POSSIBLE AUTOMATA

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Abstract This paper is concerned with the widest class of automaton structures whose semantics is compatible with our notions of state and automaton. It is first shown that the conventional spectrum of deterministic, stochastic, fuzzy and non-deterministic can be fitted into a single framework which is complete. In particular new results on the normalization of fuzzy automata and the relationship between fuzzy and stochastic automata are derived. A practical counter-example is then developed that does not fit into this spectrum, showing it to be inadequate. This is based on the richer interpretation of the notion of possibility that is required in the analysis of system stability and reliability. Finally, basic arguments are advanced to show that the structure must be at least an ordered semiring and at most a commutative ordered semiring.

1 Introduction

The concept of an automaton, or state-determined machine, has come to play a substantial role in many disparate branches of science and engineering. The joint origins of the concept in biology and computer engineering have been succintly reviewed recently by Burks [1]. In a related survey Arbib [2] criticizes the applicability of current automata theory and suggests that many new developments and extensions are required. This criticism will be echoed by those who have recognized the concepts of automata theory as relevant to their own disciplines but have been disappointed in the dearth of applicable results.

The convictions, on the one hand, that the basic concepts of automata theory are relevant but, on the other, that the present developments are not sufficiently fruitful have prompted several workers to investigate new automaton structures, e.g. Arbib's tolerance automata [3] and Zadeh's fuzzy automata [4]. Through the very diversity of interests involved automata theory has grown up piecemeal with a variety of automaton structures and semantic interpretations. The continuing intermittent addition of new structures reinforces the impression that not just the development of the subject but perhaps also its foundations are, in some sense, incomplete.

This paper was motivated by our own experience in applying algebraic system theory to problems of system identification, stability and control, where we have found it necessary to define automaton structures that do not fit the conventional spectrum of deterministic, stochastic, fuzzy and non-deterministic automata. These new structures initially appeared to be representable as automata over modal logics rather than Boolean algebra. However, the need soon became apparent for mixed logics involving continuous probability intervals as well as discrete modalities, and the variety of possibilities led us to look for some more general approach.

By suitable choice of V and ⊕ we will show that we can take the normalization for the four cases considered to be:

$$\bigoplus_S \delta(S) = 1 \qquad (1)$$

The standard forms of statement are briefly reviewed in the following sections.

2.2 Deterministic States

These express the conditions that arise when a system's behaviour is completely defined and determinate. The automaton representing it is always in a well-defined, 'sharp', state. We can express this: for each state, it is true or false that the automaton is in the state and the automaton is in precisely one state. A suitable truth set is binary, $V \equiv \{0,1\}$, with ⊕ being arithmetic +, and the normalization as in equn.1. This necessitates only one state being mapped onto 1, and hence we could express the normalization as, 'the inverse image of 1 under δ contains just one element'.

2.3 Stochastic States

These express the conditions that arise when a system's behaviour is a Markov process whose behaviour is constrained by well-defined probabilities. The probability of the automaton representing it being in a particular state is then always well-defined. That is, for each state, the probability that the automaton is in the state is defined and the automaton is in precisely one state (the probabilities over all states sum to one and the conditional probabilities of the automaton being in one state given that it is in another are all zero). A suitable truth set is a closed interval of reals, $V \equiv [0,1]$ say, with ⊕ being arithmetic +, and the normalization as in equn.1.

2.4 Fuzzy States

Zadeh's concepts of fuzzy logic [7] and fuzzy automata [4] represent an attempt to provide a calculus of approximate reasoning. Formally, a fuzzy logic is a Lukasiewicz L_{N_1} system [8 p.337] but Zadeh [9,10,11] has contributed detailed semantics which make the application of the logic attractive and practically useful, for example in pattern recognition [12] and control engineering [13,14].

Hence fuzzy states express the conditions that arise when a system's behaviour is being described by a process of approximate reasoning. The degree of membership of a particular state of the automaton representing it to being the actual state is defined. If we take the usual fuzzy logic system with the truth set being the closed interval of reals and ⊕ being a MAX operator, then $V \equiv [0,1]$ and $a \oplus b = \text{MAX}(a,b)$.

2.4.1 Normalization of Fuzzy States

The normalization of fuzzy state sets to express the condition that the automaton is actually in precisely one state requires special attention. The published semantics of fuzzy automata seem confused on this point. Wee and Fu [15] state that if a state has a degree of membership of unity then the automaton is definitely in the associated state. However, the converse is not true and it

It has been shown by Santos and Wee [5] that the main spectrum of deterministic, stochastic, fuzzy and non-deterministic automata can be fitted into a single formalism, but this is descriptive rather than axiomatic. It leaves open many questions: whether further automaton structures can be invented ad infinitum; what is the most general formulation; and so on. The search for generality is itself dubious unless backed by definite practical requirements expressed as semantic constraints. In this paper we take three distinct approaches to the problem of establishing the most general structure possible for an automaton: analysing first a sense in which the conventional spectrum of automata is already complete; secondly arguing from practical application requirements that this spectrum is inadequate; and thirdly, reversing the direction of increasing generality, to show by foundational arguments that certain quite powerful structural constraints are necessary to an acceptable concept of an automaton, i.e. that arbitrary algebraic structures formally similar to automata do not necessarily possess viable semantics.

The title of this paper is something of a play on words since we are concerned both with the weakest algebraic structures that are possible for automata, but have also chosen to develop all our exemplars within a range concerned with the expression of uncertainty, notably the fine, but important, distinction between the possible and the probable. Section 2 analyses the conventional spectrum of automata in terms of the underlying truth sets and semiring operators, and shows that it may be seen as a complete set of variants of two basic parameters. Section 3 analyses the requirements upon practical explicata of uncertainty about behaviour, and shows that they cannot be met by these variants. It develops an exemplary class of automata over a multi-valued logic of possibility and probability. The final section summarizes the extensions made and raises the converse question of the weakest possible structure that supports our concept of an automaton.

2 Conventional Automaton Structures

2.1 V-sets and Normalization

We take as our informal concept of an automaton at this stage the usual one of a machine with internal states and external inputs whose next state is a function only of its current state and current input. It is the form of this next state function, NSF, which is primarily of concern in this paper. It is only when a deterministic automaton commences in a known state that the next state itself is sharp, i.e. uniquely defined. Usually we have a statement about the current state of the automaton and the NSF enables us to infer a further statement about the next stage.

The standard forms of statement can all be represented as mappings from the set of states, S, to a truth set, V, $\delta: S \to V$; Goguen [6] calls such a mapping a V-set with S as carrier. For the purposes of describing automata states we also require normalization conditions expressing that the automaton is actually in one and only one state. We shall later take V to be a semiring with binary operations, $\oplus$ (we do not use '+' because it can be conflused with arithmetic +) which is associative and commutative, and $\odot$ which is associative, often commutative, and distributes over $\oplus$. Hence it is convenient to express the normalization condition in terms of the formal expression $\bigoplus_S \delta(S)$, meaning the result of operating over the co-domain of δ in the truth-set with $\oplus$ (i.e. 'summation' over the truth set if $\oplus$ is actually +. We assume such summation is well-defined if S should be infinite).

is possible for the rules of fuzzy logic to generate a situation in which the degrees of membership of all states are zero except one which is not unity. This is so even if the total degree of membership is "normalized" as suggested in [15] in the same way as a stochastic automaton (arithmetic sum of degrees of membership being unity). It also leaves open the meaning of two distinct states each having a degree of membership function of unity - an important case since it corresponds to the classical non-deterministic automaton.

It seems better to place the emphasis on the degree of membership of a state being zero as implying that the automaton is not in the associated state. With the usual fuzzy logic definitions of V and $\oplus$ given in section 2.4, our normalization condition of equn.1 requires only that at least one state has a degree of membership of unity. This condition is consistent with the definition of $\oplus$ in fuzzy logic, whereas the proposed "normalization" of [15] introduces arithmetic +, an operator outside fuzzy logic. Neither normalization is consistent with a degree of membership of unity implying that the automaton is definitely in the associated state, and this needs replacement.

A similar problem arises with non-determinate automaton and is clearly a semantic one to be resolved in actual applications. The formal normalization condition proposed here retains consistency between fuzzy automata and the others. We would propose the interpretation that a fuzzy automaton is definitely in a state if the truth values of all the other states are zero. The normalization then implies that the truth value of the remaining state is unity - the converse is not true.

2.5 Non-deterministic States

These might more positively be called 'possibilistic' since they express the conditions that arise when a system's behaviour is such that only the possibility and impossibility of its being in a given state can be discriminated. That is, for each state either it is possible, or impossible, that the automaton is in the state and the automaton is in precisely one state (at least one state is possible, and if only one state is possible then the automaton is in that state). A suitable truth set is binary, $V \equiv \{0,1\}$, with $\oplus$ being Boolean 'OR' which also corresponds to the MAX operation over this truth set. The normalization of equn.1 implies that the inverse image of 1 under δ contains at least one element (as it also does for fuzzy states).

2.6 From States to Transitions

For the moment we shall take it for granted that the nature of transitions can be expressed in terms of the same truth set as that for the states themselves. For example, a 'stochastic automaton' is one with stochastic states and stochastic state transitions. We can express the NSF as a function, $\delta: S \times S \rightarrow V$, which satisfies the normalization condition:

$$s \in S, \qquad \bigoplus_{S} (\delta(S) \odot \sigma(S,s)) = 1 \tag{2}$$

and where the V-set function, δ', after a transition is given by:

$$\delta': S \rightarrow V \equiv \bigoplus_{S'} (\delta(S') \odot \sigma(S,S')) \tag{3}$$

Because $\odot$ distributes over $\oplus$ we can show that the normalization of equn.1 is preserved under equn.3 provided equn.2 holds.

For the four cases discussed $\odot$ is arithmetic x (multiplication) when $\oplus$ is arithmetic + (deterministic and stochastic), and $\odot$ is Boolean 'AND', or the fuzzy equivalent 'MIN', when $\oplus$ is 'OR' ('MAX') (fuzzy and non-deterministic).

2.7 Comparisons and Contrasts

The previous sections have been phrased to bring out the similarities and differences between the four structures considered. Note that the normalization condition is uniformly that of equn.1, and the truth sets are either the entire interval, $[0,1]$, or its boundary points, $\{0,1\}$, whilst the transitions are uniformly represented by equns.3 and 2. A table of operators against truth sets:

$\oplus$	Truth set $\{0,1\}$	Truth set $[0,1]$	$\odot$
+	Deterministic	Stochastic	x
OR	Non-deterministic	Fuzzy	AND

Table I Relations Between Truth Sets and Operators for the Standard Spectra of Automata

shows that the four cases analysed encompass a complete set of variations for these truth sets and operators. This is intuitively satisfying because it gives a closure over those automata which have been most extensively studied in the past. It is an answer in this context to the question of whether we can continually invent new forms of automaton.

2.7.1 Fuzzy and Stochastic Automata

The relationships expressed in Table I between fuzzy, non-deterministic and deterministic automata, and between stochastic and deterministic automata, are well-known. However, that between stochastic and fuzzy automata is less obvious and it is worth discussing whether this is just a mathematical formality or whether it has a semantic content. Clearly the common use of the interval $[0,1]$ corresponds to quite different interpretations of the values within it - a "degree of membership" appears as a far less precise concept than a "probability". Equally the operators, + and x, appear little related to MAX and MIN. However, the following argument demonstrates a closer correspondence than might be expected.

Consider two events, A and B, with respective probabilities of occurrence, p_A and p_B. If the two events are statistically independent then the probabilities of their conjunction and disjunction are:

$$p(A \wedge B) = p_A \times p_B \tag{4}$$

$$p(A \vee B) = p_A + p_B - p_A \times p_B \tag{5}$$

Suppose, however, that A and B are not independent events but that one implies the other, $A \rightarrow B$, say. Then we have:

$$p(A \wedge B) \;=\; p_A \tag{6}$$

$$p(A \vee B) \;=\; p_B \tag{7}$$

However, the direction of implication also gives us:

$$p(A) \leq p(B) \tag{8}$$

so that equns.6 and 7 may be re-written:

$$p(A \wedge B) \;=\; MIN(p_A, p_B) \tag{9}$$

$$p(A \vee B) \;=\; MAX(p_A, p_B) \tag{10}$$

Conversely, if the "fuzzy logic" conditions of equns.9 and 10 hold for two probabilistic events, then we have:

$$p(A \wedge B) \;=\; MIN(p(A \wedge B) + p(A \wedge \bar{B}),\; p(A \wedge B) + p(\bar{A} \wedge B)) \tag{11}$$

which implies that either $p(A \wedge \bar{B}) = 0$ or $p(\bar{A} \wedge B) = 0$, i.e. either $A \rightarrow B$ or $B \rightarrow A$.

Thus we see that the applicability of the fuzzy logic operations of equns. 9 and 10 to determining the probabilities of conjunction and disjunction of two probabilistic variables is completely equivalent to their being a logical relationship of implication between the variables.

In principle therefore the fuzzy logic rules of [4] are reducible to a probabilistic logic in which all variables are connected by a chain of implication. The converse condition to that generally found useful in application of probability theory where one attempts to make variables statistically independent. The "chain" concept is intuitively significant - the MIN operation in fuzzy logic expressing that a chain is as weak as its weakest link - the MAX operation expressing that alternative chains in parallel are as strong as the strongest.

These relationships between probabilistic and fuzzy logics indicate that Table I expresses more than mathematical formalism. Clearly the relationship demonstrated between fuzzy and probabilistic logics should also extend to the richer semantics developed by Zadeh in [9,10,11]. It would also be interesting for application studies to compare probabilistic and fuzzy logics in their relative efficacies for particular situations and relate this to the presence or absence of implications between the variables involved.

In the next section we develop an argument for state specifications that go beyond those of the four so far discussed, and for automata over mixed state structures. The final section discusses the greatest generality beyond which our notion of a state-determined machine will not carry.

3 Automata Over Multi-Valued Logics of Possibility and Probability

3.1 The Need for Further Automaton Structures

Although section 2.7 gives a satisfying completeness result for the conventional spectrum of automata, it in no way implies the sufficiency of these structures to represent all possible cases of interest. That they are in fact inadequate is best seen by example, and we shall give one which is itself of particular interest in the context of calculi of possibility and probability, and of multi-valued logics.

In our studies of system stability and control we have been very concerned to embody in our formulation the distinction between possible events that may not occur and possible events that are guaranteed to occur sooner or later. The former events correspond to problems that may arise and have to be avoided. They relate to regions of states which are reachable in terms of stability analysis but not reachable in terms of control. The second type of possible event, however, is responsive to feedback control since if the situation is continually recreated in which it may occur then it eventually will occur.

Note that probability theory does not provide an explicatum of the first type of possible event. If for the purposes of analysing an uncertain system we assign an uncertain event a non-zero probability then we imply that not only may it occur but also, in a sequence of occurrences each of which may be that event, it eventually will occur with a probability arbitrarily near one. The notional assignment of a definite probability to an event also fails to provide an adequate explicatum of the second type of possible event because it has the stronger implication that the relative frequency of such events in a sequence will tend to converge to the given probability with increasing length of sequence.

Either, or both of these connotations which probability has over possibility may be too strong in practical situations where the concepts of probability theory are being used to express the effects of uncertain behaviour. For example, we are often faced with situations where an event, E, may occur, but there is no guarantee that E actually will occur, no matter how long we wait. It we ascribe some arbitrary probability to E then we certainly express that it is a possible event. However we are in a position to derive totally unjustified results based on the certainty of some eventual occurrence of E, or meaningless numeric results based on the actual 'probability' of occurrence of E.

A similar problem arises in the practical application of linear systems theory. There are many results which may be derived from the assumption of linearity (such as the complete extension of knowledge of local behaviour to that of global behaviour) which are false in most practical systems. The engineer resolves these problems in practice by using a set of 'rules-of-thumb' based on commonsense and experience to constrain the deductions he is prepared to make. Such a resolution is however extremely difficult to implement in an automated, or computer-aided, design system and becomes increasingly difficult to apply as the system involved becomes more complex.

There is a danger on the one hand that results may be derived which have no justification other than an unwarranted strength in the theory. For example, Gaines [16,17] that a two-state stochastic automaton can solve a class of control problems otherwise requiring a recursive automaton [18] and not soluble by any finite automaton

[16,18]. This powerful result is dependent on the source of uncertain behaviour being truly probabilistic, and cannot be derived if it is merely possibilistic. There is no way, however, of preventing the consequences of this result appearing in the analysis of a system in which uncertainties have been represented by probabilities rather than possibilities.

On the other hand there is a danger that significant phenomena may be overlooked because they cannot be distinguished from such spurious effects. For example, it is possible to derive deterministic results about the behaviour of automata whose transitions are indeterminate. Starting in S_o, the states of the indeterminate automaton of Fig. 1 are indeterminate. However, it is clear that if the automaton is found to be in S_4 it must have passed through S_3. If we know that the transitions of this automaton are properly probabilistic then we may also conclude that its state will eventually be in the set (S_3, S_4). If, in addition, the transition probabilities are well-defined, we may also derive the expected time for this state set to be reached. These distinct forms of implication are confounded if we merely represent the indeterminate transitions as probabilistic in all cases.

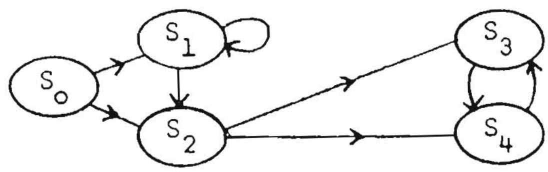

Figure 1 An indeterminate automaton

3.2 Explicata of Uncertainty

It appears that there are three distinct explicata of uncertainty, each of which has its own consequences that need clear separation:

(I) Possible Event E is possible - no reliance may be placed upon the occurrence or the non-occurrence of E. This corresponds to an interpretation of E as an event whose negative consequences must be taken into account, but whose positive consequences must not. Conventional probability theory provides no explicatum of this concept.

(II) Frequent Event E is frequent in the sense of the theory of infinite sequences, that in a sequence of events, E_i, for any N, there exists $M > N$, such that $E_M = E$, i.e. E occurs 'frequently' in the sequence E_i. This corresponds to the interpretation of E as an event whose eventual occurrence may be relied upon, but whose relative frequence of occurrence is not necessarily stable or known. A possible explicatum in probability theory is that $p(E) > 0$, the event if of non-zero probability.

(III) Probable Event E is frequent and its relative frequency of occurrence in a sequence of events converges to a definite value, p(E), its probability of occurrence.

It is clear that, in terms of our classification in the previous section, case I may be represented in terms of the non-determinate automata, and case III in terms of stochastic automata, with case II possibly being represented in terms of stochastic automata with properly probabilistic transition of unknown value. We needed, however, to encompass the cases of mixed transitions, typically situations where certain types of possible behaviour could be guaranteed but other types could not. In the following section a suitable formulation for the mixed case is developed.

3.3 A Logic of Possibility and Probability

Let us take the truth set, V, to consist of the semi-open interval, $R \equiv (0,1]$, and the elements, N, F, P, I, whose interpretation is:

N - Necessary occurrence - probability equals unity.

F - Frequent occurrence - probability unknown.

P - Possible - cannot say that it will not occur.

I - Impossible - cannot occur.

A truth value in R is a known probability of occurrence which is not zero. We shall say an event is of type R if its truth value is in R and will write R:p, where p is its probability, to emphasize this.

The ⊕ operator over V corresponds to two different routes arriving at the same state - what can we say if we know either x or y is true. A truth table for ⊕ is given in table II. The ⊙ operator over V corresponds to a state followed by a transition - what can we say if we know that y follows x. A truth table for ⊙ is given in table III.

TABLE II

⊕	N	F	R:r	P	I
N	N	N	N	N	N
F	N	F	F	F	F
R:r'	N	F	R:r+r'	F	R:r'
P	N	F	F	P	P
I	N	F	R:	P	I

TABLE III

⊙	N	F	R:r	P	I
N	N	F	R:r	P	I
F	F	F	F	P	I
R:r'	R:r'	F	R:rr'	P	I
P	P	P	P	P	I
I	I	I	I	I	I

Consider first the structure with R taken as a single logic variable, i.e. V ≡ {N,F,R,P,I}, which allows for all the explicata of uncertainty developed in section 3.2 Note that, without R, the tables for ⊕ and ⊙ are simply those of a 4-value Post logic, and hence can be mapped onto a fuzzy logic. R, however, behaves anomalously in that R⊕P = F whereas R⊙P = P. It has been suggested by Brown that the V-set of a fuzzy logic be taken to be a distributive lattice [19]. However the interaction of R and P is inconsistent with ⊕ and ⊙ being lattice operations. This is a concrete example of the need for more general truth sets discussed by Goguen [6].

If we now consider the full truth set as first specified in which R is actually a semi-open interval, then the logic is now a mixed continuous discrete structure which can, however, still be neatly represented in the 'truth tables'. Such structures are both theoretically interesting and practically necessary to obtain rich enough explicata of the behaviour of uncertain systems.

It will be noted that the diagonals of the two tables show the idempotency of the elements, and the wider significance of this may be raised. However, the individual elements of R are clearly not idempotent in general ($p+p \neq p$, and $p \times p \neq p$, in general), and if we consider a variant on F, such as G interpreted as 'properly probabilistic' (unknown probability in the open interval, (0,1)), then idempotency can be seen to fail even for a discrete element (G⊕G = F).

4 Possible Automata

4.1 Semirings

We have noted that the truth set need not be a fuzzy set or a distributive lattice, and that the elements need not be idempotents under ⊕ or ⊙. In the example of the previous section it can be seen that ⊕ and ⊙ are both associative and commutative and that ⊙ distributes over ⊕, i.e. together they give the truth set the structure of a commutative semiring. It is also apparent that this semiring is positive [19, p.125] in that if we consider the zero element (I in Tables II and III) then:

$$a \oplus b = I \rightarrow a = I = b \qquad (12)$$

and:

$$a \odot b = I \rightarrow a = I \text{ or } b = I \qquad (13)$$

In this example we have shown that a stronger structure would be too restrictive. However, the question remains of whether a positive commutative semiring is still too strong a structure on which to base automata theory. The following notes outline arguments to show on fundamental, and intuitively satisfying, grounds that at least an ordered semiring is necessary.

First consider the operator, ⊕, which represents the combination of different trajectories to the same state. Trajectories may be combined in pairs so that this gives the truth set the structure of a partial groupoid (partial because some pairs of values may not arise and hence their result is undefined, e.g. probabilities of 1 and 1). However, we must also take into account the independence of trajectories, that they represent alternative paths and there should be no effect of order or grouping when combining them. This implies that ⊕ is necessarily commutative and associative, and hence defines a partial commutative semigroup over the truth set (it may be taken as a partial monoid by adding the null trajectory as an identity element). We may drop the term "partial" in general by noting that the "don't care" conditions can always be fitted in to complete the monoid.

Even these constraints do not fully represent the necessary structure since each trajectory terminating in a state can only add to our knowledge about the automaton being in that state. There can be no cancellation of information obtained by considering independent trajectories. One possible expression of this is to require the monoid to be positive, so that:

$$\forall \quad a,b \in V, \quad a \oplus b = 0 \rightarrow a = 0 = b \tag{14}$$

where 0 is the identity element of the monoid written additively. It can readily be seen (by adding a or b to each side of the left equn. of (14)) that if the elements of the monoid are idempotent (14) automatically holds. Idempotency also implies the natural order on the monoid is a semi-lattice. That is defining a relation, $\geq$, on V in terms of $\oplus$:

$$\forall \quad a,b \in V, \quad a \geq b \leftrightarrow \exists\, c : a = b \oplus c \tag{15}$$

Unfortunately the positivity condition of (14) alone does not guarantee that this is even a partial order, and it seems that the best statement of the constraint upon the monoid is that the natural pre-order on it defined by (15) is actually a partial order. This itself implies that the monoid is positive and is implied if the elements are idempotent. Intuitively, this order relation corresponds to our having two independent sources of information about a state which cannot cancel - taken together they must give at least as much information as either alone.

The operator $\odot$ presents more interesting problems since it represents the interaction between states and transitions, and there is no a priori reason to suppose that they can be expressed in the same language. Let us start with the more general assumption that the transitions are drawn from a set of functions, $F \equiv \{f: V \rightarrow V\}$. Considering the same argument as for $\oplus$, it can be seen that the result of applying a function to each individual trajectory separately (and then combining them) must be the same as applying it to them already combined - i.e. the functions must distribute over $\oplus$:

$$\forall \quad f \in F, \quad a,b \in V, \quad (a \oplus b)f = (af) \oplus (bf) \tag{16}$$

The implications of distributivity are not intuitively obvious and they may be expressed more meaningfully in terms of the order relation of (15), since (16) shows that f must be isotone with respect to $\geq$. Again we may argue that a transition cannot in itself increase information about a state so that f must be isotone non-increasing (this also makes it a residuated mapping in the sense of [21]).

The isotone non-increasing mappings over the truth set clearly form a semi-group which can be extended to be a semiring by the definitions:

$$\forall \quad f,g,h \in F, \quad h = f \odot g \leftrightarrow \forall a \in V, \quad ah = afg \tag{17}$$

and:

$$\forall \quad f,g,h \in F, \quad h = f \oplus g \Leftrightarrow \forall a \in V, \quad ah = af \oplus ag \tag{18}$$

The partial order defined by (15) has a natural extension to F in terms of (18) and this in turn implies that F under $\oplus$ and $\odot$ is a positive semiring [20]. There are a number of possible injections of V into F such that:

$$af = a \odot f \tag{19}$$

and hence the entire structure of the monoid and its relevant endomorphisms may be represented in terms of a positive semiring.

It will be noted that the examples given previously are such that $\odot$ is commutative whereas no informal arguments have been put forward here to suggest that this is true in general. It is easy enough to generate simple structures in which $\odot$ is not commutative but all our other requirements are satisfied. We have yet to find a semantics for such structure to show that they are necessary. Conversely there appears to be no argument on the lines of those advanced to suggest that such a semantics is not possible.

4.2 The Role of Idempotency

If one accepts the informal arguments of the previous section in terms of the monoid over V representing "information" about the automaton being in a state then it would be natural to assume that its elements were idempotents, i.e. that getting the "same" information a second time contributed nothing extra. Only the probabilistic case gives a counter-example, and here the "information" is a value rather than a datum.

Suppose however that instead of considering the probabilities themselves one considers the underlying Borel set structure of the σ-algebra for the probabilities. Then the "information" consists of disjoint sub-sets whose measures correspond to the probabilities and if $\oplus$ is regarded as the union operation on the sub-sets it is, of course, idempotent.

In this case our semiring becomes a lattice, as it was for all the non-probabilistic examples given. Thus it might well be that an intuitively satisfying axiomatization of automata theory could lead to the stronger structure of a lattice, rather than a semiring, providing one is prepared to carry the full structure of a measure algebra when carrying results over to probabilistic automata.

This suggestion throws further light on the relationship between fuzzy and probabilistic automata. The normalization conditions are the same in that the joins of the truth values for all the states should be units, but the fuzzy truth values must form a linearly-ordered chain (a "vertical" section), whereas the probabilistic truth values must form a totally unordered set (a "horizontal" section) whose meets are zero.

5 Conclusions

This paper is exploratory and intended to 'open up' certain aspects of automata theory and of the logic of uncertainty. We have been concerned to stay close to the semantic roots of these topics and avoid over-emphasis on mathematical formalism. Automata theory to a large extent, and probability theory to a lesser extent, have evolved pragmatically with new constructs being introduced to satisfy new requirements. It seems appropriate now to return to fundamentals and examine the minimum underlying sub-structure common to all our concepts of automata and uncertainty. Goguen [6] has given an extremely clear and coherent account of the logic of uncertainty in category-theoretic terms. The present paper may be seen as a further exploration within the same framework, illustrating on one hand the need for the systems within that framework that go outside the conventional spectrum of automata, and on the other hand defining the boundaries of that framework beyond which the basic connotations of a structure being an automaton are lost.

6 References

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