General fuzzy logics

B.R. GAINES
University of Essex, Colchester, UK

1. What is a fuzzy logic?

One aspect of the tremendous growth of interest in fuzzy systems and fuzzy reasoning [1, 2] is the development of *fuzzy logics* [3-12] and their relationships [11, 12] to standard multivalued logics (MVLs) [13]. I have noted previously [12] the varying usage of the term 'fuzzy logic' which may be classified in three broad categories:

(a) A basis for reasoning with vague statements. This general definition is consistent with the colloquial meaning of 'fuzzy', and also with its use in a technical sense different from, but related to [14, 15] that of Zadeh's 'fuzzy sets theory'.

(b) A basis for reasoning with vague statements using fuzzy set theory for the fuzzification of logical structures. This restricted form of (*a*) seems closest to Zadeh's own usage of the term 'fuzzy logic' [3] and his general use of the term 'fuzzy' as a qualifier.

(c) A multivalued logic in which truth values are in the interval [0, 1] and the valuation of a disjunction is the maximum of the disjuncts, whilst that of a conjunction is the minimum of the conjuncts. This narrow definition encompasses the population stereotype of a 'fuzzy logic' [4-9]. It is interesting that most infinitely-valued MVLs [13] have min/max connectives for conjunction/disjunction and hence are 'fuzzy logics' on this definition.

One may note some scope for confusion between these three definitions because:

(i) There is some disagreement about the basic conjunction/disjunction connectives of fuzzy logic [8, 9], i.e. neither (*b*) and (*c*) are necessarily accepted;

(ii) even if the min/max connectives of fuzzy logic are accepted in the sense of (*c*), the further connectives of equivalence/implication/negation are left undefined [7], or defined [3, 4, 5] or assumed [6], in different forms;

(iii) fuzzification of the classical propositional calculus (PC) in the sense of (*b*) gives a fuzzy logic in the sense of (*c*) but one with an inappropriate form of implication [10, 12, 6] in which the assertion of $A \supset B$ does not necessitate the degree of membership of B being greater than that of A;

(iv) for his models of truth values in human verbal reasoning Zadeh [3] fuzzifies in terms of (*b*) Lukasiewicz infinitely-valued logic [13] (here abbreviated to L_1), a logic which is itself a 'fuzzy logic' in the sense of (*c*).

There is no sense in which one would wish to legislate in favor of one of the three defintions—all are appropriate in their proper contexts. However, one may also note that at the level of definition (*a*), there has been much previous work on practical reasoning with vague data under the auspices of *probability theory*. I have previously suggested [10-12] both formal and semantic links between fuzzy and probability logics that provide foundation for a general *logic of uncertainty* encompassing these logics, many classical MVLs, and some modal logics.

The following section briefly presents such a logic of uncertainty, a general fuzzy logic, or a *basic probability logic* as I have previously termed it. Section 3 elucidates the effects of imposing the constraint of definition (*c*) upon the logic, and Section 4 gives a semantics for the general logic which illuminates the formal relationship between fuzzy and probability logics.

2. A general logic of uncertainty

To integrate together the various logics of uncertainty, including fuzzy and probability logics, it is essential to make a clear initial distinction between the (algebraic) structure of *propositions* and the ascription to these propositions to *truth values* (making them into *statements*). Indeed, it will be shown that fuzzy logics may be uniquely distinguished from other logics of uncertainty by the irrelevance, only in their case, of propositional structures when assigning truth values to compound statements.

The natural and conventional algebraic semantics for a propositional calculus is a lattice structure: $L(X, F, T, \vee, \wedge)$, generated by a set of elements, X, under two (idempotent, commutative) monoid operations, $\vee$ (disjunction), $\wedge$ (conjunction), with maximum element, T, and minimum element, F, i.e. L satisfies:

$$\forall\ x \in L \qquad x \vee x = x \wedge x = x \qquad (1)$$

$$\forall\ x, y \in L \qquad x \vee y = y \vee x,\ x \wedge y = y \wedge x \qquad (2)$$

$$\forall\ x,y,z \in L \qquad x \vee (y \vee z) = (x \vee y) \vee z,\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad (3)$$

$$\forall\ x,y \in L \qquad x \wedge (x \vee y) = x,\ x \wedge (x \wedge y) = x \qquad (4)$$

$$\forall\ x \in L \qquad x \vee F = x,\ x \wedge F = F,\ x \vee T = T,\ x \wedge T = X \qquad (5)$$

the idempotent, commutative, associative, and adsorption postulates, together with a definition of the minimal and maximal elements [16]. The usual order relation is also defined:

$$\forall\ x,y \in L \qquad x \leqslant y \Leftrightarrow \exists z \in L : y = x \vee z \quad (6)$$

It is possible to make a case for weaker structures (e.g. dropping idempotency) but, for present purposes, this will be taken as an unreasonably wide generalization of our concepts of conjunction and disjunction. Now suppose that every element of L is assigned a 'truth-value' (for different applications different terminologies might be more appropriate, 'probability', 'degree of knowledge', 'level of belief', etc.) in the closed interval, [0, 1] by a continuous, order-preserving function $p: L \rightarrow [0, 1]$, with the constraints:

$$p(F) = 0,\ p(T) = 1 \qquad (7)$$

$$\forall x, y \in L,\ p(x \vee y) + p(x \wedge y) = p(x) + p(y) \quad (8)$$

i.e. p is a continuous, order-preserving, *valuation* [16] on L. Note that, for p to exist, the lattice must be modular, and that we have:

$$p(x \wedge y) \leqslant \min(p(x), p(y)) \leqslant \max(p(x), p(y)) \leqslant p(x \vee y) \qquad (9)$$

To complete the definition of an MVL one needs values for equivalence, $\equiv$, implication, $\supset$, and negation, $\bar{\ }$, of propositions. These may be defined naturally by noting that the equivalence relation on L, $\equiv$, defined by:

$$x \equiv y \Leftrightarrow p(x \vee y) = p(x \wedge y) \qquad (10)$$

is a congruence on L, and that the deviation from equality in (10) defines a *metric* on L under this congruence [16]. Hence it is reasonable to define:

$$p(x \equiv y) = 1 - p(x \vee y) + p(x \wedge y) \qquad (11)$$

as a measure of the degree of equivalence between x and y.

Implication may be defined in terms of equivalence by noting that, according to the usual lattice semantics, we require:

$$\forall\ x, y \in L,\ x \leqslant y \Leftrightarrow p(x \supset y) = 1 \qquad (12)$$

but that for x, y satisfying this we have: $x \vee y = y$, $x \wedge y = x$. Thus it is natural to define implication as the degree to which these equivalences hold:

$$p(x \supset y) = p((x \vee y) \equiv y) = 1 - p(x \vee y) + p(y)$$
$$= 1 - p(x) + p(x \vee y) = p((x \vee y) \equiv x) \quad (13)$$

Negation may also be defined in terms of equivalence in the usual way:

$$p(\bar{}x) = p(x \equiv F) = p(x \supset F) = 1 - p(x) \quad (14)$$

I have previously called the logic thus defined a *basic probability logic* (BPL) because it satisfies the usual definitions of a probability logic (PL) [13], or probability over a language [17], except for the law of the excluded middle (LEM). In a BPL the law of contradiction and LEM are not necessarily theses, but if one is then so is the other, i.e. we have (from (8) and (14)):

$$p(x \vee \bar{}x) + p(x \wedge \bar{}x) = 1 \quad (15)$$

so that: $p(x \vee \bar{}x) = 1 \Leftrightarrow p(x \wedge \bar{}x) = 0$ (16)

The form of implication in a BPL has the property, from (13), that:

$$p(y) = p(x \vee y) - 1 + p(x \supset y) \geqslant p(x) - (1 - p(x \supset y)) \quad (17)$$

which enables a lower bound to be placed on the truth value of y given those of x and $x \supset y$. Thus it satisfies the normal requirement [5, 6] that the assertion of $x \supset y$ may be used to infer that $p(y) \geqslant p(x)$, and hence also that $p(y) \geqslant \max(p(x_i))$ where y is constrained by 'rules' of the form $x_i \supset y$, a common pattern of inference in applications of fuzzy reasoning [18].

3. Truth functionality in BPLs

A BPL is not truth-functional (TF) in that the truth-values of the binary connectives, conjunction/disjunction/equivalence/implication, are not uniquely defined in terms of those of the two connected propositions. Note, that there is only one degree of freedom in that fixing the value of any of these connectives fixes that of all of them. There have been many debates in philosophical logic about truth-functionality but, particularly in the context of a logic of uncertainty, there seems to be no fundamental basis on which to demand truth-functionality, quite the contrary. However, a great deal may be learned, and many interesting logics derived, by considering various ways in which a BPL may be made TF:

(i) *A BPL with binary truth values is precisely PC.* If the truth values in a BPL are restricted to the end points of [0, 1] then it becomes TF with truth tables for all the connectives precisely as in the classical propositional calculus. Thus a BPL, and all its derivatives, are proper extensions of PC.

(ii) *A strongly truth-functional BPL is the 'fuzzy logic' L_1.* I have called a logic *strongly TF* [12] if there is a single equational definition for each of its binary connectives giving their truth values in terms of those of the connected propositions *regardless of the propositional structures* (e.g. having generating elements in common). The arguments of Bellman and Giertz [7] may be used to show that a strongly TF BPL is necessarily a 'fuzzy logic' (in sense (*c*) of Section 1), with the min/max bounds of (*a*) being attained.

Thus a fuzzy logic is a limiting case of a BPL and has the important computational property of allowing one to drop, without loss of information, the propositional structure of a statement and retain only its truth value. This property also holds for PC and hence it is a natural assumption in an MVL. However, its very strength is also its weakness because it implies that for any two statements, x and y, either $p(x \supset y) = 1$, or $p(y \supset x) = 1$, i.e. the lattice, L, reduces to a *chain* of propositions mutually connected by implication. This very strong requirement is unlikely in general, although there are situations in which it becomes a reasonable hypothesis (see Section 4).

(iii) *A BPL with LEM is Rescher's probability logic.* Adding the law of the excluded middle (and hence also the law of contradiction) to a BPL gives a classical probability logic [13]. The PL is still not truth-functional but the demand for LEM makes it impossible for it to be strongly TF. In many practical cases the semantics require LEM and one is led to consider weaker forms of truth-functionality in which the computation of truth values of the binary connectives requires both the values *and* the propositional structure of the connected propositions:

(iv) *A BPL with LEM may be made truth-functional by an equational definition of the connectives for conjunction or disjunction which is commu-*

tative, associative and such that LEM or the law of contradiction applies, and is applied to pairs of propositions with no common elements. Notice that it is now essential to retain propositional structures in order to use the lattice laws and definition of a valuation to evaluate connectives in terms of pairs of components that have no common element. However, the resultant logics can now be made consistent with a far wider range of semantics, essentially now applying to the generating set of basic propositions, X, e.g.:

(v) *Assuming the truth-value of conjunction in X is the minimum of the truth-values of the conjuncts gives a 'fuzzy logic' in which the truth-value of a disjunction in X is the maximum of those of the disjuncts.* Thus min/max connectives are not incompatible with LEM. They imply that the generating propositions form a chain but that their negations form a separate chain, thus enabling LEM to apply.

(vi) *Assuming the truth-value of conjunction in X is the product of the truth-values of the conjuncts gives a logic of statistical independence.* Thus is the common assumption made by system analysts and engineers in order to make an uncertain system truth-functional.

(vii) *Assuming the truth-value of disjunction in X is the sum of the truth-values of the disjuncts gives a logic of mutual exclusion.* This is another common assumption, justified when the generating elements represent events, such as being in different states, that cannot occur together.

Thus a BPL may be made TF in a variety of ways of which only a few 'pure' examples have been given. In practice different propositions in the generating set may be connected in different ways and it makes more sense to reverse the definitions and consider *which* propositions are mutually exclusive, statistically independent, fuzzily related, etc., i.e. to classify the structure of the particular propositional calculus encountered in each practical situation. This concept will be further clarified in the semantic examples of the next section.

4. Semantics for BPLs

The close relationship established in Sections 2 and 3 between fuzzy and probability logics may evoke suspicion since we know that in many applications a fuzzy 'degree of membership' is most definitely *not* associated with a (physical) probability. For example, the man who has a degree of membership (dm) .5 to the class of short men or the woman who has dm .7 to the class of beautiful woman do not necessarily represent samples from a population (they may be the *only* people). Neither is there a sampling distribution in our own measurements that makes the man appear smaller than 5 feet on 50% of the occasions we measure his height—indeed, for beauty we possess no physical measuring rule!

Thus the formal relationship of probability theory to fuzzy logic may appear as spurious. However, this would be to adopt too narrow a view of probability theory, taking a strict 'physical frequentist' interpretation when there are well-established alternative semantics for probability in terms of 'subjective probability' [19-21], 'belief' [22, 23], etc., that are closely related to both classical and computational-complexity-based probability [24, 25]. There is a common semantics for all these interpretations of a BPL in terms of the binary responses of a population that shows that the formal relationships established are more than a mathematical artifice.

Consider a population each member of which can 'respond' to certain questions with a binary, yes or no, reply. The forms of question will involve evaluating a statement which belongs to the generating set, X, of a lattice, L, as defined in Section 2. For example, 'is this statement, $x \in X$, true or false, or reasonable or unreasonable, or generally believed, etc.'. The valuation of x is defined to be the proportion of the population replying yes to the question. A compound statement in L is given a valuation in terms of the proportion of the population who say yes to each of x and y for terms of the form, $x \wedge y$, or who say yes to either x or y for terms of the form, $x \vee y$, and similarly for more complex combinations of conjunction and disjunction.

This is essentially a set-theoretic model for L as a lattice of sub-sets of the population and (1) through (10) are clearly valid. A distance measure and hence valuations of logical equivalence, implication, and negation, may be defined as in (11) through (14). Thus, for any given population whose members are able to give one of two responses to a question about each element of X, there is a simple and well-defined procedure for ascertaining the valuation of any arbitrary statement in L, involving, conjunction, disjunction, equivalence, implication and negation, which is consistent with (1) through (14). Thus such a population is a model for a (distributive)

BPL.

Returning to the initial examples, one may now suppose that 50% of some test 'population' agree that the man is 'short' whilst 70% agree that the woman is 'beautiful'. If the 'population' was one of measuring instruments then the results express the effects of physical 'noise'. If the 'population' is one of people then this is a social acceptance model of linguistic usage, a reasonable model of Zadeh's 'fuzzy reasoning' based on human linguistic behavior. If the 'population' is one of 'neurons' then this is a model of individual decision-making. If we allow metalinguistic statements about the value of $p(x)$ to be made by members of the population then this is a model of 'subjective probability' or 'belief', and so on.

Consider now the additional constraints that must be placed upon the behavior of the population to correspond to result (iii) and (ii) of Section 3. Rescher's probability logic is obtained if someone who says 'yes' to x must say 'no' to $\sim x$. Lukasiewicz's L_1 is obtained if members of the population each evaluated the evidence for x in the same way but applied differing thresholds of acceptance. The member with the lowest threshold would then always respond with 'yes' when any other member did, and so on up the scale of thresholds, thus giving the required relation of implication between propositions. This model, although unusual, has its intuitive attractions, e.g. Reason [26] has shown that the threshold applied by human being in coming to a binary decision on an essentially analog variable seems to be associated with personality factors and a trait of the individual. If so, human populations would tend to show more a fuzzy, than a stochastic, logic in their decision making.

Similarly populations showing the 'statistical independence' of (vi) or the 'mutual exclusion' of (vii) may be defined. However, rather than argue the case for one type of population or another, one can now envisage that logics based on a real population will be of mixed type and hence it is more interesting to insert the concepts and talk in terms of a 'fuzzy', 'probabilistic', 'independence', 'exclusion', etc. *relationship* between propositions. Such relationships are mainly of interest to the extent that they are *necessary* and hence would appear as modal operators over a family of possible p's or L. In terms of our example so far it seems unlikely that anyone would argue for the logical necessity of semantics that make it possible to compute the truth value of 'the man is short and the woman is beautiful' on a truth-functional basis. However there would be reasonable grounds for the fuzzy TF of 'the man is short and he is not heavy'. A BPL and associated population semantics can encompass all these possible variants on a general logic of uncertainty.

5. Conclusions

In conclusion, there is no one logical system that stands out clearly as *the* logic of vagueness, uncertainty or fuzzy reasoning. It has been shown that a non-functional basic probability logic provides a formal foundation for a general logic of uncertainty encompassing both fuzzy and probability logics. Classical probability logic is obtained by adding the law of the excluded middle. The fuzzy logic L_1 is obtained by demanding strong truth-functionality. Since both LEM and TF have been subject to philosophical debate over many centuries one is unlikely to choose between them on general grounds!

However, it has also been shown that LEM is consisten with weaker forms of TF leading to partially 'fuzzy logics' (with min/max connectives between primitive propositions), logics of statistical independence, mutual exclusion, etc. The 'population model' semantics given show that these formal relationships between various logics of uncertainty carry over to an intuitively satisfying model of uncertain reasoning. The model also clarifies the distinction between fuzzy 'degree of membership' and conventional 'probability', showing it to be one of detailed semantic interpretation rather than one of logic or basic semantics.

Finally, the characterization of fuzzy logic as bieng *strongly TF* highlights its unique computational advantages. They are not so much ones of numerical simplicity (of min/max operations) as ones that stem from the *memory-reduction* possible through the irrelevance of propositional structures when computing truth-values. In any other logic of uncertainty it is necessary to know the actual structure (in terms of primitive propositions in the generating set) of propositions, x and y, when computing $x \wedge y$, $x \supset y$, etc., *whereas in fuzzy logic it is necessary only to remember the truth values,* $p(x)$, $p(y)$. Thus, regardless of whether fuzzy logic is *correct* in a given application, it is *easy to apply,* requiring a substantially lower memory load in generating or following complex arguments. This is not only practically important but may also be very relevant to the role of fuzzy logic in modelling human reasoning where memory resources are notoriously weak. It may, for example, explain Edwards [27] results that humans, whilst being good probability estimators, do not use the information efficiently in Bayesian computations (requiring a logic of statistical independence). A fuzzy logic is easier to apply, but is equivalent in this context to throwing away information.

Thus, the wider framework for logics of uncertainty described in this chapter establishes a close link between fuzzy logic and probability theory, to the mutual advantage of both fields. It also makes clear the unique computational advantage of fuzzy logic derived from its strong truth-functionality.

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References

1. ZIMMERMANN, H.J., 'Bibliography: Theory and application of fuzzy sets', Lehrstuhl fur Unternehmensforschung, RWTH Aachen, West Germany (October 1975).
2. GAINES, B.R. and KOHOUT, L.J., 'The fuzzy decade: a bibliography of fuzzy systems and closely related topics', *International Journal of Man-Machine Studies* 9, pp.1-68 (1977).
3. ZADEH, L.A., 'Fuzzy logic and approximate reasoning', *Synthese* 30, pp.407-428 (1975).
4. GOGUEN .J.A., 'The logic of inexact concepts', *Synthese* 19, pp.325-373 (1969).
5. LEE, R.C.T. and CHANG, C.L., 'Some properties of fuzzy logic', *Information and Control* 19, pp.417-431 (1971).
6. LEE, R.C.T., 'Fuzzy logic and the resolution principle', *JACM* 19, pp.109-119 (1972).
7. BELLMAN, R. and GIERTZ, M., 'On the analytic formalism of the theory of fuzzy sets', *Information Sciences* 5, pp.149-156 (1973).
8. KOCZY, L.T. and HAJNAL, M., 'A new fuzzy calculus and its application as a pattern recognition technique', Proc. 3rd Int. Congr. WOGSC, Bucharest (1975).
9. SANFORD, D.H., 'Borderline logic', *American Philosophical Quarterly* 12, pp.29-39 (1975).
10. GAINES, B.R., 'Multivalued logics and fuzzy reasoning', Lecture Notes of AISB Summer School, Cambridge, UK, pp.100-112 (1975).
11. GAINES, B.R., 'Stochastic and fuzzy logics', *Electronics Letters* 11, pp.188-189 (1975).
12. GAINES, B.R., 'Fuzzy reasoning and the logics of uncertainty', Proc. 6th Int. Symp. Multiple-Valued Logic, IEEE 76CH1111-4C, pp.179-188 (1976).
13. RESCHER, N., *Many-valued logic*, McGraw-Hill, New York (1969).
14. GOGUEN, J.A., 'Concept representation in natural and artificial languages: axiom, extensions and applications for fuzzy sets', *International Journal of Man-Machine Studies* 6, pp.513-561 (1974)
15. ARBIB, M.A. and MANES, E.G., 'A category-theoretic approach to systems in a fuzzy world', *Synthese* 30, pp.381-406 (1975).
16. BIRKHOFF, G., *Lattice Theory*, Rhode Island: American Mathematical Society (1948).
17. FENSTAD, J.E., 'Representations of probabilities defined on first order languages', in J.N. Crossley, *Sets, Models and Recursion Theory*, Amsterdam: North-Holland, pp.156-172 (1967).
18. MAMDANI, E.H. and ASSILIAN, S., 'An experiment in linguistic synthesis with a fuzzy logic controller', *International Journal of Man-Machine Studies* 7, pp.1-13 (1975).
19. SAVAGE, L.J., 'Elicitation of personal probabilities and expectations', *J. American Statistical Association* 66, pp.783-801 (1971).
20. FINETTI, B. de, *Probability, Induction and Statistics*, London: John Wiley (1972).
21. MENGES, G., 'On subjective probability and related problems', *Theory and Decision* 1, pp.40-60 (1970).
22. GROFMAN, B. and HYMAN, G., 'Probability and logic in belief system', *Theory and Decision* 4, pp.179-195 (1973).
23. MITROFF, I.I., 'On the problem of representation in Lockean and dialectical belief systems: a systems approach to policy analysis', *International Journal of General Systems* 4, pp.75-85 (1975).
24. FINE, T.L., *Theories of Probability*, NY: Academic Press (1973).
25. GAINES, B.R., 'Behavior/structure transformations under uncertainty', *International Journal of Man-Machine Studies* 8, pp.337-365 (1976).
26. REASON, J.T., 'Motion sickness—some theoretical considerations', *International Journal of Man-Machine Studies*, 1, pp.21-38 (1969).
27. EDWARDS, W., PHILLIPS, L.D., HAYES, W.L., and GOODMAN, B.C., 'Probabilistic information processing systems: design and evaluation', *IEEE transactions on Systems Science and Cybernetics*, Vol.SSC-4, pp.248-265 (1968).