Memory Minimisation in Control with Stochastic Automata

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Stochastic automata have been shown to require less states than deterministic automata in the solution of certain recognition and hypothesis-testing problems. This letter extends the result to a class of control problems involving the regulation of a discrete dynamical system.

One of the major properties of stochastic machines is that they offer the possibility of memory savings over equivalent deterministic machines. Rabin (1963, theorem 4) has shown that there are regular events accepted by a 2-state stochastic automaton which are only accepted by a deterministic automaton with an arbitrarily large number of states. Gaines (1969) has shown that there are discriminations which cannot be learnt by an adaptive-threshold-logic element with discrete bounded weights, unless the weight changes are non-deterministic. Hellman and Cover (1971) have shown that, for deterministic automata of arbitrary size, there exist hypothesis-testing problems which they cannot solve, but for which a 2-state stochastic automaton has an arbitrarily small error.

The present letter extends the class of problems for which stochastic behaviour is known to be advantageous by obtaining a similar result for certain abstract regulatory control problems. We shall consider an abstract formulation of the problem of regulating a discrete dynamical system to maintain its state within a prescribed region. Consider the finite automaton $(I, P, S, \delta, \lambda)$, where I is a finite set of inputs, $P \equiv (0,1)$ is a binary set of possible outputs, S is a finite set of states, $\delta: S \times I \to S$ *is* the next-state function, and $\lambda: S \to P$ is the output function. λ is to be regarded as a performance function, since we shall consider the problem of regulating the inputs to the automaton to cause its output to become, and remain, 1.

Suppose that there is some distinguished element $\Lambda \in I$ (the 'zero' input for the autonomous system), and consider the sets of states:-

$$W \equiv \{s: \lambda(s) = 1\}$$
$$A \equiv \{s: \forall n \geq 0, \delta(s, \Lambda^n) \in W\}$$
$$B \equiv \{s: \exists n: \delta(s, \Lambda^n) \in A\}$$

W is the subset of S in which it is desired that the state should reside, A is a weak attractor within W, and B *is* its region of attraction (Bhatia and Szego, 1967). We shall assume that both B and S-B are non-empty, so that the autonomous system has a region of local asymptotic stability but is not asymptotically stable in the large, and consider the family of control automata, whose inputs are from P and whose outputs are in 1, which induce global stability. For this family to exist, it is necessary that B be *reachable* (Arbib, 1965) from S, i.e.:-

$$\forall s \in S, \exists u \in F_1: \delta(s,u) \in B$$

where F_1 is the free semigroup generated by I.

Consider, as a regulator for the dynamical system (plant), the 2-state stochastic automaton whose states are. a_0 and a_1, whose inputs are from *P*, whose outputs are in *I*, whose state transitions are determined by the following table of probabilities that the next state will be a_1:-

	Previous state		
Input		0	1
	a_0	0.5	0.5
	a_1	0.0	1.0

and whose output is □ in state a_1 and randomly chosen uniformly from *I* when in state a_0. If the automata are coupled, it is clear that the condition in which the state of the plant is within *A* and that of the stochastic regulator is a_1 is a stable one. However, there is finite probability of attaining this condition from any other of the coupled automata, and hence the probability of the automata being in this condition tends to unity; we may say that the coupled system is stochastically asymptotically stable in the large (Kushner, 1967).

Consider now the regulator to be an arbitrary deterministic automaton with *n* states ($P, I, S_c, \delta_c, \lambda_c$), with inputs from *P*, outputs in *I*, a state set S_c, a next-state function $\delta_C : S_c \times P \to S_c$, and an output function $\lambda : S_c \to I$. Suppose now that the plant is such that there is a 1:1 mapping $\eta : S_c \to S$, which is not onto in that *S* has *n* + 1 elements so that there is one state of the plant, a say, which is not the image of any element of S_c. Suppose also that the next-state function of the plant is selected to be such that:-

$$\forall s_c \in S_c, \forall i \in I, \delta\{\eta(s_c), i\} = \begin{cases} \eta\{\delta_c(s_c, 0)\} & i = \lambda_c(s_c) \\ \alpha & \text{otherwise} \end{cases}$$

$$\forall i \in I, \delta(\alpha, i) = \alpha$$

The output function is selected to be such that:-

$$\forall s \in S \quad \lambda(s) = \begin{cases} 1 & s = \alpha \\ 0 & \text{otherwise} \end{cases}$$

and that, if the initial state of the regulator is s_c, the initial state of the plant is chosen to be $\eta(S_c)$.

This type of plant will be termed a 'frustration automaton' for the deterministic regulator, in that it ensures that the plant output remains zero when coupled to the regulator, while also satisfying the reachability criterion for global stability to be possible with the stochastic regulator. Indeed the plant is such that *any* control policy other than that of the regulator in use ensures global stability!

Having established the existence of a frustration automaton for a regulator of known initial state and structure, it is immediately possible to generalise the result to. show that an automaton exists which is a universal frustration automaton for all deterministic regulators with *n* states or less. The plant will have an initial transient in which its output is zero and in which it performs state and structure identification experiments (Moore, 1956). These are trivial in that the input of the regulator (output of plant) is to be maintained zero for all time, and hence the behaviour of the regulator will be cyclic with period less than or equal to *n*. The plant will then assume the form of a frustration automaton for the particular regulator which it has identified.

An explicit construction of such a universal frustration automaton for regulators with n or less states is as follows. Let the plant be connected to the regulator at time t = 0, and let the output of the regulator at time t = i be $R(i)$, while the output of the plant is $P(i)$ $(i = 1, 2, 3, \ldots)$. Let the output of the plant be defined to be:-

$$P(i) \equiv \begin{cases} 0 & 1 \le i \le 2n \\ P(i-1) & i > 2n \text{ and } R(i) = R(m-n+i) \\ 1 & \text{otherwise} \end{cases}$$

where $m \geq 0$ is defined to be the minimum integer such that:-

$$R(n+i) = R(m+i) \quad (1 \le i \le n)$$

Such an m must exist in the range $0 \le m < n$, since the regulator is driven by a constant input of 0 and hence its output must become periodic after a time of at most n.

This plant can be realised as a finite-state deterministic machine (it need only remember the last $2n$ outputs of the regulator), and its overall behaviour is to identify the regulator over a period of $2n$ and then switch its output to 1 if and only if the output of the regulator once deviates from its identified cycle. In these terms, the success of the stochastic regulator in dealing with the same plant may be seen to be due to the inherent acyclicity, or unpredictability, of its output.

Thus we have the result that there exists for any n, a discrete dynamical system which cannot be regulated by any deterministic automaton with n states or less, whereas it can be (stochastically) regulated by a stochastic automaton with two states.

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