BCS AISB Summer School, Cambridge, UK, July 1975.

MULTIVALUED LOGICS AND FUZZY REASONING

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1 Shallow and Exact Reasoning

These notes are concerned with recent developments in multivalued logic, particularly in fuzzy logic and its status as a model for human linguistic reasoning. This first section discusses the status of formal logic and the need for logics of approximate reasoning with vague data. The following sections present a basic account of fuzzy sets theory; fuzzy logics; Zadeh's model of linguistic hedges and fuzzy reasoning and finally a bibliography of all Zadeh's papers and other selected references.

Models of the human reasoning process are clearly very relevant to artificial intelligence (AI) studies. Broadly there are two types: psychological models of what people actually do; and formal models of what logicians and philosophers feel a rational individual would, or should, do. The main problem with the former is that it is extremely difficult to monitor thought processes - the behaviorist approach is perhaps reasonable with rats but a ridiculously inadequate source of data on man - the introspectionist approach is far more successful (e.g. in analysing human chess strategy) but the data obtained is still incomplete and may not reflect the actual thought processes involved.

Formal models of reasoning avoid these psychological problems and have the attractions of completeness and mathematical rigour, hopefully proving a normative model for human reasoning. However, despite tremendous technical advances in recent years that have greatly increased the scope of formal logic, particularly modal logic (Snyder 1971), the applications of formal logic to the imprecise situations of real life are very limited. Some 50 years ago, Bertrand Russell (1923) noted:

"All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence ..... logic takes us nearer to heaven than other studies".

The attempts of logicians to rectify this situation and broaden the scope of logic to cover various real-world problems has been surveyed recently by Haack (1974), and the role of modern developments in philosophical logic in AI has been excellently presented by McCarthy & Hayes (1969). These present notes are concerned with an area of massive recent development not covered by either of these references, that of 'fuzzy logic' and approximate reasoning initiated by Lofti Zadeh.

It is no coincidence that Zadeh's previous work had been concerned with successively improved refinement in the definitions of such terms as 'state' and 'adaptive' in systems engineering. It was dissatisfaction with the decreasing semantic content of such increasingly refined concepts that led to his (1972 "Fuzzy languages") remarks that:

"In general, complexity and precision bear an inverse relation to one another in the sense that, as the complexity of a problem increases, the possibility of analysing it in precise terms diminishes". "Thus, 'fuzzy thinking' may not be deplorable, after all, if it makes possible the solution of problems which are much too complex for precise analysis".

During recent years Zadeh (see bibliography) has developed in detail a model for approximate reasoning with vague data. Rather than regard human reasoning processes as themselves "approximating" to some more refined and exact logical process that could be carried out perfectly with mathematical precision, he has suggested that the essence and power of human reasoning is in its capability to grasp and use inexact concepts directly. He argues that attempts to model, or emulate, it by formal systems of increasing precision will lead to decreasing validity and relevance. Most human reasoning is essentially 'shallow' in nature and does not rely upon long chains of inference unsupported by intermediate data - it requires, rather than merely allows, redundancy of data and paths of reasoning - it accepts minor contradictions and contains their effects so that universal inferences may not be derived from their presence.

The insight that Zadeh's arguments give into the nature of human thought processes and, in particular, to their support of replication in the computer, are of major importance to a wide range of theoretical and applied disciplines - particularly to the role of formalism in the epistemology of science. The arguments have become associated with 'fuzzy sets theory' (Zadeh 1965) and this does indeed provide a mathematical foundation for the explication of approximate reasoning. However, it is important to note that Zadeh's analysis of human reasoning processes and his exposition of fuzzy sets theory are not one and the same - indeed they are quite distinct developments that must be separated, at least conceptually, if a full appreciation is to be had of either. As analogies one may conceive that fuzzy sets are to approximate reasoning what Lebesgue integration is to probability theory; what matrix algebra is to linear systems theory; or what lattice theory is to a propositional calculus.

The table below was compiled from an up-to-date bibliography on fuzzy systems containing some 300 references (Gaines & Kohout 1976) and demonstrates the growth of such work in recent years:

65	66	67	68	69	70	71	72	73	74	75
2	5	4	11	16	17	31	46	58	64	31 (at May 75)

<u>Table of papers on fuzzy systems by year of publication</u>

The relevance of this work to AI is indicated by its many recent applications to subject areas such as: pattern recognition (Siy & Chen 1974); taxonomic clustering (Bezdek 1974); analysis of line drawings (Chang 1971); robot planning (Goguen 1974, Kling 1973, LeFaivre 1974); medical diagnosis (Albin 1975); engineering design (Becker 1973); systems modelling (Fellinger 1974); process control (Mamdani & Assilian 1975); and management information systems (Wenstop 1975). The remainder of these notes are concerned with fuzzy sets theory, fuzzy reasoning, and its relations to developments in multivalued logics.

2 Fuzzy Sets Theory

Zadeh (1965) first developed the concept of a fuzzy set as an extension of that of a standard set in which the characteristic function, $A(x)$, of an element, x, of a set, A, was allowed to take not only the values 0 (not a member) and 1 (a member), but also to range anywhere between these values - the semantics were to be consistent with the natural order on the unit interval, e.g. that $A(x)=0.6$ denotes a greater 'degree of membership' than does $A(x)=0.4$. To correspond to the natural concepts of intersection and union it would be expected that the degrees of membership to fuzzy subsets, A and B, would not be decreased in their union nor increased in their intersection. Zadeh postulates that the resultant values are the lowest and the highest possible, respectively:

$$C = A \cup B \;\rightarrow\; C(x) = \max(A(x), B(x)) \tag{1}$$

$$C = A \cap B \;\rightarrow\; C(x) = \min(A(x), B(x)) \tag{2}$$

It remains to define the complement of a fuzzy set, and Zadeh postulates that:

$$B = \bar{A} \;\rightarrow\; B(x) = 1 - A(x) \tag{3}$$

All these definitions reduce to the standard case when the characteristic function is restricted to its usual binary values. However, it would be fallacious to assume that the extension outlined is the only one with this property. For example, whilst the definitions of union and intersection use naturally defined extreme values, that of negation may seem more arbitrary. Any antitone mapping of the unit interval into itself that inverted 0 & 1 would also be consistent with both the binary case and the semantics of the ordering of truth values. For example, an alternative negation, A, might be:

$$B = \hat{A} \;\rightarrow\; B(x) = \begin{cases} 1 \text{ if } A(x)=0 \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

This has the property that, in general, $\hat{\hat{A}} \neq A$, which is desirable in modelling the intuitionistic propositional calculus (IPC - section 3.1) where inferences from negative data are disallowed. Zadeh has discussed alternatives to definitions (1) through (3), as have many other authors - the particular 'max' and 'min' rules of fuzzy sets theory are not fundamental to its application to approximate reasoning. However, they are the most widely used bases for fuzzy logic in the literature.

Given these basic definitions it is possible to 'fuzzify' any domain of mathematical reasoning based on set theory by assuming that variables do not take specific values but instead have a separate 'degree of membership' to each possible value. That is, instead of having a sharp value, a variable is fuzzily restricted to a domain of values. The definition of the 'value' of a function of many variables may now be extended to fuzzified variables in a natural way - if in the standard case y=f(x1,x2,...), and u(x1) is the degree of membership of a particular 'value' to x1, then:

$$u(y) = \begin{cases} \underset{x}{\mathrm{MAX}} \; (\min(u(x1),u(x2),..)) \\ 0, \text{ if no } x \text{ exists} \end{cases} \qquad (5)$$

where x=(x1,x2,...) is any n-tuple such that y=f(x1,x2,...). That is: with each argument to the function is assoicated a degree of membership that is the lowest of those of each of its components; and with each value of the function is associated a degree of membership that is the highest of all the arguments resulting in that value.

In the same way that probability distributions are normalized to sum to unity and this is preserved under transformations, there is a natural normalization of the degrees of membership of a variable that is preserved under fuzzification. A fuzzy variable is said to be 'normalized' if at least one value has a degree of membership of unity. It is readily seen that a function, fuzzified as in equn.(5), of normalized variables is itself normalized (there must be at least one argument with degree of membership 1 and this will give a value with the same membership).

Zadeh's 1965 paper was presented as an extension of set theory and there has been a great deal of literature concerned with the technicalities of fuzzifying various mathematical structures, topologies, automata, etc., and determining what theorems remain proveable in the essentially generalized structure. Such work underpins the foundations of any future applications of fuzzy sets theory and is included in the bibliography. However it is the semantics of the theory applied to vague reasoning that there is most of relevance to AI.

3 Fuzzy and Other Multivalued Logics

Any logical structure may be fuzzified by considering propositions to have degrees of membership to truth values. If we take the conventional propositional calculus (PC) with truth values 0 & 1, then after fuzzification each statement, A, will be represented by a pair of values, (a1,a2), representing its degree of membership to falsity and truth, respectively. For example, fuzzifying the truth table for implication, $\supset$, in PC gives the following expression:

If C = A$\supset$B then (c1,c2)=(min(a2,b1),max(min(a1,b1),min(a1,b2),min(a2,b2))) (6)

Similar expressions may be derived for fuzzifying the truth tables of negation, $\sim$, disjunction, $\vee$, conjunction, $\wedge$, and equivalence, $\equiv$, but they are more meaningfully obtained by noting that fuzzification preserves the relations giving interdefinability of the connectives of PC. That is, if F is any false proposition (i.e. (f1,f2)=(1,0)), then we may write:

$$\sim A \quad \text{for} \quad A \supset F \tag{7}$$

$$A \vee B \quad \text{for} \quad \sim A \supset B \tag{8}$$

$$A \wedge B \quad \text{for} \quad \sim(\sim A \vee \sim B) \tag{9}$$

$$A \equiv B \quad \text{for} \quad (A \supset B) \wedge (B \supset A) \tag{10}$$

Equn.(7), for example, when substituted in (6) gives us:

$$\text{If } B = \sim A \text{ then } (b1,b2)=(a2,a1) \tag{11}$$

and, similarly, expressions may be derived for the other connectives.

If we assume the fuzzy variables are normalized then, as there is only one non-zero component, there is a 1-1 correspondence with the unit interval that simplifies the above expressions. Let:

$$a = (1-a1+a2)/2 \tag{12}$$

and so on for the other variables (this transformation can be inverted given that one of a1 and a2 must be 1). Then the equations for the logic operations become:

$$C = A \supset B \quad \rightarrow \quad c = \max(1-a, b) \tag{13}$$

$$B = \sim A \quad \rightarrow \quad b = 1-a \tag{14}$$

$$C = A \vee B \quad \rightarrow \quad c = \max(a,b) \tag{15}$$

$$C = A \wedge B \quad \rightarrow \quad c = \min(a,b) \tag{16}$$

$$C = A \equiv B \quad \rightarrow \quad c = \min(\max(1-a,b),\max(1-b,a)) \tag{17}$$

This set of simpler equations is what a number of authors have proposed as a 'fuzzy logic' (e.g. Lee 1972), probably not deriving them as a fuzzification of PC but instead as a direct set-theoretic interpretation of a logic based on equns.(1) through (3). The relation between equns.(3) & (14) is particularly interesting since fuzzification does not involve the complement operation, and hence the coincidence of definitions shows that Zadeh's definition of a fuzzy complement is a natural one for PC.

3.1 Relationship to VSS and Godel & Lukasiewicz Logics

Equns.(15) & (16) are valid for the disjunction and conjunction connectives of a wide range of multivalued logics (Rescher 1969), and it is interesting to examine the relationship of the system of equns.(13) through (17) to such logics. It turns out to be identical to the infinitely valued version of the 'variant-standard sequence' (VSS) investigated by Dienes (Rescher 1969 p.49) - i.e. VSS is exactly the fuzzification of PC. This logic has a defect in its semantics of inference, as noted by Lee (1972), that the assertion that A implies B (with value 1) does not necessitate that $b \geq a$, the truth value of B is greater than or equal to that of A. This seems a natural requirement in terms of our interpretation of the natural ordering of 'degrees of membership', and is implicitly assumed in most practical applications of fuzzy logic (e.g. Mamdani & Assilian 1975). It enables the assertion of a rule of the form, $A \supset B$, to be interpreted that B has a truth value in a particular instance at least equal to that of A, and hence greater than or equal to the maximum of any A1, A2, etc., that imply B.

If we require that the truth value of $A \supset B$ is 1 when $b \geq a$ then this may be used to define a variant of VSS based on some subset of definitions (9) through (17). To complete the definition of implication we must define the truth value of $A \supset B$ when $b < a$. Two possible definitions are:

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geq a, \quad c = b \text{ otherwise} \tag{18}$$

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geq a, \quad c = 1-a+b \text{ otherwise} \tag{19}$$

so that, when the implication is not absolute, the truth value is that of the implied proposition (equn. 18), or (equn. 19) it is a function of the difference between the two. If we couple each of these definitions with (7) for negation, (10) for equivalence, (15) for disjunction, and (16) for conjunction, we get two important systems: equn.(18) gives Godel's infinitely valued logic (Rescher 1969 p.45) which has a negation similar in form to the complement of equn.(4) and is closely related to the intuitionistic propositional calculus; equn.(19) gives Lukasiewicz's infinitely valued logic (Rescher 1969 p.37) which is the one used by Zadeh for statements involving truth and falsity in linguistic reasoning.

3.2 Relationship to Probability Logic

Other multivalued logics, some with connectives other than those of equns.(15) & (16) for disjunction and conjunction, may be derived from other subsets of these definitions - only the semantics of particular classes of situation can determine whether one system is more appropriate than another. The only other one to which I shall draw attention is that of 'probability logic' (PL). Rescher (1969) shows that the standard axioms for unconditional probability may be regarded as defining a logic which is closely related to the modal logic S5. PL is not truth-functional in that the truth value of a proposition is not uniquely defined by those of its components. Gaines (1975 "Stochastic ...") has shown that PL may be made truth functional in two distinct ways: (a) By assuming statistical independence between atomic components, a common assumption in systems engineering; (b) By assuming that of any two atomic components one must imply the other, giving a fuzzy logic satisfying equns.(15) & (16).

The equivalents of equns.(15) & (16) for a PL with assumed statistical independence are:

$$C = A \vee B \rightarrow c = a+b-ab \tag{20}$$

$$C = A \wedge B \rightarrow c = ab \tag{21}$$

Gaines (1975 "Stochastic ...") has re-analysed Mamdani & Assilian's (1975) data on experiments with a fuzzy logic linguistic controller using this form of connective and shown that it makes no difference to the results - the 'fuzzy reasoning' used is robust to changes in the form of 'fuzzy logic' on which it is based (more information on this controller is given in my notes on "Control Engineering & AI").

Giles (1975) has given a model for various forms of multivalued and probability logics as a dialogue between two participants, in essence a game-theoretic semantics. Gaines (1975 "Fuzzy ...") has given an alternative model that also encompasses both probability and fuzzy logics in terms of the responses of a population (e.g. people or neurons). Atomic propositions are modelled as questions to which each member of the population makes a binary, yes/no, response - the truth value of a proposition is the proportion of 'yes' responses, and that of compound propositions is determined by counting those who say yes to both A & B for terms of the form, $A \wedge B$, and so on for more complex compounds. This is essentially a set-theoretic model of a general logic and different specialized forms may be obtained by adding further constraints to it:

(i) If we assume that a 'yes' to A implies a 'no' to ~A then we obtain Rescher's probability logic;

(ii) If further we assume that the responses are independently distributed in the population we obtain what Gaines (1975) terms a 'stochastic logic' satisfying equns.(20) & (21);

(iii) If we assume instead that members of the population each evaluate any questions according to the same criteria but each require a different, individual 'weight of evidence' to reply 'yes', then we obtain a fuzzy logic satisfying equns.(15) & (16).

This last assumption, so different from the conventional one of statistical independence, also has its intuitive attractions. Reason (1969) has shown that the threshold applied by people in coming to a binary decision on an essentially analog psychophysical variable seems to be associated with personality factors and is characteristic of the individual. If so, human populations would tend to show a more fuzzy than stochastic logic in their overall decision making. Similarly the concept of uniformity in information-processing but varying thresholds of sensitivity is a reasonable one for populations of cells. Note that both the Giles and Gaines models give the pure forms of the logics as extreme cases - the most reasonable general assumption is a mixed form of probability/fuzzy logic.

Thus developments in 'fuzzy logic' and 'fuzzy reasoning' may be related both to classical multivalued logics and to classical probability theory. One suspects that there must be some underlying unifying structure that would form a better basis for modelling human reasoning than any of these particular logics alone - certainly no one of them has a claim at present to be the one correct logic for reasoning under uncertainty.

4 Linguistic Variables, Hedges and Fuzzy Reasoning

Whilst the technical aspects of both fuzzy sets and fuzzy logics have attracted much attention and are fascinating and significant in their own right, it is in their application to linguistics and approximate reasoning that their practical importance lies. It is not possible to do justice in these notes to Zadeh's prodigious output and detailed arguments, or to the application studies of recent years. The following extracts are intended to give a feel for the approach and motivate further reading of the literature in the bibliography. A good general introduction is given in Zadeh (1973 "Outline of ..."); Lakoff (1973) gives a linguistic introduction; Goguen (1974) is more technical but relates categories and concepts; Kling (1973) and Lefaivre (1974) have developed a version of planner capable of fuzzy reasoning; Albin (1975) and Wenstop (1975) have used models of fuzzy reasoning in studies of medical diagnosis and management information systems, respectively; and so on - the subject area now has a high semantic content in addition to its technical attractions.

Three illustrations will serve to define the type of problem with which Zadeh is concerned:

(1) Reasoning with 'linguistic variables' such as: "young", "middle-aged", "tall", or "rich", rather than precise quantities such as: "12 years old", "45 years old", "6 feet tall", or "having $1M";

(2) The effect of general linguistic 'hedges' upon such variables, e.g. "very small", "more or less tall", "fairly rich", etc., which allow a single concept to be extended in a standard way to cover many more situations;

(3) Syllogisms for approximate reasoning with linguistic variables, e.g. "John is very old - Charlie is about the same age as John - so Charlie is old".

Zadeh represents the meaning of a linguistic variable as a 'compatibility function' or 'fuzzy restriction' assigning a degree of membership to each possible value of the variable. For example, "older" might correspond to degrees of membership commencing at 0 for age 0 and increasing very slowly to 0.1 at age 25, to 0.3 at age 40, and then more rapidly to 0.9 at age 65, and then more slowly, asymptotic to unity. The numerical forms of such functions do not matter a great deal since it is the order relations that play most part in the later development. MacVicar-Whelan (1974) has performed some psychological experiments on their form and Lakoff (1973) reports similar experiments. Individuals <u>do</u> find it natural to assign such numerical values to the degree of compatibility of a particular value with a concept. Alternatively one may think of a population model in which the compatibility is measured in terms of the proportion of people who say, "yes, a young man may be 25 years old". Many models are possible and it is useful to have one in mind, but again much of the development of a theory of linguistic reasoning is independent of the exact model.

Zadeh has given a detailed account of how, given the compatibility function for a single linguistic variable such as "young", the compatibility functions may be calculated for the same variable subject to linguistic hedges, "not very young", "more or less young", etc. He shows how complex hedges may be decoded by a standard syntax into a number of elementary operations on compatibility functions and gives approximate forms of such operations as arithmetic operators. These definitions give a superficial appearance of mathematical precision to the effect of hedges. However Zadeh introduces the notion of 'linguistic approximations' in which compatibility functions resulting from a process of fuzzy reasoning are described by the closest reasonably simply-hedged linguistic variable. This process means that the reasoning itself is essentially approximate, 'shallow reasoning' that loses information at each stage, and may therefore consist only of comparatively short chains.

As noted in Section 3, the logic which Zadeh chooses to fuzzify for linguistic statements involving truth or falsity is one of Lukasiewicz's multivalued logics with connectives defined by equns.(10), (14), (15), (16) & (19). Hence the form of implication used is not that of PC which, when fuzzified, gives counter-intuitive results. This is not really surprising in that there are philosophical objections to the implication of PC as an explication of "if ... then" in ordinary language. Lukasiewicz originally developed his logic in 3-valued form to allow for the status of future contingent propositions, and later extended it to have the semantics of a "modal" logic.

5 Conclusions and Background References

The classical formal logics such as PC may be seen as expressing idealized, precise 'reasoning', such as that of the digital computer at a hardware logic level. AI research may be seen as an attempt to replicate the less formal linguistic reasoning with vague and imprecise rules and data, actually adopted by human beings. This is not in itself a new problem - in "A System of Logic" published in 1843, John Stuart Mill commences with the remark:

"Since reasoning, or inference, the principal subject of logic, is an operation which usually takes place by means of words, and in complicated cases can take place in no other way: those who have not a thorough insight into both the signification and purpose of words, will be under chances, amounting almost to certainty, of reasoning or inferring incorrectly".

(The rest of this fascinating book is also worth reading - there are few problems of knowledge and its acquisition about which Mill has no perceptive comments - it is a pity that he did not have access to a PDP10 !). He criticizes the weakness of formal logic in explicating linguistic reasoning but, like most work since, attempts to bridge the gap linguistically rather than develop a new basis in logic. Zadeh's use of fuzzy logic to model natural linguistic reasoning may be viewed as a more direct response to Mill's argument above some 130 years later.

Apart from papers so far reference, I would recommend anyone interested in this area to have at hand: Rescher's (1969) book on multi-valued logics; Snyder's (1971) book on modal logics as an introduction and Hughes & Creswell (1968) as a reference; Creswell's (1973) book on logic and language as an alternative modern approach to linguistic semantics; Fillmore & Langendoen (1971) and Hockney et al (1975) as basic references on the same; and Krantz et al (1971) for alternative approaches to partially qualitative description. McCarthy & Hayes (1969) is well worth reading first, followed by Lakoff (1973) and any (or all !) the Zadeh references.

Having quoted so many eminent authorities I may as well end with a quote from the most venerable of them all - Lazarus Long, the senior, was over 1,000 years old when he wrote:

"The difference between science and the fuzzy subjects is that science requires reasoning, whilst those other subjects merely require scholarship" (R. Heinlein, "Time Enough for Love", NEL 1974).

Hopefully the direction of the work described in these notes indicates that the scholarship of multivalued logic has a part to play in the science of reasoning about (rather fuzzy) human linguistic behaviour!

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