Digital Signal Processing Introduction

CPSC 501: Advanced Programming Techniques
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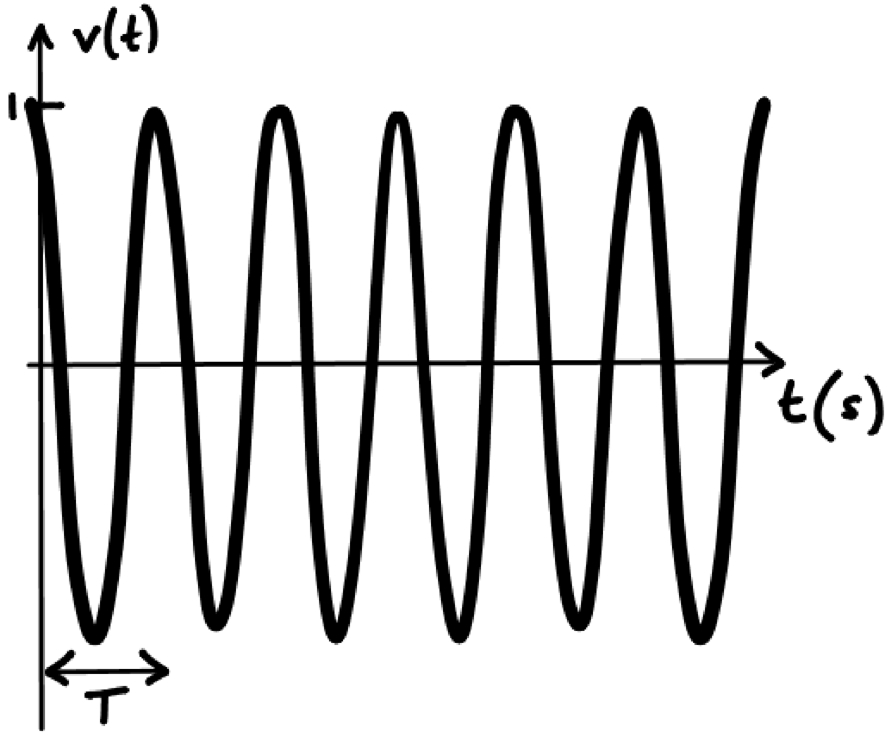
Signals

How to we get a signal

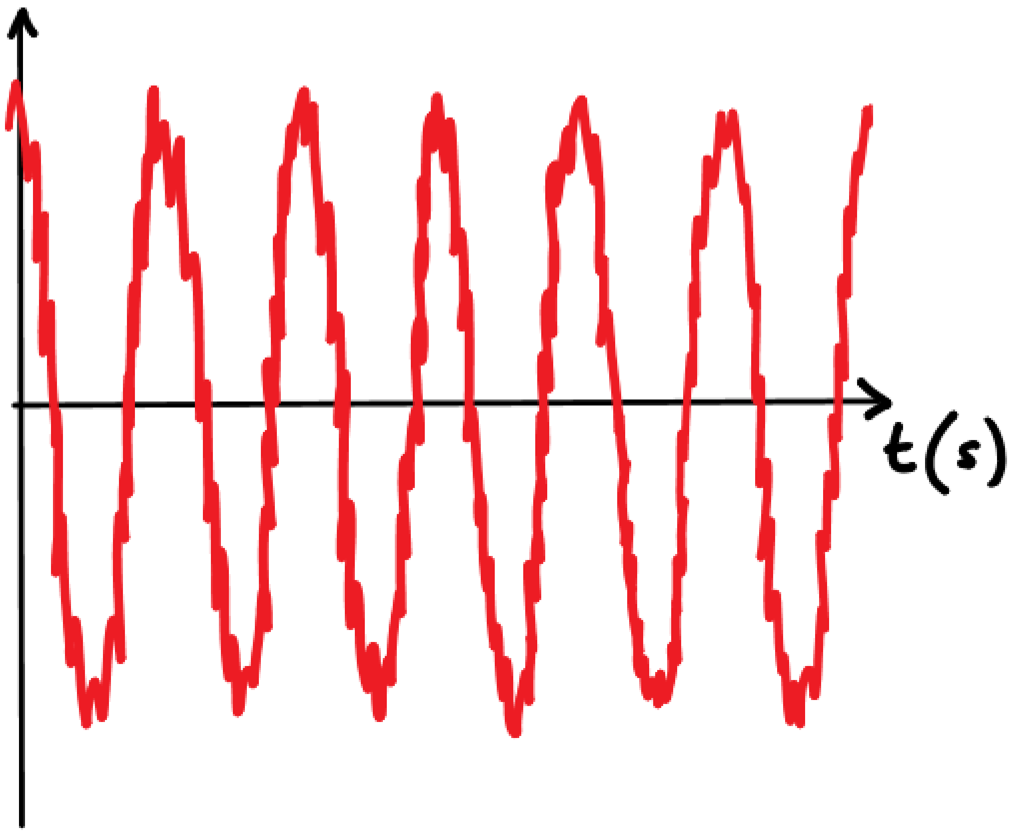
Analog Signal

- Analog signal. This signal v(t)=cos(2πft) could be a perfect analog recording of a pure tone of frequency f=1 Hz.
- The period T=1/f is the duration of one full oscillation.



Noisy Signal

- Noisy analog signal. Noise degrades the sinusoidal signal.
- It is often impossible to recover the original signal exactly from the noisy version



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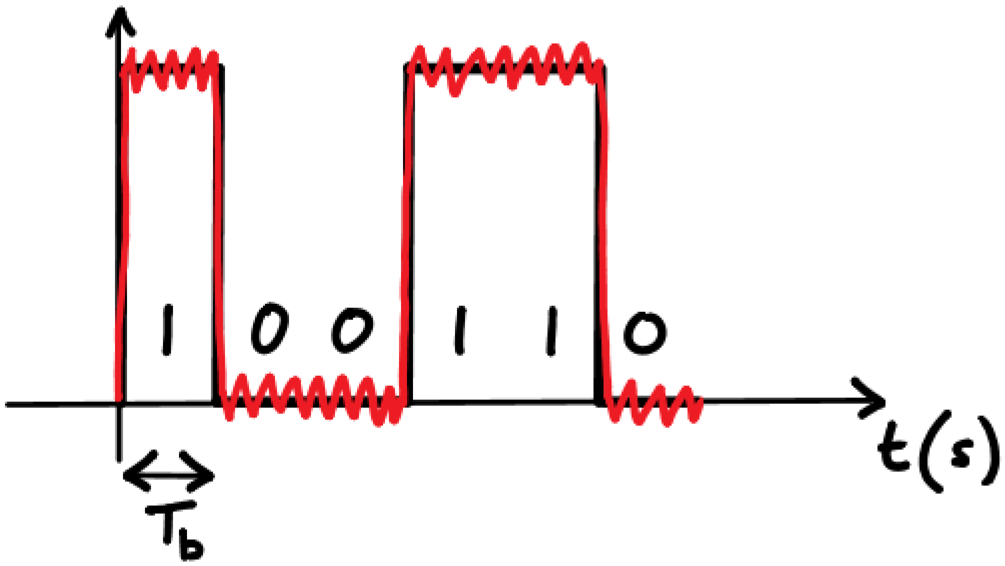
Digital Signal

- Analog transmission of a digital signal.
- Consider a digital signal 100110 converted to an analog signal for radio transmission.
- The received signal suffers from noise, but given sufficient bit duration T_b, it is still easy to read off the original sequence 100110 perfectly.



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Sampling

- A continuous signal may be sampled

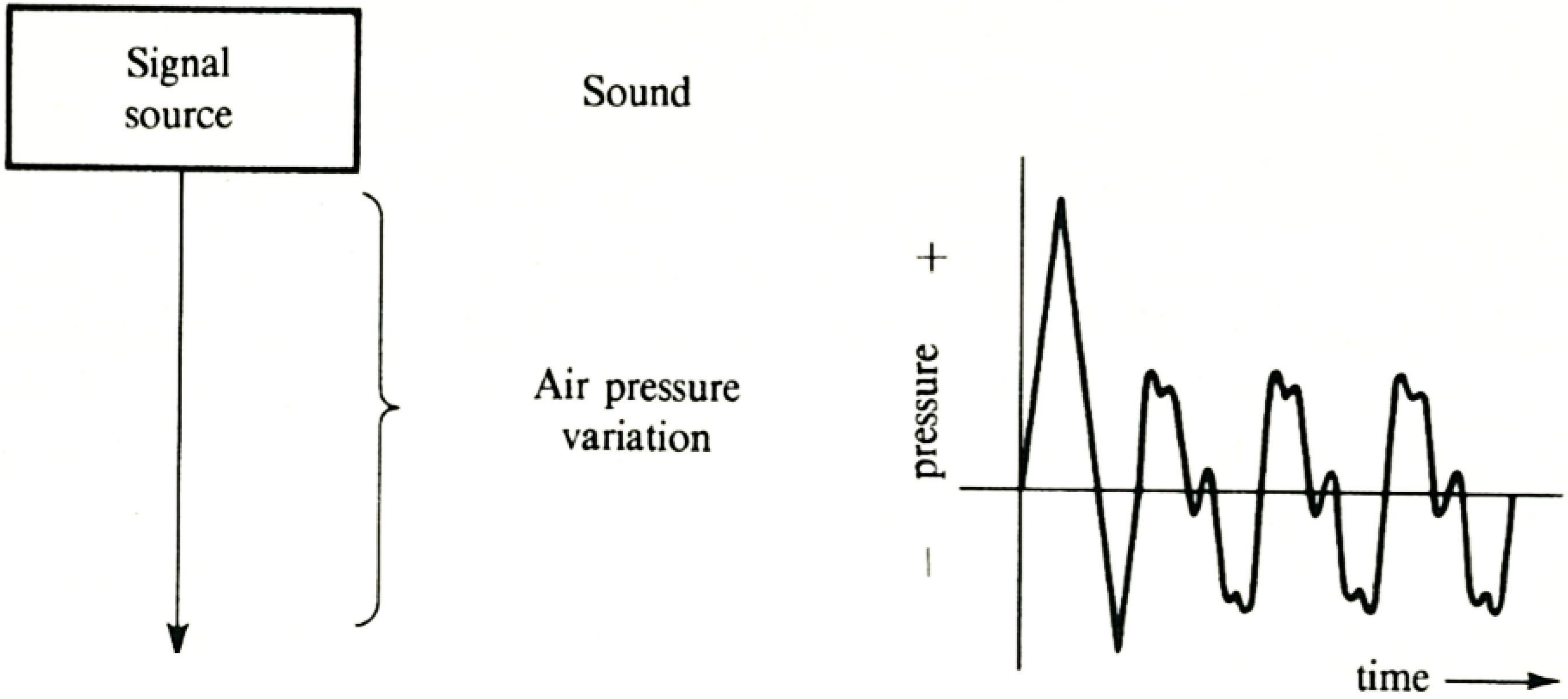
  - i.e. measured periodically at small intervals of time, and converted into a series of numbers (samples)
    - Such a series is a digital signal

  - An **analog-to-digital converter** does the sampling
    - E.g. Sampling an audio signal

From Voice to Bits

How to we get a signal

Original Signal



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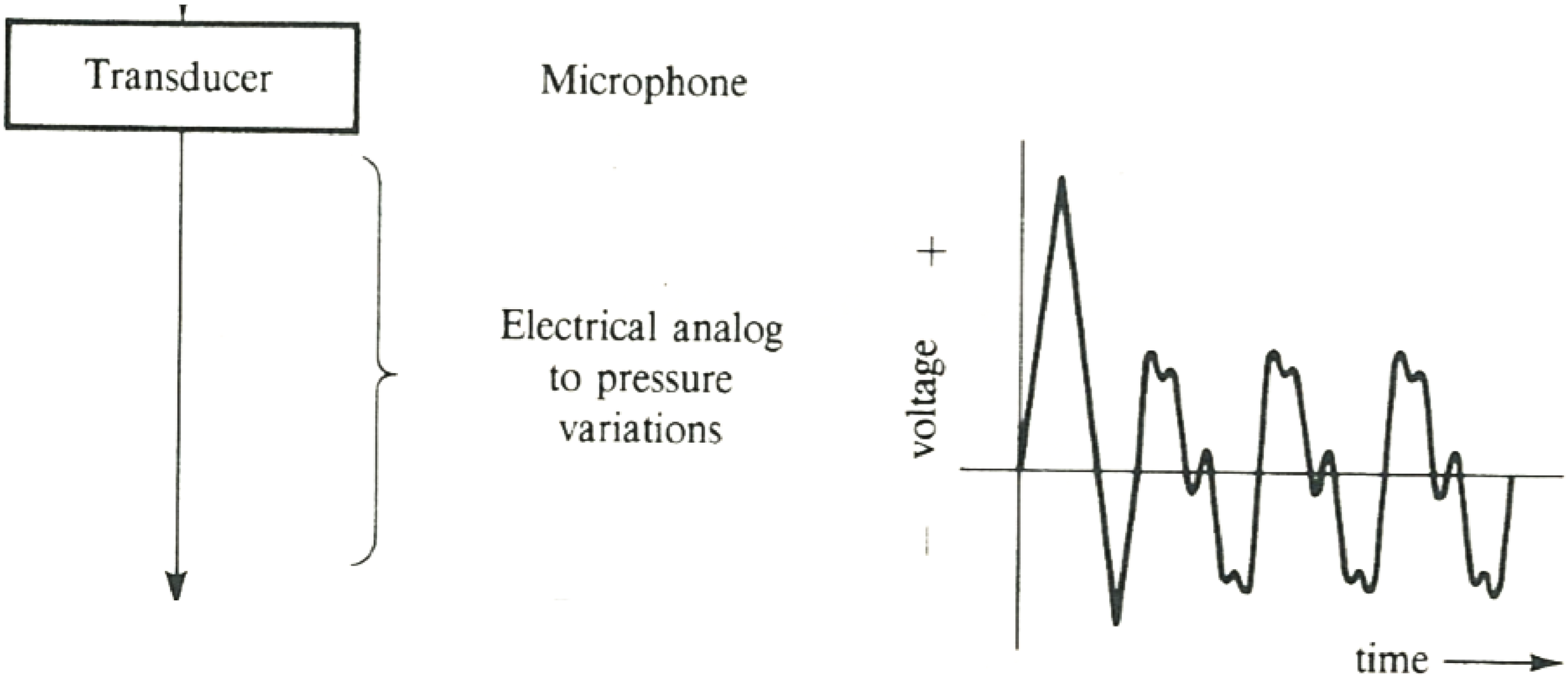
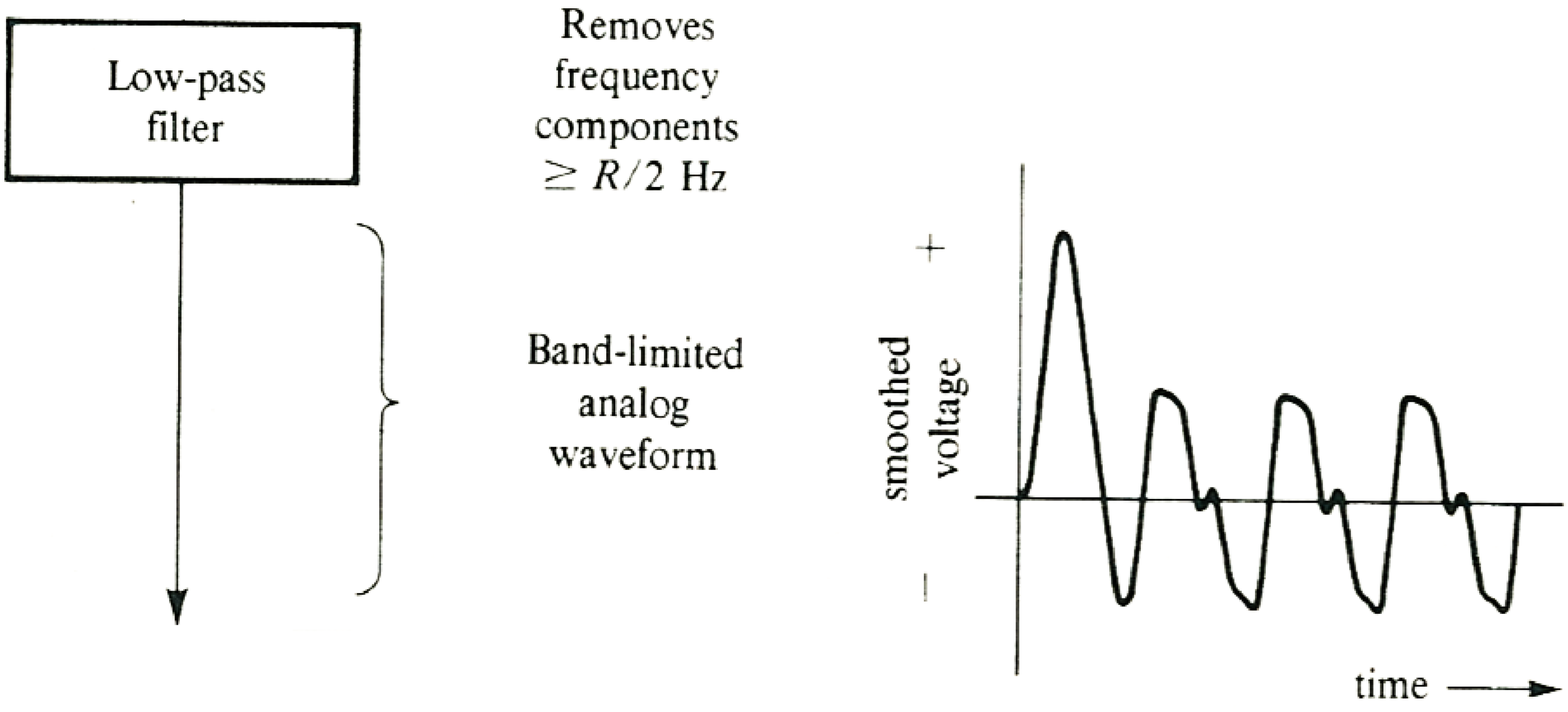
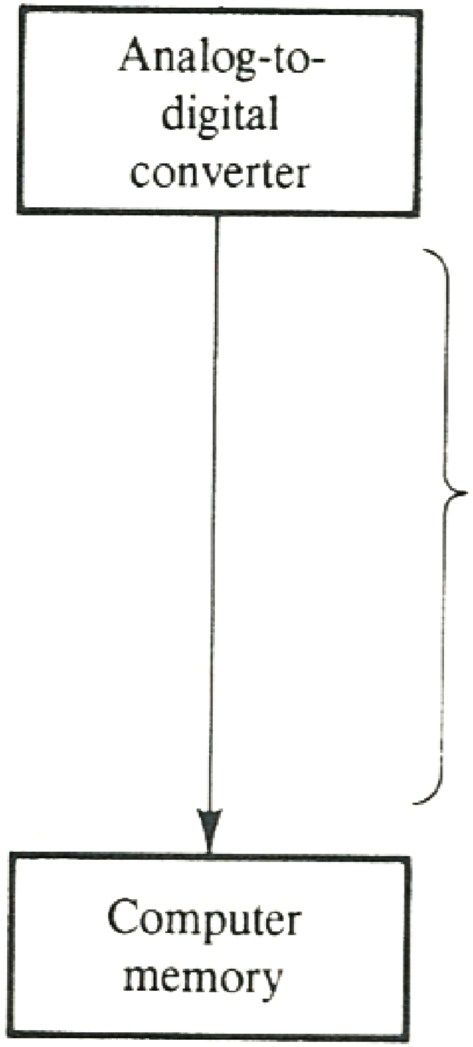
Analog Signal: Via Transducer



Analog Signal Cleaning



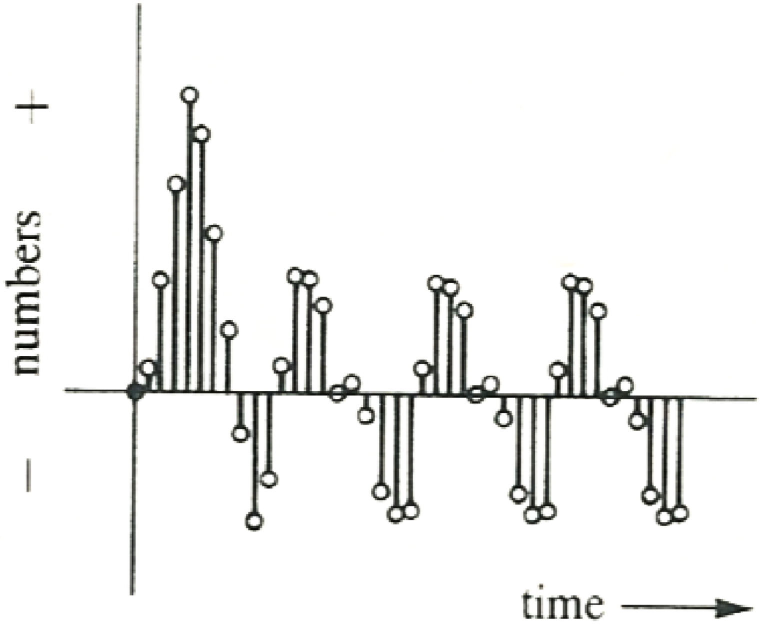
Analog to Digital



Samples at R Hz and quantizes to B bits

Stores complete representation as sequence of binary numbers

Discrete representation of band-limited analog waveform (digital signal)



And back again

Sampling and Quantization

- A **digital-to-analog converter** converts the digital signal back into an analog signal

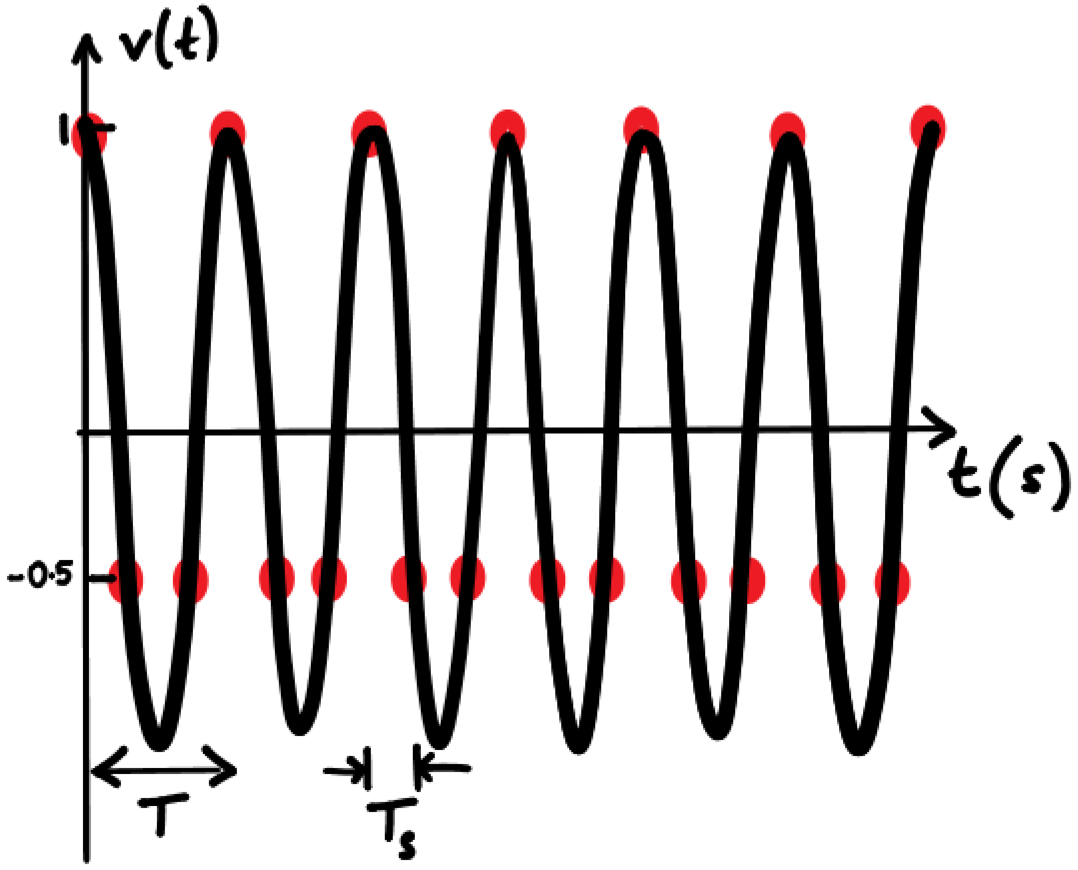
Sampling

Sampling

- Sampling is the process of recording an analog signal at regular discrete moments of time.
- The sampling rate f_s is the number of samples per second.
- The time interval between samples is called the sampling interval $T_s = \frac{1}{f_s}$.

Original signal

- The signal v(t)=cos(2πft) is sampled uniformly with 3 sampling intervals within each signal period T.
- Therefore, the sampling interval T/3 and the sampling rate 3f.
- Notice that there are three samples in every signal period T.



Sample points

- To express the samples of the analog signal x(t), we will use the notation x[n] for example
  - integer values of n index the samples

- Typically, the n=0 sample is taken from t=0

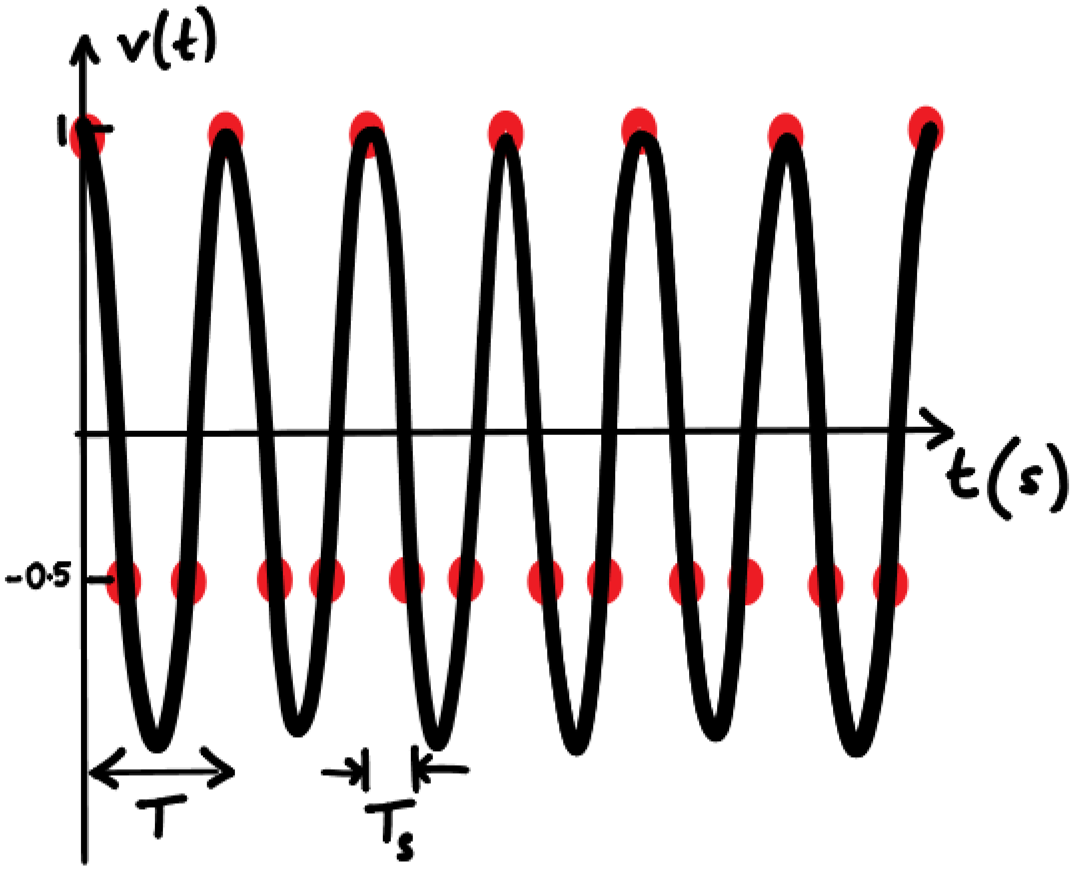
- Consequently, the n=1 sample must come from the $t = T_s$ time point, exactly one sampling interval later; and so on.

- sequence of samples can be written as

$$x[0] = x(0), x[1] = x(T_s), x[2] = x(2T_s), \ldots$$

Store sample: extracted from formula

- $x[n] = x(nT_s)$ for integer n
- Our signal was
- $x(t) = \cos(2\pi f t)$
- $x[n] = \cos(2\pi f n T_s)$
- $x[n] = \cos(2\pi f n \frac{T}{3})$with $T_s = \frac{T}{3}$
- $x[n] = \cos(\frac{2\pi n}{3})$ as $\mathrm{T} = \frac{1}{f}$



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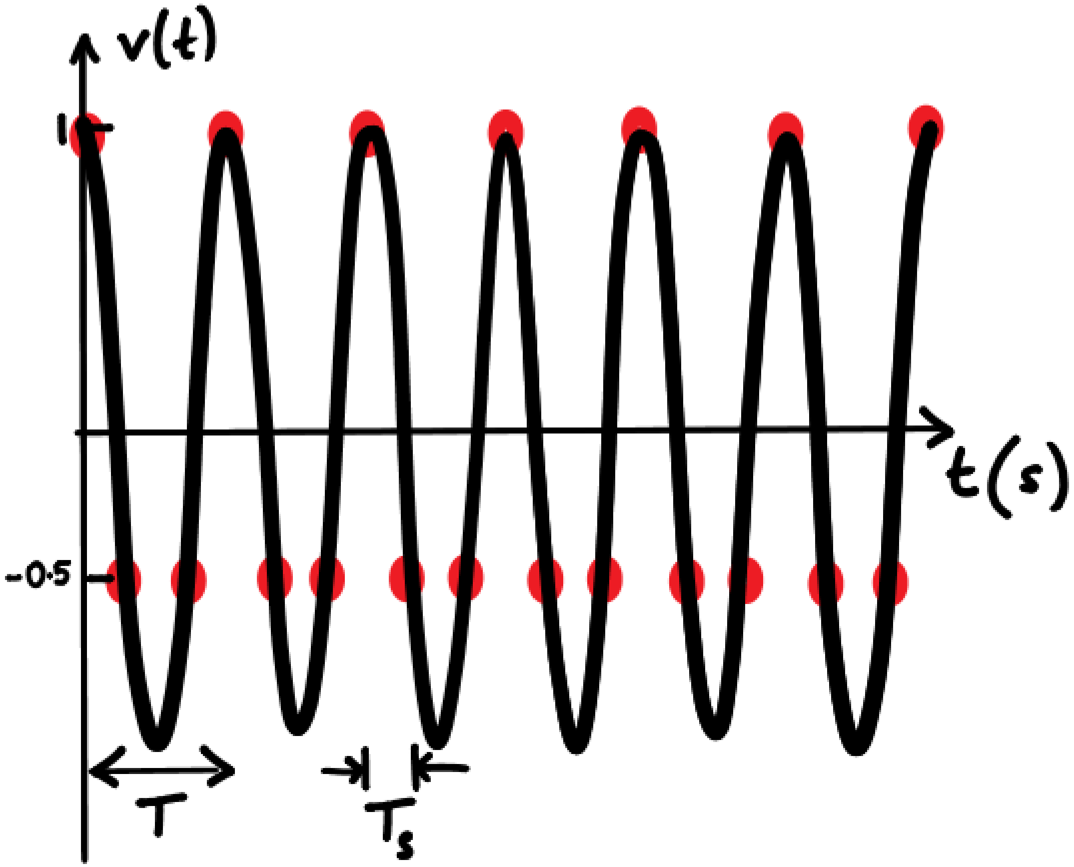
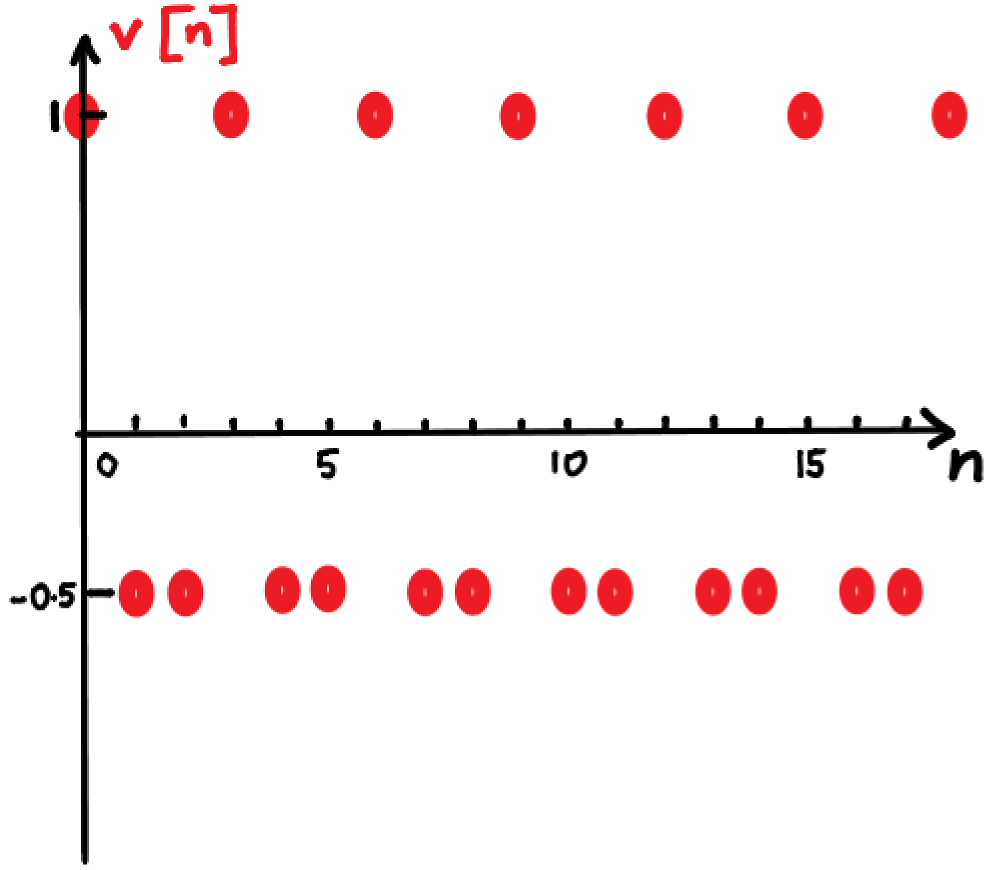
Stored sample: if measured

- $x[n] = \cos(\frac{2\pi n}{3})$
- $x[0] = \cos(0) = 1$
- $x[1] = \cos\left(\frac{2\pi}{3}\right)$ =-0.5
- $x[2] = \cos\left(\frac{4\pi}{3}\right)$ =-0.5
- $x[3] = \cos(2\pi) = 1$



Can we rebuild it?

- $x[n] = \cos(\frac{2\pi n}{3})$
- $x[0] = \cos(0) = 1$
- $x[1] = \cos\left(\frac{2\pi}{3}\right)$ =-0.5
- $x[2] = \cos\left(\frac{4\pi}{3}\right)$ =-0.5
- $x[3] = \cos(2\pi) = 1$



Sampling rate

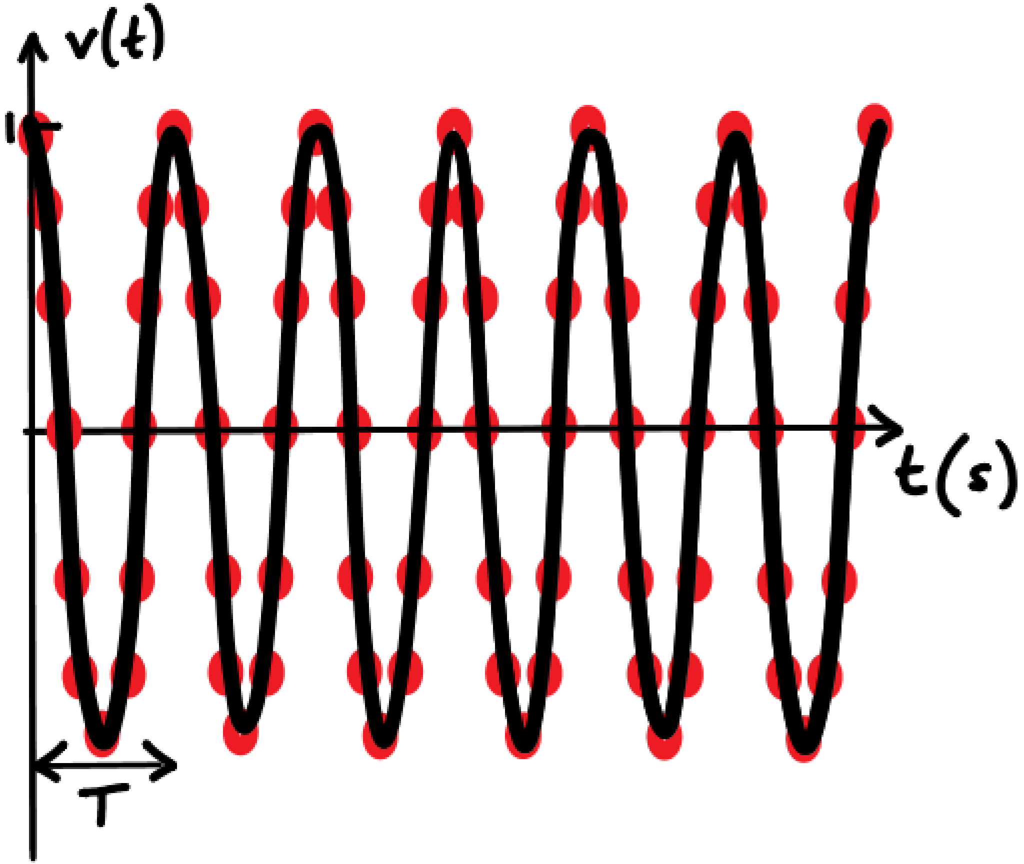
Higher rate sampling

Sampling at a high rate.

The signal v(t)=cos(2πft) is sampled uniformly with 12 sampling intervals within each signal period T.

The sampling interval $T_s = \frac{T}{12}$ and the sampling rate $f_s = 12f$.

The original signal x(t) can be recovered from the samples by connecting them together smoothly.



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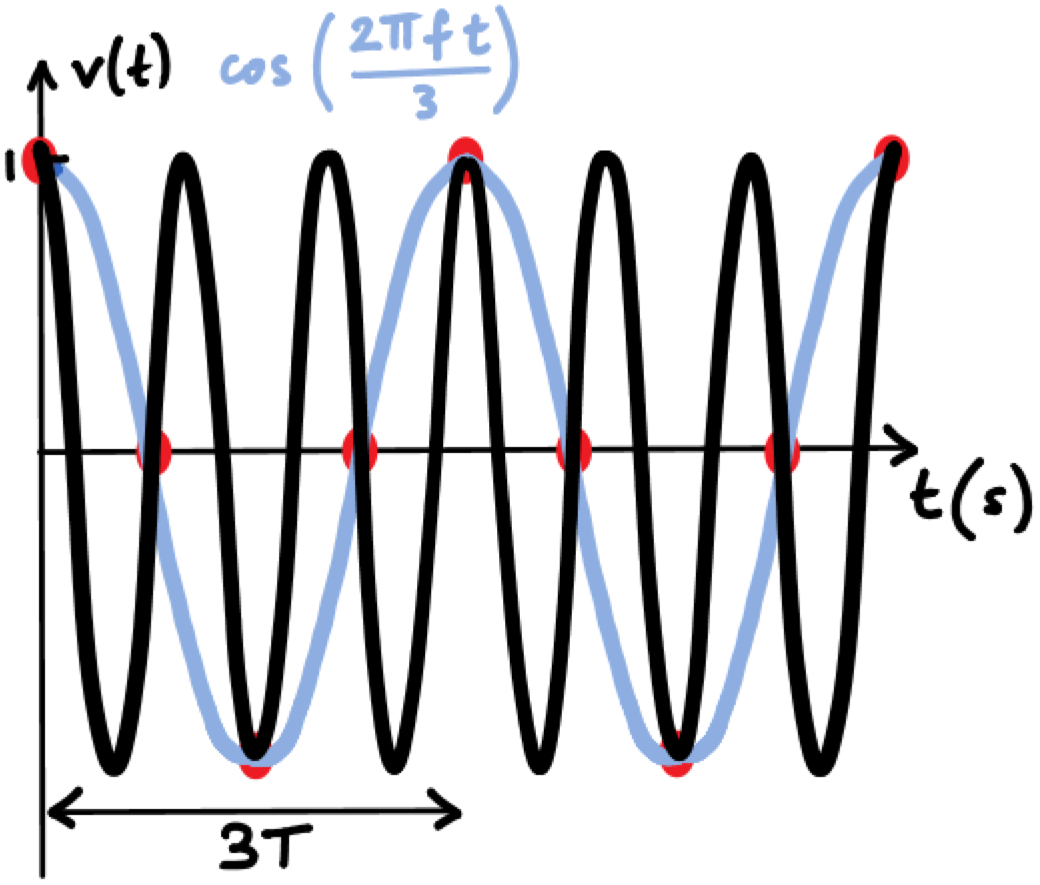
Lower rate sampling

In contrast, if a sinusoidal signal is sampled with a low sampling rate, the samples may be too infrequent to recover the original signal.



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Best sample rate

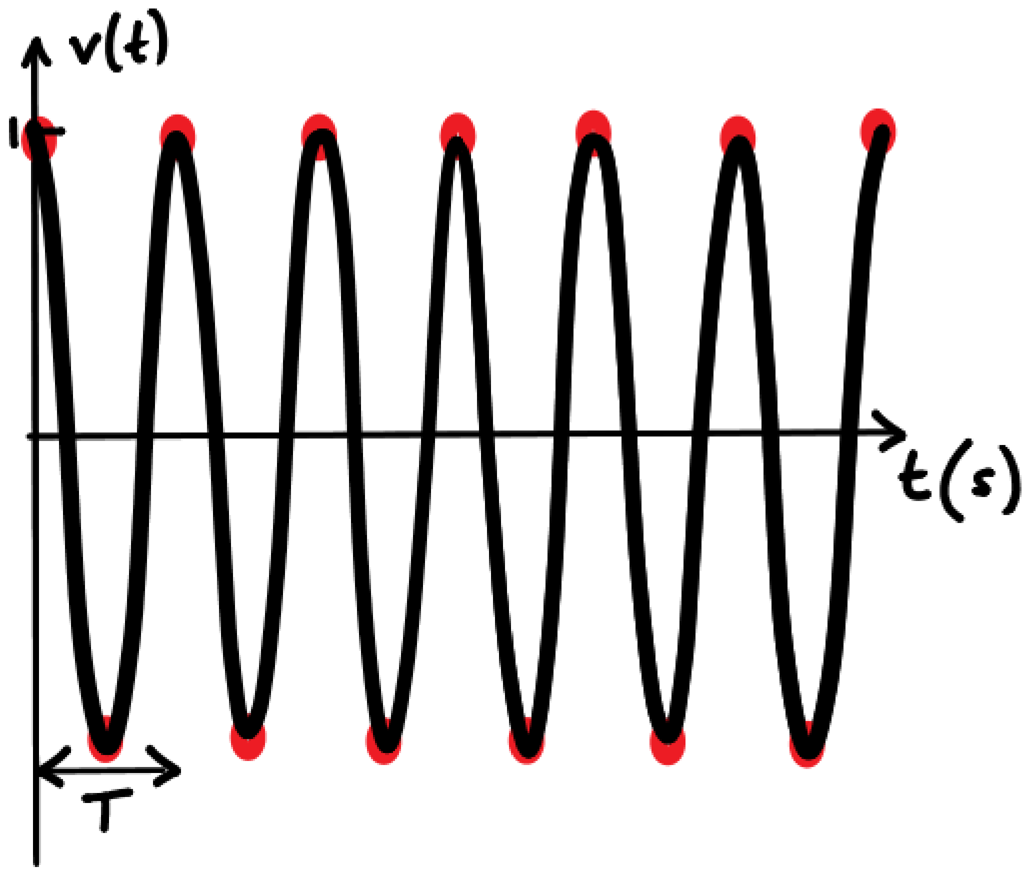
Sampling a cosine at $f_s = 2f$.

The signal v(t)=cos(2πft) is sampled uniformly with 2 sampling intervals within each signal period T.

sampling interval $T_s = \frac{T}{2}$ and the sampling rate $f_s = 2f$.

sample at every peak/trough of the sinusoid, there is no lower frequency sinusoid that fits these samples.

x(t) can be recovered exactly from the samples by ideal low pass filtering.



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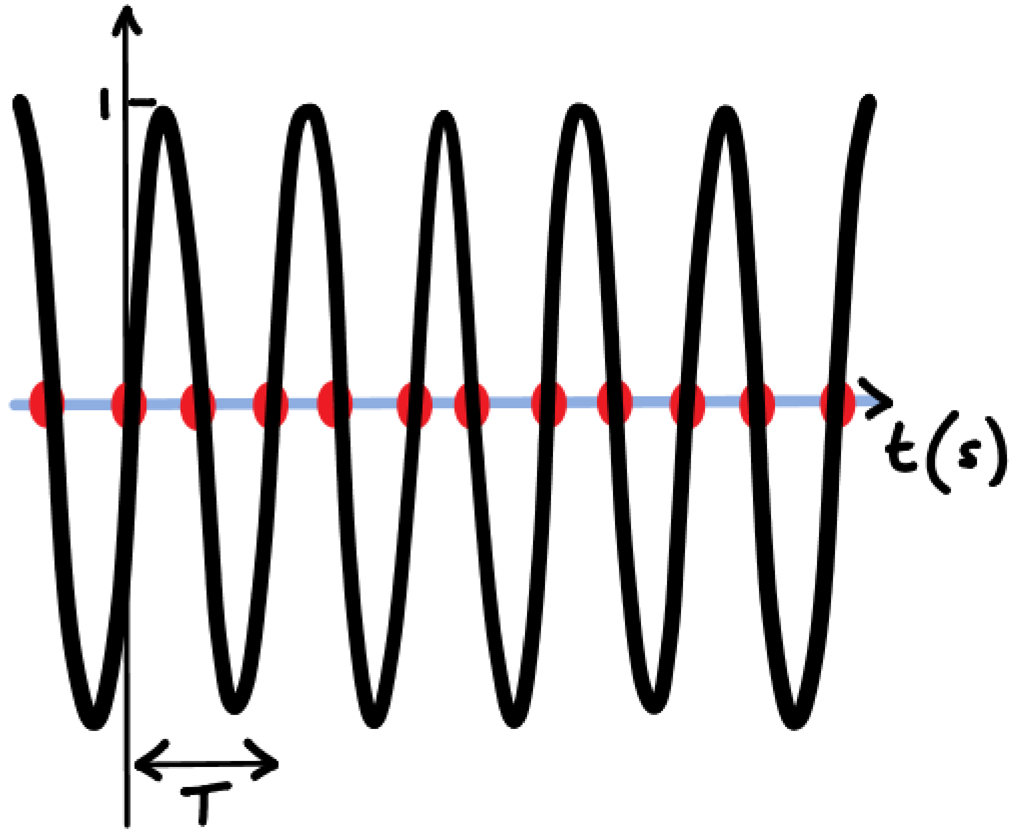
Worst case sample rate

The signal sin(2πft) is sampled uniformly with 2 sampling intervals within each signal period T.

Since all the samples are at the zero crossings, ideal low pass filtering produces a zero signal instead of recovering the sinusoid.



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So how do we decide sample rate

Nyquist-Shannon theorem

The Nyquist-Shannon sampling theorem

The sampling rate for exact recovery of a signal composed of a sum of sinusoids is larger than twice the maximum frequency of the signal.

This rate is called the Nyquist sampling rate $f_{Nyquist}$

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Terminology reminder

- Sampling is the process of recording an analog signal at regular discrete moments of time.
- The sampling rate f_s is the number of samples per second.
- The time interval between samples is called the sampling interval $T_s = \frac{1}{f_s}$.

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Theroem basics

- The sampling theorem:
  - **The sampling frequency must be greater than twice the bandwidth of the signal in order to recreate it perfectly**
    - $f_h < R/2$, where f_h is the frequency of the highest component of the signal, and R is the sampling rate
  - If you sample at too low a rate, **aliasing** or **foldover distortion** results

Details

Sampling and Quantization

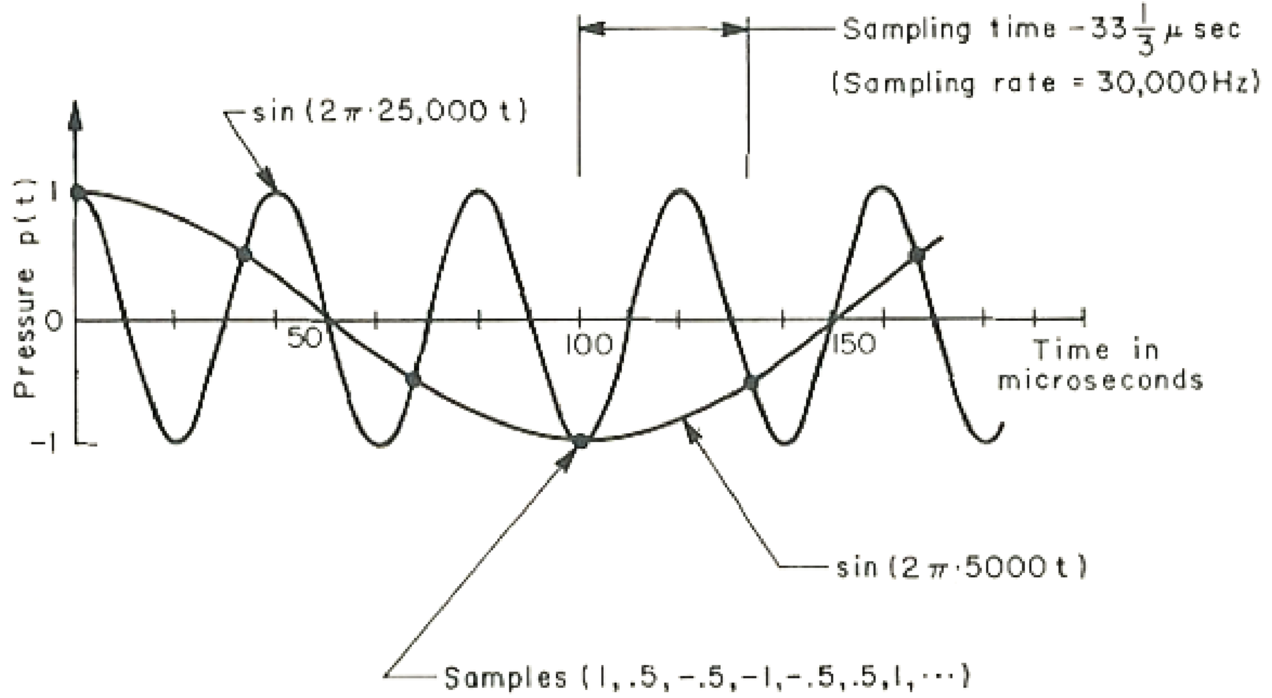

Fig. 5. Example of high-frequency (25,000 Hz) and foldover frequency (5000 Hz) resulting from low sampling rate (30,000 Hz).

Max Mathews: *The Technology of Computer Music*. MIT Press, 1969

Sampling and Quantization

- The frequency of the alias is calculated with:

$$F_a = \left| F - \frac{(k+1)R}{2} \right|, \qquad \frac{kR}{2} \leq F \leq \frac{(k+2)R}{2} \qquad (1.1)$$

where

F_a is the "apparent" frequency in Hz,
F is the actual frequency in Hz,
R is the sampling rate in Hz (samples per second), and
k is any *odd* integer which satisfies the inequality.

Example

- The frequency of the alias is calculated with:

$$F_a = \left| F - \frac{(k+1)R}{2} \right|, \qquad \frac{kR}{2} \leq F \leq \frac{(k+2)R}{2} \qquad (1.1)$$

where

F_a is the "apparent" frequency in Hz,
F is the actual frequency in Hz,
R is the sampling rate in Hz (samples per second), and
k is any *odd* integer which satisfies the inequality.

- If F = 25000 Hz, and R = 30000 Hz

Example (cont'd)

- The frequency of the alias is calculated with:
  - If F = 25000 Hz, and R = 30000 Hz
  - $\frac{kR}{2} \leq F \leq \frac{(k+2)R}{2}$
  - $\frac{k*30000}{2} \leq 25000 \leq \frac{(k+2)*30000}{2}$
  - $k * 30000 \leq 50000 \leq (k + 2) * 30000$
  - $k \leq 5/3 \leq (k + 2)$
  - $k \leq 5/3 \leq (k + 2)$

Example (cont'd)

- The frequency of the alias is calculated with:
  - If F = 25000 Hz, and R = 30000 Hz

  - $k \leq 5/3 \leq (k+2)$
  - $k \leq 1.6666\ldots \leq (k+2)$
  - $1 \leq 1.6666\ldots \leq 3$ when k =1

Example (cont'd)

- The frequency of the alias is calculated with:
  - If F = 25000 Hz, and R = 30000 Hz, then k = 1

  - $1 \leq 1.6666 \ldots \leq 3$ when k =1
  - $F_a = \left|F - \frac{(k+1)R}{2}\right|$
  - $F_a = \left|30000 - \frac{(2)20000}{2}\right|$
  - $F_a = 5000$

Low-pass filtering

- To avoid aliasing, the signal is low-pass filtered before A/D conversion, eliminating any frequency components above R/2

More information

Some rates used

- Common audio sample rates:
  - CD: 44.1 kHz
    - Note: range of human hearing is 20 Hz to 20 kHz
- Pro audio: 48 kHz, 96 kHz, 192 kHz
- Speech codecs: 8000 Hz
- Apple lossless (maximum 384 kHz)
- Streaming music 44.1 kHz (some of this limit is contractual)

Digital form?

- The A/D converter quantizes the instantaneous amplitude of each sample
  - i.e. represents it using N-bit binary number
    - Normally a signed integer
  - The more bits the better, to improve the signal-to-noise ratio
    - E.g. 16 bits gives SNR of about 96 dB

Commons sample bit sizes

- Common sample sizes:
  - CD/Stream: 16-bit
  - Pro audio (subscriber streams): 20-bit, 24-bit
  - Speech codecs: 8-bit, 12-bit

Onward to … spectral analysis.

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