Digital Signal Processing: Convolution

CPSC 501: Advanced Programming Techniques
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Convolution
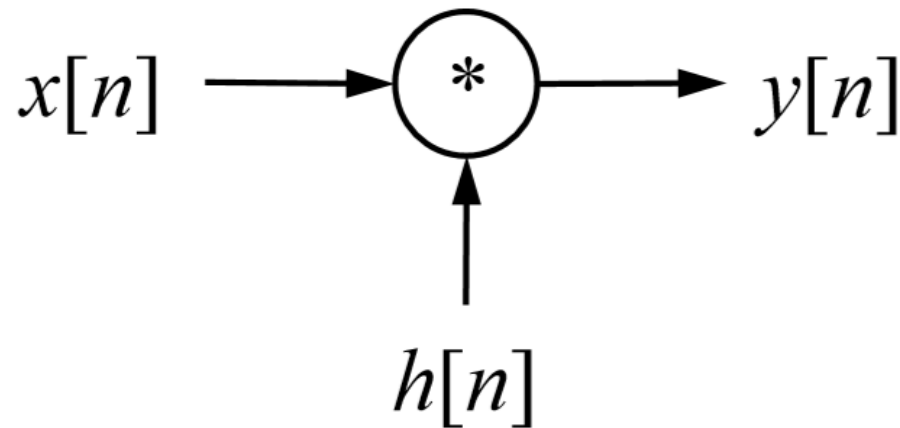
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Time-Domain Convolution

- Convolution is mathematical operation that takes 2 input signals, x[n] and h[n], to produce a third signal, y[n]
  - Represented mathematically as:

  x[n] * h[n] = y[n]
  - Note: * means convolution, not multiplication
  - Represented graphically:

$$x[n] \longrightarrow \circledast \longrightarrow y[n]$$

$$h[n]$$

Time-Domain Convolution

- Input Side Algorithm (array implementation):

```
void convolve(float x[], int N, float h[], int M, float y[], int P) {
    int n, m;
    /*  Outer loop:  process each input value x[n] in turn  */
    for (n = 0; n < N; n++) {
            /*  Inner loop:  process x[n] with each sample of h[n]  */
            for (m = 0; m < M; m++)
                    y[n+m] += x[n] * h[m];
    }
}
```

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Time-Domain Convolution

- Note:
  - N, M, and P are the number of elements in each array
  - P must equal N + M – 1
  - The signals x[n], h[n], and y[n] are f.p. numbers scaled to the range:
    - -1.0 to +1.0
  - You may need to convert to/from signed integers
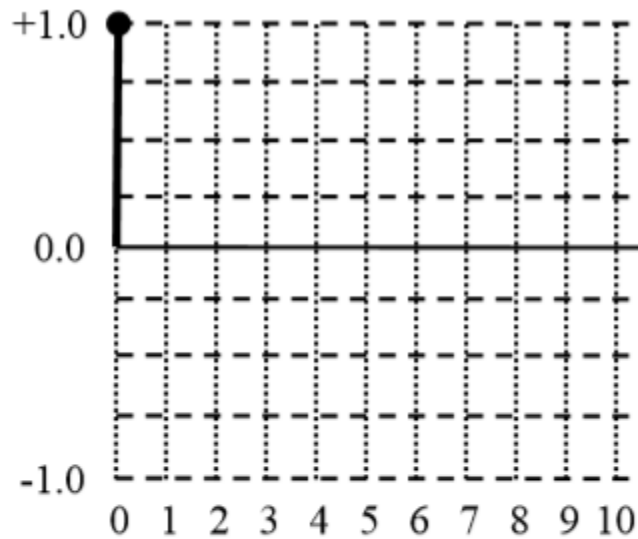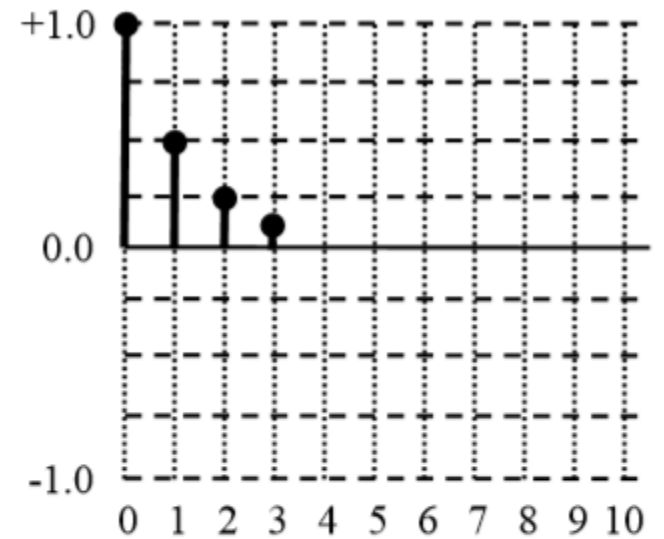 - This algorithm can be adapted to deal with samples stored in audio files

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Examples

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Example convolutions:
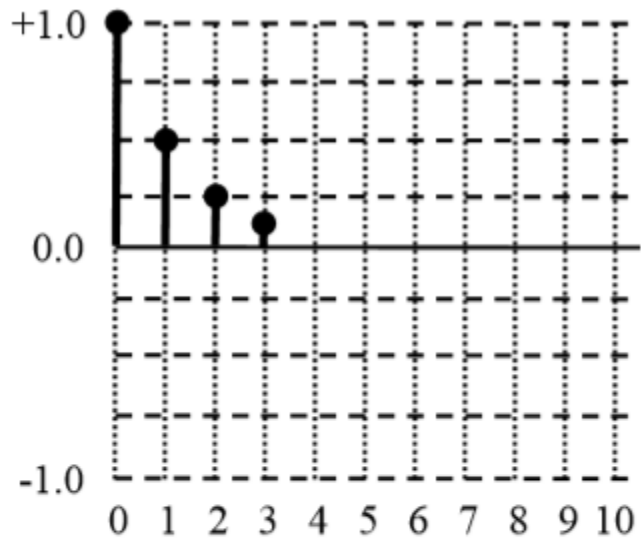
$$x[n] \qquad * \qquad h[n] \qquad = \qquad y[n]$$
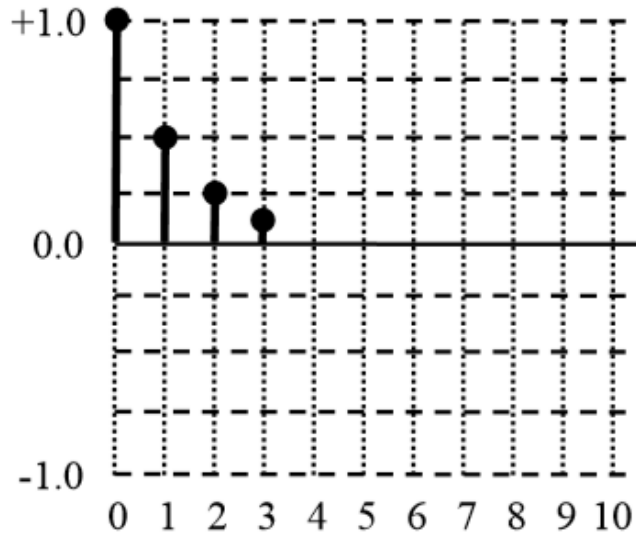
"identity"
impulse response

Example convolutions:
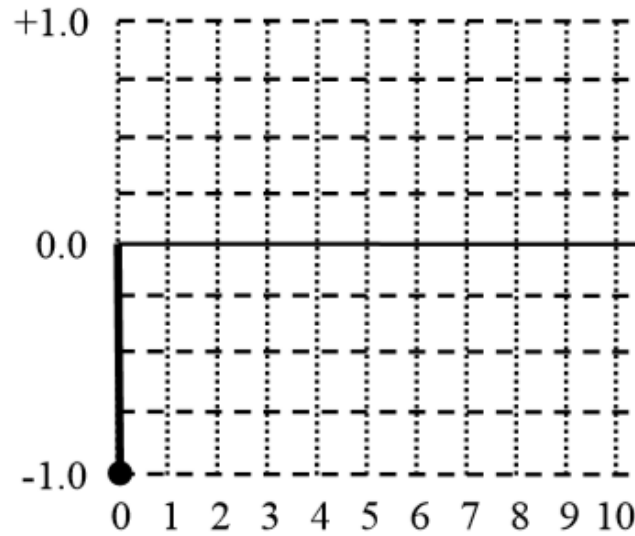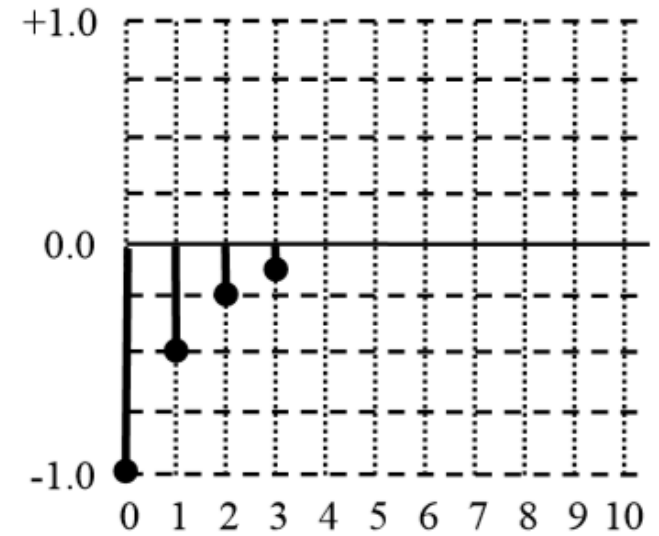
$$x[n] \qquad * \qquad h[n] \qquad = \qquad y[n]$$

"inverse" IR

Example convolutions:
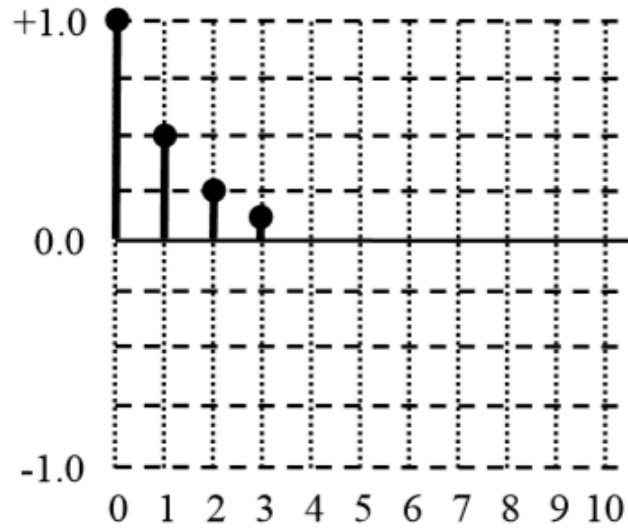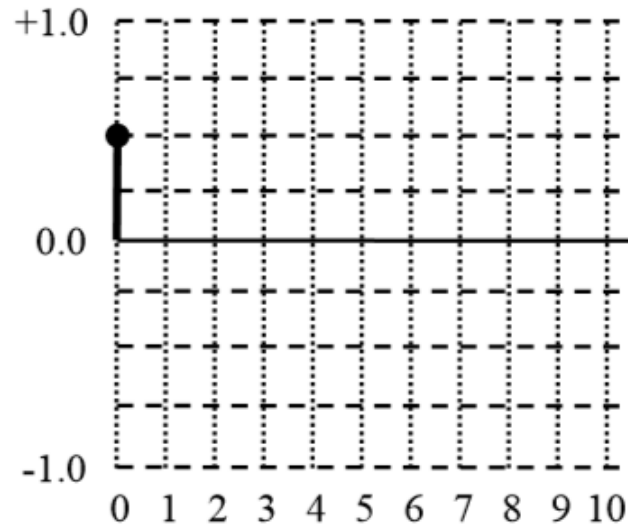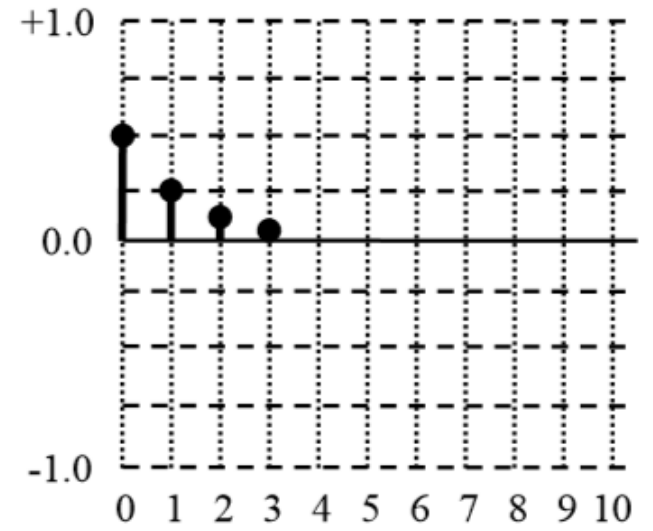
$$x[n] \quad * \quad h[n] \quad = \quad y[n]$$

"scaling" IR

Example convolutions:
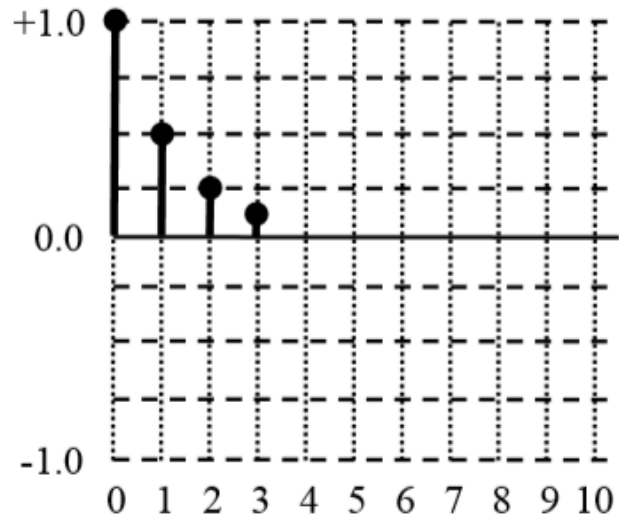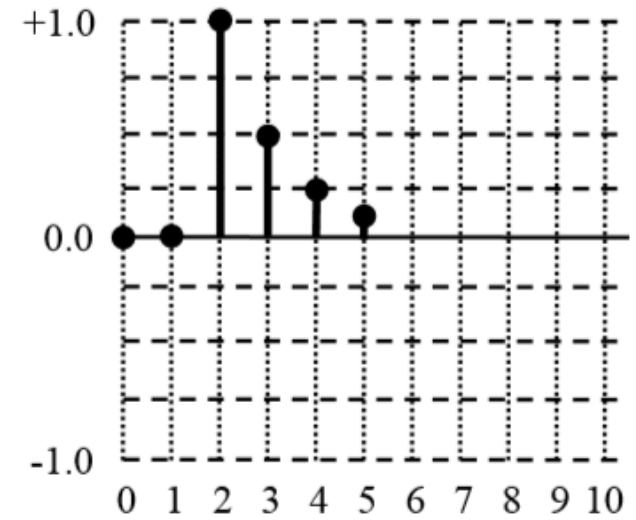
Example convolutions:

$$x[n] \quad * \quad h[n] \quad = \quad y[n]$$

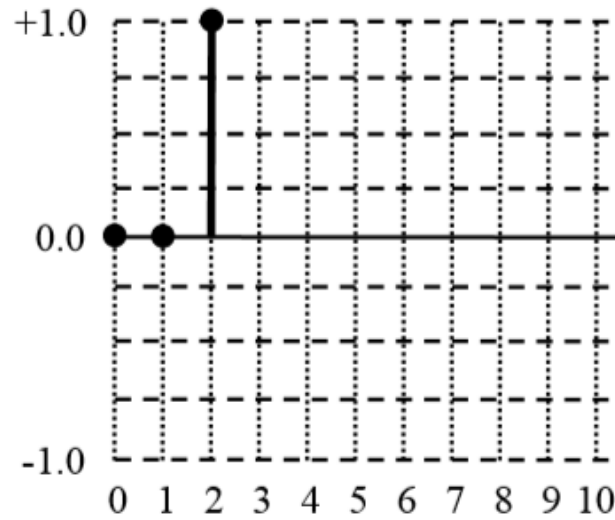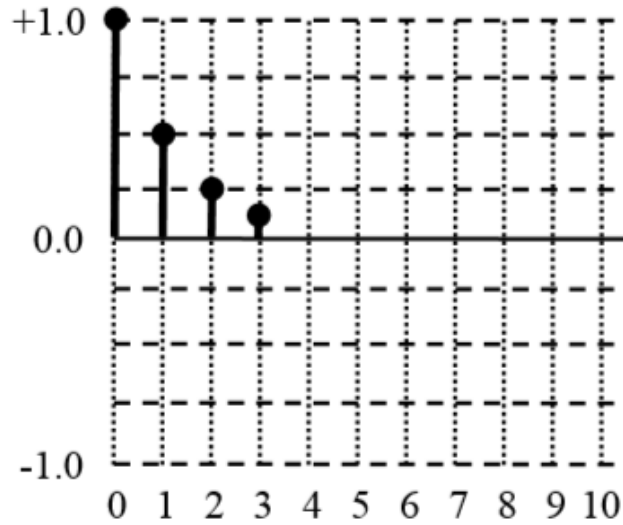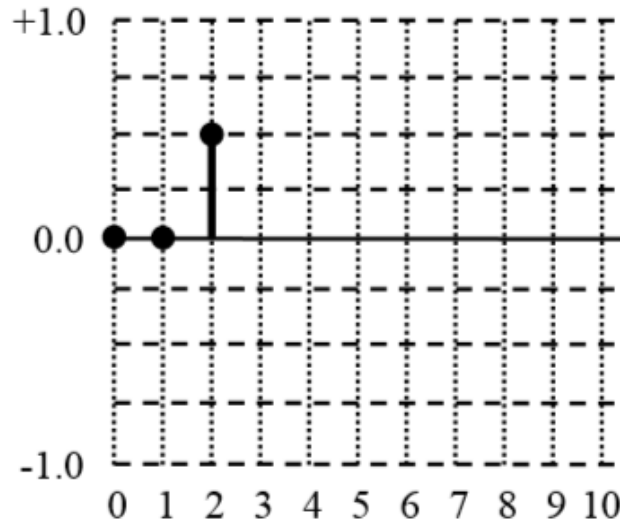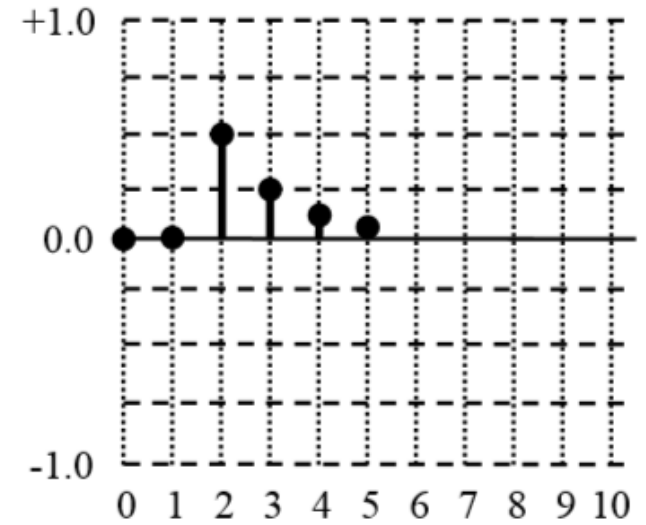"delay & scaling" IR

Example convolutions:
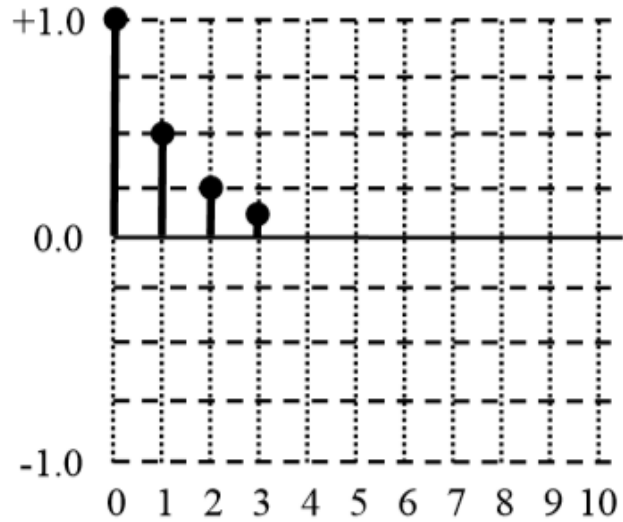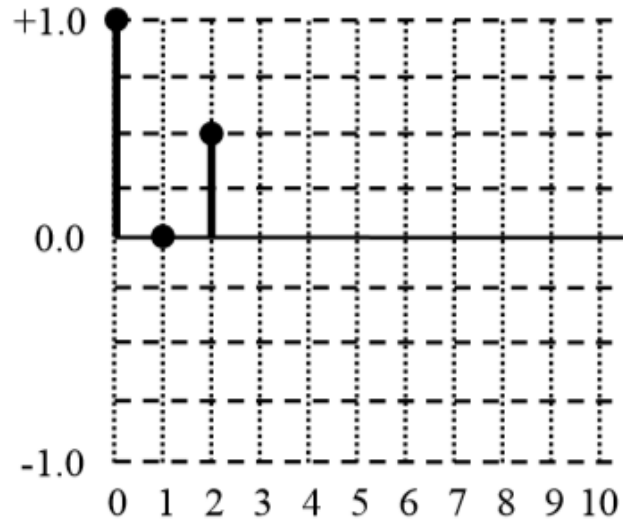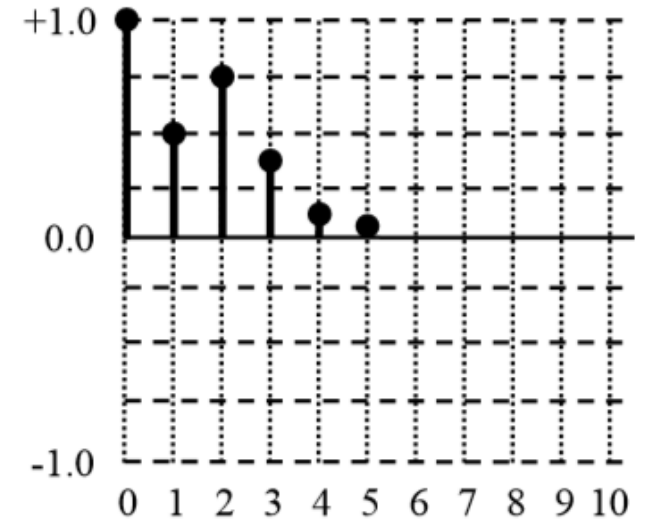
$x[n]$ * $h[n]$ = $y[n]$

"overlapping echo" IR

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Example convolutions:

$$x[n] \qquad * \qquad h[n] \qquad = \qquad y[n]$$

"non-overlapping echo" IR

Example convolutions:

- Convolution is a commutative operation
  - That is: x[n] * h[n] = h[n] * x[n] = y[n]
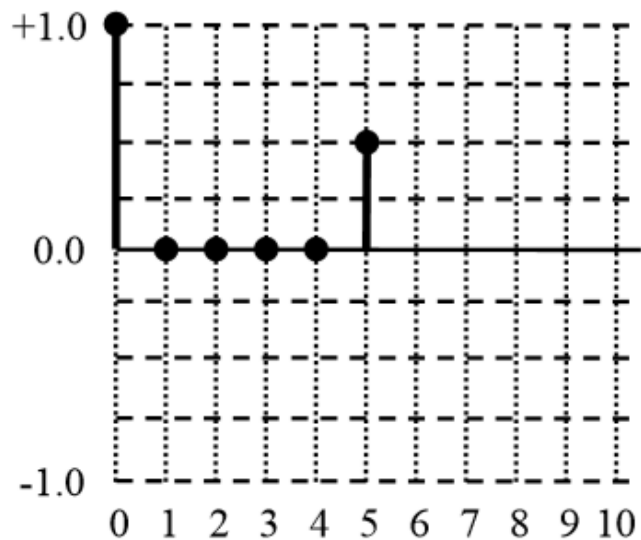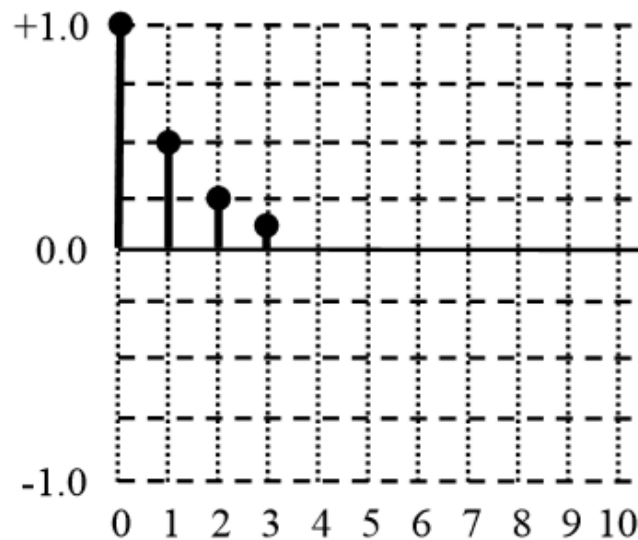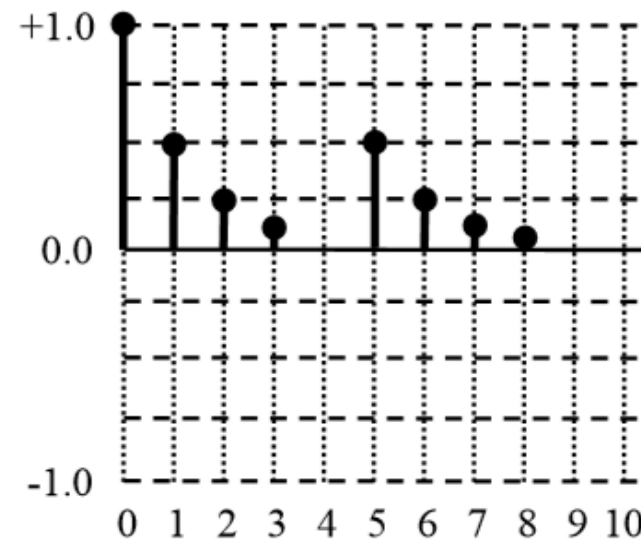
$$h[n] \qquad * \qquad x[n] \qquad = \qquad y[n]$$

Time Domain Convolution

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Time-Domain Convolution

- The signals x[n] and h[n] can be any arbitrary kind of signal
  - However, x[n] is usually an input signal that you want to process
  - And, h[n] is the impulse response (IR) of a system you want mode

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Impulse Response

- The IR of a system is found by inputting a normalized impulse (delta function) into the system, and measuring or calculating the output

$\delta[n]$ → System → $h[n]$

delta function   impulse response

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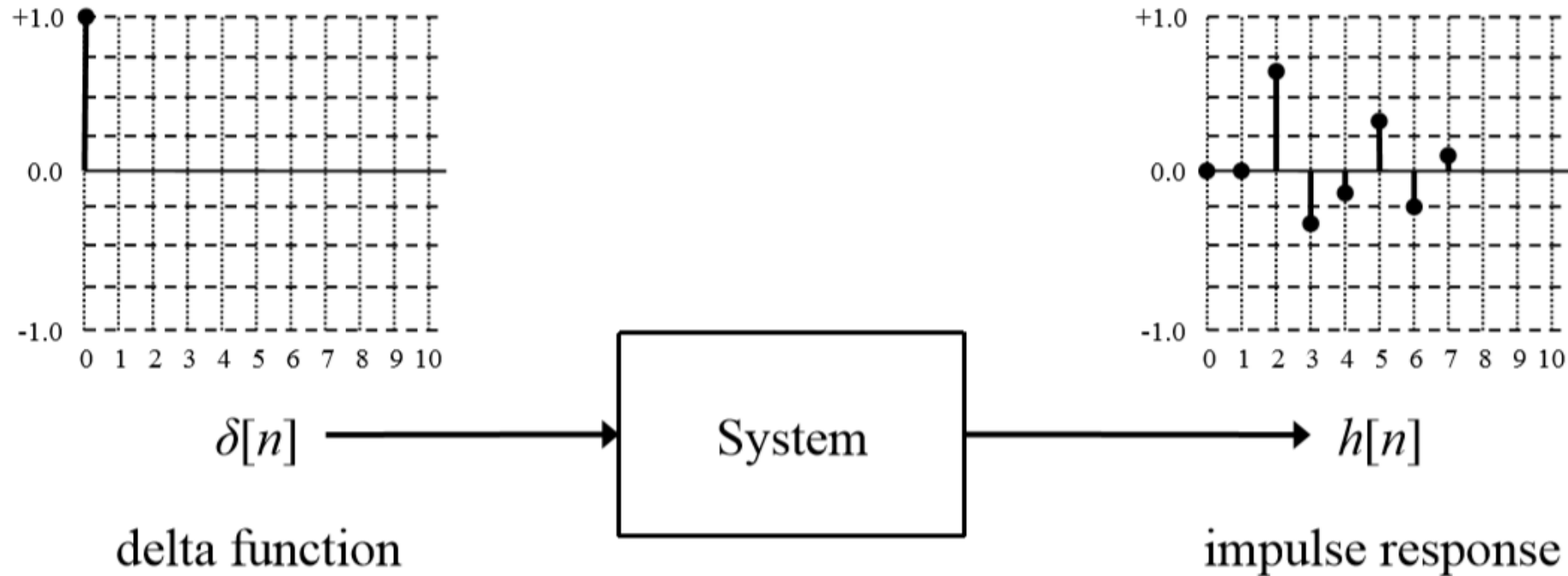
Model the System as it applied to signal change

- If the system is linear and time-invariant (LTI), the impulse response perfectly characterizes the system
  - That is, we know how the system will respond to any input it receives

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Simulate the system as convolution

- We can thus simulate how a system would respond to an input signal by convolving the input signal with the system's IR

- That is, the following are equivalent:

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$x[n] \longrightarrow \circledast \longrightarrow y[n]$$
$$h[n]$$
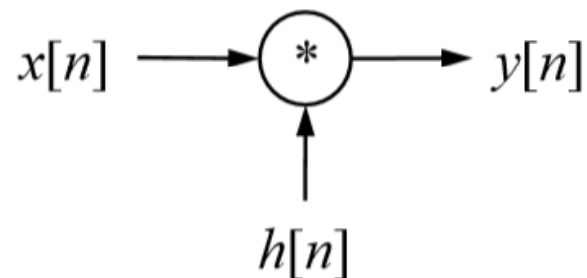
Application: Concert Hall

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Time-Domain Convolution

- The IR of a concert hall can be found by firing a starting pistol (an approximate delta function), and using a microphone to record to the system's response
  - Convolving a "dry" recording of an instrument with the IR results in an output signal where it sounds like the instrument is playing in the hall
    - This is the basis for a convolution reverb digital effect

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Convolution using the FFT

- Time-domain convolution can be very slow, especially if h[n] is large
  - Is O(N × M), where N and M are the sizes of x[n] and h[n]
- High-speed or FFT convolution uses the principle that convolution in the time domain corresponds to multiplication of spectra in the frequency domain

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Convolution using the FFT

- This is the convolution theorem, and is expressed mathematically as:

$$x[n] * h[n] \quad \underset{IDFT}{\overset{DFT}{\rightleftarrows}} \quad X[k] \cdot H[k]$$

Time domain   Frequency domain

- Where X[k] and H[k] are the spectra of x[n] and h[n]:
  - X[k] = DFT[x[n]]
  - H[k] = DFT[h[n]

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Convolution using the FFT

- We can calculate the convolution by:
  - Transforming both x[n] and h[n] to the frequency domain using the FFT
    - Note: zero-padding will have to be used so that x[n] and h[n] have the same length, the length is a power of 2 (needed for the FFT), and is long enough to avoid circular convolution (wrap around)
  - Multiplying X[k] by H[k] point by point
    - Note: this will be complex multiplication (imaginary/real)
  - Converting the result back to the time domain using the inverse FFT (IFFT)

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Convolution using the FFT
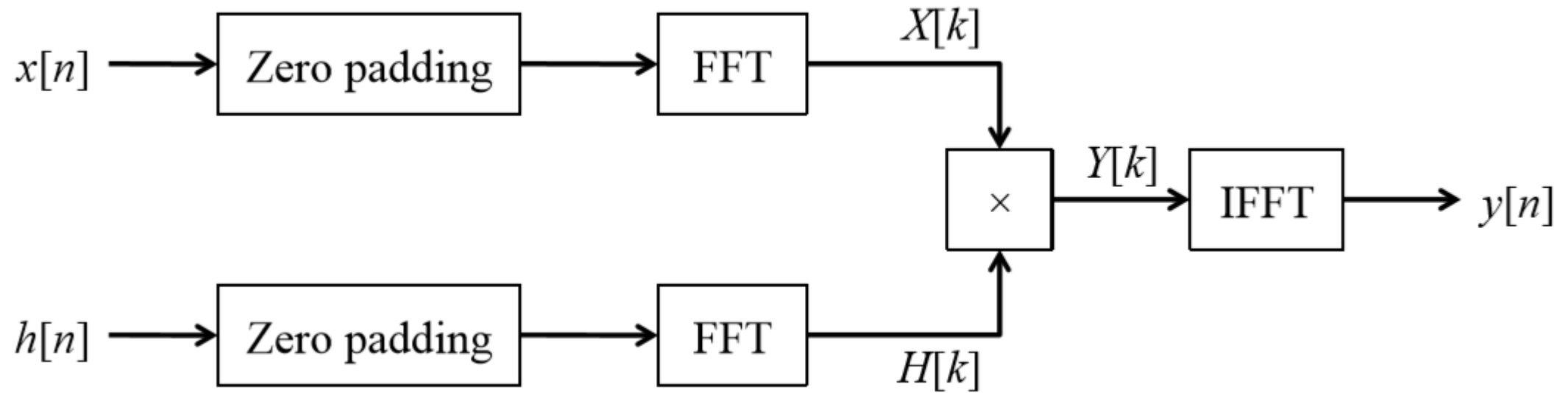
Convolution using the FFT

- Note: the output from either the FFTs or the IFFT (but NOT both) will have to be scaled by dividing by each data point by N, if the FFT/IFFT algorithm you are using doesn't do scaling

- E.g.

```
for (k = 0, i = 0; k < N; k++, i += 2) {
    /*  Scale the real and imaginary parts of a data point */
    x[i] /= (double)N;
    x[i+1] /= (double)N;
}
```

Onward to ...
next topic.

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