ;$

${}^{8}_{3} \otimes I\ 15;\ {}^{6}_{15} \otimes I\ 16;\ 16 \otimes E\, {}^{17}_{18};$

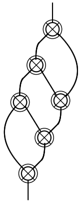
$18 \otimes E\, {}^{19}_{20};\ 19 \otimes E\, {}^{21}_{22};\ {}^{22}_{20} \otimes I\ 23;$

${}^{21}_{23} \otimes I\ 24;\ {}^{17}_{24} \otimes I\ 25 \qquad \Rightarrow \ldots$

The Pentagon, step by step 3



$\Rightarrow$



$1 \otimes E\,{}^{2}_{3};\ 2 \otimes E\,{}^{4}_{5};\ 4 \otimes E\,{}^{6}_{7};\ {}^{7}_{5} \otimes I\ 8;$

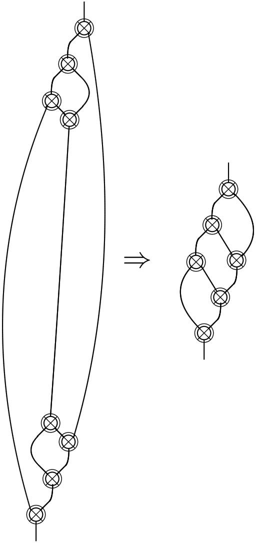
$\emptyset^{9=11,6=13}_{3=12,8=14}\ ;$

${}^{8}_{3} \otimes I\ 15;\ \emptyset^{6=13=17}_{15=18};$

$15 \otimes E\,{}^{19}_{20};\ 19 \otimes E\,{}^{21}_{22};\ {}^{22}_{20} \otimes I\ 23;$

${}^{21}_{23} \otimes I\ 24;\ {}^{6}_{24} \otimes I\ 25 \qquad \Rightarrow \ldots$

The Pentagon, step by step 4



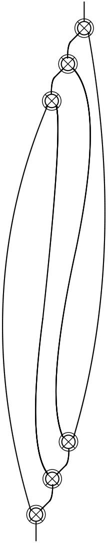
$1 \otimes E\,{}^{2}_{3};\ 2 \otimes E\,{}^{4}_{5};\ 4 \otimes E\,{}^{6}_{7};\ {}^{7}_{5} \otimes I\ 8;$

$\emptyset^{9=11,6=13}_{3=12,8=14};\ \emptyset^{6=13=17,8=19}_{15=18,3=12=20};$

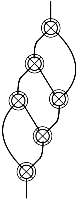
$8 \otimes E\,{}^{21}_{22};\ {}^{22}_{3} \otimes I\ 23;$

${}^{21}_{23} \otimes I\ 24;\ {}^{6}_{24} \otimes I\ 25 \qquad \Rightarrow \dots$

The Pentagon, step by step 5



$=$



$1 \otimes E^{2}_{3}; 2 \otimes E^{4}_{5}; 4 \otimes E^{6}_{7};$

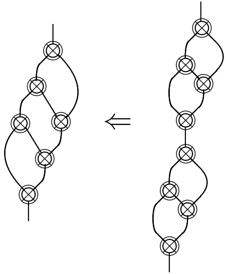
$\emptyset^{6=13=17,8=19,9=11,7=21}_{3=12=20,5=22,8=14,15=18,};$

${}^{5}_{3} \otimes I\,23; {}^{7}_{23} \otimes I\,24; {}^{6}_{24} \otimes I\,25$

$=$ (renaming bound variables)

$1 \otimes E^{2}_{3}; 2 \otimes E^{4}_{5}; 4 \otimes E^{6}_{7}; {}^{5}_{3} \otimes I\,8; {}^{7}_{8} \otimes I\,9; {}^{6}_{9} \otimes I\,10$

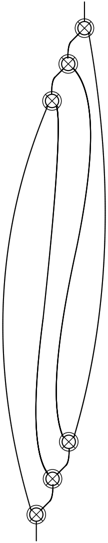
The Pentagon, step by step 6



$$1 \otimes E\,{}^{2}_{3};\, 2 \otimes E\,{}^{4}_{5};\, {}^{5}_{3} \otimes I\,6;\, {}^{4}_{6} \otimes I\,7;\, 7 \otimes E\,{}^{8}_{9};$$

$$8 \otimes E\,{}^{10}_{11};\, {}^{11}_{9} \otimes I\,12;\, {}^{10}_{12} \otimes I\,13 \qquad \Rightarrow \dots$$

The Pentagon, step by step 7



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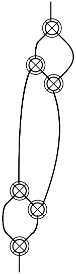


$1 \otimes E\,^{2}_{3}; 2 \otimes E\,^{4}_{5}; {}^{5}_{3} \otimes I\;6; \emptyset^{4=8}_{6=9}; 4 \otimes E\,^{10}_{11};$
${}^{11}_{6} \otimes I\;12; {}^{10}_{12} \otimes I\;13$

$=$ (exchanging non-interacting subcircuits)

$1 \otimes E\,^{2}_{3}; 2 \otimes E\,^{4}_{5}; 4 \otimes E\,^{10}_{11}; {}^{5}_{3} \otimes I\;6;$
${}^{11}_{6} \otimes I\;12; {}^{10}_{12} \otimes I\;13$

$=$ (renaming bound variables)

$1 \otimes E\,^{2}_{3}; 2 \otimes E\,^{4}_{5}; 4 \otimes E\,^{6}_{7}; {}^{5}_{3} \otimes I\;8; {}^{7}_{8} \otimes I\;9; {}^{6}_{9} \otimes I\;10$

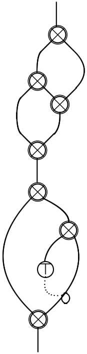
The Unit Coherence, step by step 1

We extend our simplified notation to include thinning links:

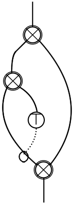
$[A, \top]\top E^R[A]$ becomes ${}^{1}_{2}\top I\,{}^{1}_{\circ}$ and note that $2 = \top$

$[\top, B]\top E^L[B]$ becomes ${}^{1}_{2}\top E\,{}^{\circ}_{2}$ and note that $1 = \top$

$[]\top I[\top]$ becomes $\top I\, 1$ and note that $1 = \top$

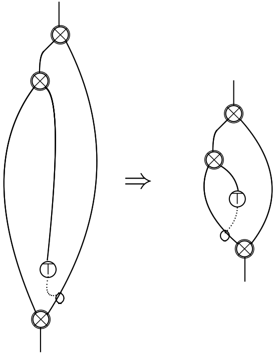


$\Rightarrow$



$$ {}^{1}_{2} \otimes I\, 3; 2 \otimes E\, {}^{4}_{5}; {}^{5}_{3} \otimes I\, 6; {}^{4}_{6} \otimes I\, 7; 7 \otimes E\, {}^{8}_{9}; 9 \otimes E\, {}^{10}_{11}; {}^{10}_{11}\top E\, {}^{\circ}_{11}; {}^{8}_{11} \otimes I\, 12 \quad \Rightarrow $$

The Unit Coherence, step by step 2



$$\tfrac{1}{2} \otimes I\ 3; 2 \otimes E\ {}^{4}_{5}; {}^{5}_{3}\top E\ {}^{\circ}_{3}; {}^{4}_{3} \otimes I\ 12$$
$$= \tfrac{1}{2} \otimes I\ 3; 2 \otimes E\ {}^{4}_{5}; {}^{4}_{5}\top I\ {}^{4}_{\circ}; {}^{4}_{3} \otimes I\ 12$$

Using

$$A, \top, B \colon \left\langle x, z, y \mid [z, y] \top E^{L}[y]; [x, y] \otimes I[w] \mid w \right\rangle : A \otimes B$$
$$= A, \top, B \colon \left\langle x, z, y \mid [x, z] \top E^{R}[x]; [x, y] \otimes I[w] \mid w \right\rangle : A \otimes B$$

i.e. ${}^{5}_{3}\top E\ {}^{\circ}_{3}; {}^{4}_{3} \otimes I\ 12 = {}^{4}_{5}\top I\ {}^{4}_{\circ}; {}^{4}_{3} \otimes I\ 12$

Extensions

This notation was originally developed for linearly distributive categories, and so handles par as well as tensor.[1] We also extended it to "Full intuitionist linear logic"[2], which showed how to include scope or functor boxes. The idea is simple enough (though the full notation does get to be a handful!), and I'll leave that to your own bedtime reading . . .

[1] See BCST [JPAA 1996]
[2] See CS [TAC 1997]