FUZZY REASONING AND THE LOGICS OF UNCERTAINTY

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This paper is concerned with the foundations of fuzzy reasoning and its relationships with other logics of uncertainty. The definitions of fuzzy logics are first examined and the role of fuzzification discussed. It is shown that fuzzification of PC gives a known multivalued logic but with inappropriate semantics of implication and various alternative forms of implication are discussed. In the main section the discussion is broadened to other logics of uncertainty and it is argued that there are close links, both formal and semantic, between fuzzy logic and probability logics. A basic multivalued logic is developed in terms of a truth function over a lattice of propositions that encompasses a wide range of logics of uncertainty. Various degrees of truth functionality are then defined and used to derive specific logics including probability logic and Lukasiewicz infinitely valued logic. Quantification and modal operators over the basic logic are introduced. Finally, a semantics for the basic logic is introduced in terms of a population (of events, or people, or neurons) and the semantic significance of the constraints giving rise to different logics is discussed.

1 Introduction

Since Zadeh's[1] introduction of fuzzy set theory some ten years ago there has been a steady growth of interest in this approach to an imprecise system theory[2]; a recent bibliography[3] lists over 600 references of which many are application studies of 1974-75 vintage. Whilst this work is clearly related to past studies of multivalued logics and may be viewed as a logical continuation of many of the studies of the Polish logic school[4] of 1920-39, and the fuzzy models of linguistic hedges have obvious antecedents in modal studies of vague quantifiers[5], tense logics[6] and language in general[7], in fact this work did not so arise. It has its roots in the need in systems engineering[8-11] to be able to cope formally with concepts such as stability, feedback, adaption, etc., that are widely applicable and conceptually powerful yet have an inherent vagueness that must be captured in any formalism that does not destroy their utility through excessive strictness in their definition. Sanford[12] wryly remarks on how a branch of philosophical logic has become the province of engineers. However, it is the very need of systems engineering[13], both in its theoretical constructs and in the computer implementation of algorithms, that offers a new and interesting application area for this branch of philosophical logic.

Whilst, in the long run, it is such applications of fuzzy reasoning that are of greatest interest, there are impediments to these in the short term that stem both from the novelty of the approach and from some lack of clarity in the foundations of fuzzy logic[14]. For example: the implication function in fuzzy reasoning has been left undefined[15] or defined[16-18], or assumed[19], in different forms; there are even differences of opinion about the more basic logical connectives[12,20]; fuzzy reasoning is being applied to situations where probability theory would conventionally be applied and yet there is no formal theory of the relationship between the two approaches[21]; and so on. No one would expect a new and rapidly developing subject area not to show inconsistencies, and it would be both unreasonable and unwise to attempt to remove them by the imposition of rigid definitions at too early a date. Nevertheless it seems appropriate to explore at this time the sources of inconsistency and the links with more formal developments and alternative approaches with a view to strengthening and broadening the foundations of what appears to be a development of major importance for systems engineering.

This paper is concerned with these problems, the foundations of fuzzy reasoning and its relationships with other logics of uncertainty. In section 2 the definition of a fuzzy logic is first examined and the role of fuzzification discussed. It is shown that the fuzzification of PC gives a known multivalued logic but with inappropriate semantics of implication, and various alternative forms of implication are discussed. Section 3 broadens the discussion to encompass other logics of uncertainty and argues that there is a close relationship between fuzzy and probability logics. A basic multivalued logic is then developed in terms of a truth function over a lattice of proposition that encompasses a wide range of logics of uncertainty. Various degrees of truth-functionality are then defined and used to derive specific logics including probability logic and Lukasiewicz infinitely valued logic. Quantification and modal operators over the basic logic are also discussed. Finally, in section 4 a semantics for the basic logic in terms of a population (of events, or people, or neurons, etc.) is outlined and the semantic significance of the constraints discussed.

2 Fuzzy Logics

2.1 What is (or are) Fuzzy Logic(s) and Fuzzy Reasoning?

Zadeh[1] defined fuzzy sets and the operation of union, intersection and complementation on them. He did not then, and so far as I am aware, has not later, defined a "fuzzy logic" although his later papers on fuzzy reasoning[16,22] include the concept of fuzzified truth values in the infinitely valued Lukasiewicz logic (here abbreviated to L_1). "Fuzzification" is a well-defined technique for extending a precise mathematical function in many variables to apply to fuzzily restricted, rather than precisely defined, values of the variables. Zadeh fuzzifies a variety of mathematical structures to provide more appropriate models of their use in the process of reasoning with vague data carried out, linguistically, by human beings. L_1 is only one example of a structure to be fuzzified and Zadeh does not treat the model of linguistic terms denoting truth such as "very true" any differently from terms denoting other, less abstract, concepts such as "very high" or "very hot". Indeed in two distinct journal publications[11,23] he emphasizes "much of the logic behind human reasoning is not the traditional two-valued or even multivalued logic, but a logic with fuzzy truths, fuzzy connectives and fuzzy roles of inference".

There has been a rich development of Zadeh's approach to fuzzy reasoning with vague concepts and many important applications[2,3,14,24,25]. However, work has also been carried out on "fuzzy logics" which are not those of Zadeh's "fuzzy reasoning" but are instead multivalued logics (MVLs) based on a natural analogy with fuzzy set theory in that the truth value of a conjunction is taken as the minimum of those of the conjuncts, and that of disjunction as the maximum of those of the disjuncts. Such min/max connectives are common to virtually all of the truth-functional MVLs studied in the literature[26] and since, as noted previously, the fuzzy implication connective is variously defined, or left undefined, virtually any such MVL could be called a "fuzzy logic".

Thus, there are at least three potential definitions of "fuzzy logic":

(a) "A basis for reasoning with vague statements" - this very general definition is consistent with the colloquial use of the term "fuzzy" and its use in technical senses different from that of Zadeh[27], or in more general formulations[28,29]. "Fuzzy" becomes a modern term replacing previous usage in the literature on measurement and philosophical logic of terms such as "inexact"[30,31] and "vague"[32,33]. It has the advantage of not being negative in its connotations, which itself is significant since one suspects that the study of uncertain reasoning would have developed more extensively earlier without these connotations[34,35].

(b) "A basis for reasoning with vague statements using fuzzy set theory for the fuzzification of logical structures" - this more restricted form of definition (a) comes closest to being the intensive form of that given extensively by the contents of Zadeh's own papers.

(c) "A multivalued logic in which truth values are in the interval [0,1], and the valuation of a disjunction is the maximum of those of the disjuncts, and that of a conjunction is the minimum of those of the conjuncts" - this is close to being the population stereotype of a fuzzy logic currently. It may be generalized to truth values in a lattice[18,36,37] or specialized to include the truth value of negation as being one minus the truth value of the statement negated. However, all variants of this definition require statements to have truth values in an ordered set, and define the logical connectives in terms of the order relation.

There is scope for confusion in the level of discourse between definitions (b) and (c) because Zadeh fuzzifies in terms of (b) logics which are "fuzzy" in terms of (c). It would be better to use the term MVL, or "truth-functional MVL over an order relation" for the so called "fuzzy logics" of definition (c), and retain the adjective "fuzzy" as indicating any mathematical structure (including a MVL) that has been fuzzified. However, the state of the literature already makes such terminological exactitude a difficult objective, and one has to accept the distinct uses of the term "fuzzy".

In the next section I shall consider to what extent "fuzzy logics" in terms of definition (c) can be taken as fuzzifications of other logics in terms of definition (b). In the following section I shall consider the desirable properties of "fuzzy logics" in terms of definition (c) if they are to be the fuzzified "base logics" for fuzzy reasoning in terms of definitions (b) and (a).

2.2 The Multivalued Logic VSS as the Fuzzification of PC

Consider standard fuzzy set theory[1] in which for any possible element, x, of a set A, the usual binary characteristic function A(x) is generalized to take any "degree of membership" in the closed unit interval, [0,1], rather than just its end points, and the degrees of membership for union and intersection of sets are given by:

$$C = A \cup B \rightarrow C(x) = \max(A(x), B(x)) \quad (1)$$

$$C = A \cap B \rightarrow C(x) = \min(A(x), B(x)) \quad (2)$$

Given these basic definitions it is possible to "fuzzify" any domain of mathematical reasoning based on set theory by assuming that variables do not take specific values but instead have a separate "degree of membership" to each possible value. That is, instead of having a sharp value, a variable is fuzzily restricted to a domain of values. The definition of the "value" of a function of many variables may now be extended to fuzzified variables in a natural way - if in the standard case y = f(x1,x2,...), and u(x1) is the degree of membership of a particular "value" to x1, then:

$$u(y) = \begin{cases} \max\limits_{x} (\min(u(x1), u(x2), \ldots)) \\ 0, \text{ if no x exists} \end{cases} \quad (3)$$

where x = (x1,x2,...) is any n-tuple such that y = f(x1,x2,...). That is: with each argument to the function is associated a degree of membership that is the lowest of those of each of its components; and with each value of the function is associated a degree of membership that is the highest of all the arguments resulting in that value. Note that fuzzification does not involve the complementation of a fuzzy set which is advantageous since this operation is not as well-defined as the union and intersection[15].

In the same way that probability distributions are normalized to sum to unity and this is preserved under transformations, there is a natural normalization of the degrees of membership of a variable that is preserved under fuzzification. A fuzzy variable is said to be "normalized" if at least one value has a degree of membership of unity. It is readily seen that a function, fuzzified as in eqn.(3), of normalized variables is itself normalized (there must be at least one argument with degree of membership 1 and this will give a value with the same membership).

Any logical structure may be fuzzified by considering propositions to have degrees of membership to truth values. If we take the conventional propositional calculus (PC) with truth values 0 and 1, then after fuzzification each statement, A, will be represented by a pair of values, (a1,a2), representing its degree of membership to falsity and truth, respectively. For example, fuzzifying the truth table for implication, ⊃, in PC gives the following expression:

If C = A ⊃ B

$$\text{then } (c1,c2) = (\min(a2,b1), \max(\min(a1,b1), \min(a1,b2), \min(a2,b2))) \quad (4)$$

Similar expressions may be derived for fuzzifying the truth tables of negation, ~, disjunction, ∨, conjunction, ∧, and equivalence, ≡, but they are more meaningfully obtained by noting that fuzzification preserves the relations giving interdefinability of the connectives of PC. That is, if F is any false proposition (i.e. (f1,f2) = (1,0)), then we may write:

~A	for A ⊃ F	(5)
A ∨ B	for ~A ⊃ B	(6)
A ∧ B	for ~(~A ∨ ~B)	(7)
A ≡ B	for (A ⊃ B) ∧ (B ⊃ A)	(8)

Eqn.(5), for example, when substituted in (4) gives us:

If $B = \sim A$

then $(b1,b2) = (a2,a1)$ (9)

and, similarly, expressions may be derived for the other connectives.

If we assume the fuzzy variables are normalized then, as there is only one non-unity component, there is a 1-1 correspondence with the unit interval that simplifies the above expressions. Let:

$$a = (1-a1+a2)/2 \qquad (10)$$

and so on for the other variables (this transformation can be inverted given that one of a1 and a2 must be 1). Then the equations for the logic operations become:

$$C = A \supset B \rightarrow c = \max(1-a, b) \qquad (11)$$

$$B = \sim A \rightarrow b = 1-a \qquad (12)$$

$$C = A \vee B \rightarrow c = \max(a,b) \qquad (13)$$

$$C = A \wedge B \rightarrow c = \min(a,b) \qquad (14)$$

$$C = A \equiv B \rightarrow c = \min(\max(1-a,b)\max(1-b,a)) \qquad (15)$$

This system of equations gives a MVL that Rescher[26] calls the infinitely valued form of a variant standard sequence (VSS) and attributes to Dienes[38] Thus normalized fuzzified PC is precisely Dienes VSS. Note that the connectives of equns.(12) through (14) are those assumed in papers on fuzzy logic and, particularly, that the 1-a definition of negation arises naturally from PC and was not introduced as a fuzzy complement operation.

2.3 Implication in the Base MVL for Fuzzy Reasoning

The natural way that VSS arises through the fuzzification of PC makes it attractive to consider this MVL as a basis for fuzzifying yet again to get Zadeh's model of hedged linguistic truth values[16]. That is, one would consider linguistic truth values such as "true" or "very true" as having a separate degree of membership to each truth value in the interval [0,1]. However, the implication function of VSS has a defect in that it does not allow the assertion of A B to be used to infer that $b \geqslant a$, i.e. the truth value of B is at least that of A. This defect was noted by Lee[19], who assumes a PC form of implication ($\bar{A} \vee B$) in his "fuzzy logic", and also runs counter to the constraints on implication discussed by Lee and Chang[17], and used in such practical applications as the fuzzy controller of Mamdani and Assilian[39]. In such applications, rules of inference such as: if A1 then B or if A2 then B, etc., may be interpreted as: $A1 \supset B$, $A2 \supset B$, etc., and used to infer in a particular instance that the truth value of B is at least equal to the maximum of all A1, A2, etc. that imply B.

If we require that the truth value of $A \supset B$ is 1 iff $b \geqslant a$ then this may be used to define a variant of VSS based on some subset of definitions (11) through (15). To complete the definition of implication we must define the truth value of $A \supset B$ when $b < a$. Three possible definitions are:

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geqslant a,\ c = b \text{ otherwise} \qquad (16)$$

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geqslant a,\ c = b/a \text{ otherwise (with 0/0 taken as 1)} \qquad (17)$$

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geqslant a,\ c = 1-a+b \text{ otherwise} \qquad (18)$$

so that, when the implication is not absolute, the truth value is that of the implied proposition (equn. 16), or (equn.17) it is the ratio between the two, or (equn.18) a function of the difference between the two. If we couple each of these definitions with (5) for negation, (8) for equivalence, (13) for disjunction and (14) for conjunction we obtain three distinct "fuzzy logics" that are related to important MVLs: equn. (16) gives Godel's infinitely valued logic[26] in which negation has the form -

$$B = \sim A \rightarrow b = \begin{cases} 1 & \text{unless } a = 1 \\ 0 & \text{if } a = 1 \end{cases} \qquad (19)$$

and is closely related to the intuitionistic propositional calculus (IPC - it has same axioms with addition of $(A \supset B) \vee (B \supset A)$); equn.(17) gives another MVL with negation as in equn.(19), closely related to IPC and analysed by Goguen[18]; and equn.(18) gives Lukasiewicz's infinitely valued logic[26] which has a negation of the form:

$$B = \sim A \rightarrow b = 1-a \qquad (20)$$

the same as that of fuzzified PC (equn.12), and is that proposed by Zadeh[16] as a base logic for fuzzification as a model of linguistic truth values.

Note that the procedure for inference from rules of the form $A1 \supset B$, $A2 \supset B$, etc., previously discussed is available in all three of these logics. The crucial difference really shows up only in the form of negation. The fuzzy complementation-like negation of equn.(20) is not as naturally determined[15] as the basic max/min connectives of disjunction/conjunction, and has problematic semantics[40]. The negation of equn.(19) has the IPC property that asserting the double negative of a proposition does not imply that proposition which is desirable in situations where inference is not possible from negative instances. All these logics (and some other MVLs) are suitable as base logics for fuzzification on models of fuzzy reasoning but will have differing semantics related to statements using "not" as a modifier.

This section has classified the somewhat varied use of the term "fuzzy logic" and shown that there are various MVLs satisfying definition (c) and which can also act as base logics for fuzzification in terms of definition (b). In the following section I shall attempt to give an integrated approach that develops all these possibilities from a common foundation and, in particular, incorporates previous approaches to uncertainty and vagueness through probability logics.

3 An Integrative Approach to Logics of Vague Reasoning

The type of technical discussion found in sections 2.2 and 2.3 can be carried on ad infinitum. It is of interest to show that the fuzzification of PC leads to a known MVL but one that has an implication function that is inappropriate to the required semantics of vague inference. However, the lines of argument and discussion are reminiscent of studies some 40 to 50 years ago[4,41] when similar discussion of implication in MVLs took place[42]. Has there been a change or are we just seeing a revival of technical interest in some aspects of infinitely valued logics?

What Zadeh has provided is fresh semantics for MVLs in terms of the "degree of membership" to a fuzzy set, and what the many application studies of fuzzy reasoning[2,3] are providing is detailed structure to those semantics, an extensive definition of what they need to be. Salomaa[43] in his survey of MVL nearly

20 years ago remarks on the difficulty of providing interpretations of MVLs with many truth values. Indeed, apart from the many studies in philosophical logic of 3 and 4-valued systems one would be fair in assuming that most other studies of MVLs have been more to provide exemplary matrices for such purposes as demonstrating independence of axioms in various logical systems than for their own interest.

Thus the new direction, strength, motivation, etc. is in the application to reasoning with uncertain data, and fresh progress must surely come by stress upon the semantics of uncertainty rather than upon technical considerations. However, here one comes up against another anomaly in the fuzzy reasoning literature. For the systems scientist, probably for most of us, the natural tool to use in reasoning under conditions of uncertainty has always been probability theory. Indeed the literature of inference and decision-making under uncertainty is both contained in, and the prime motivation behind, that of probability theory. The non-probabilistic studies of uncertainty are minute in comparison and, morever, do not conflict with probability theory[31-32,44]. In recent years there have been major developments in probability theory[45] such as those stemming from Carnap's study of "logical probability",[46-47], the more recent advances by Savage[48] and de Finnetti[49] in laying rigorous foundations for "subjective probability", and the studies by many workers[50-53] of new, operational foundations for probability in work on "computational complexity", and its close relationship to "subjective probability"[64-55]

Yet there has been a definite and intentional rejection in the fuzzy logic literature of any links between it and probability theory. Bellman and Zadeh [56], Goguen[18], and Lee[19], for example, all place great emphasis on fuzziness not being probabilistic in nature and "degree of membership" not being a probability. Much of this emphasis can be attributed to tutorial exaggeration - it was necessary in early papers to make it clear that a man 5 feet tall might be said to have a degree of membership of .2 to the fuzzy set of "tall men" without there being any implication that he was a sample drawn from a population with probability .2, or when you measured the height of a "tall" man there was .2 probability that it would be under 5 foot, etc. Probabilistic noise on a signal makes it "fuzzy" in some sense, but it had to be emphasized that such is not the nature of all "fuzziness".

However, the authors go beyond mere emphasis in important respects:

(a) "furthermore the correspondents of a+b and ab, ... are the simpler operations Max(a,b) and Min (a,b)" (Ref 50 p.142) - it is implicitly assumed that events are mutually exclusive (for conjunction) or statistically independent (for disjunction) - these are the natural assumption of the applied mathematician or engineer - probability logic is not truth-functional and such assumptions are necessary to make it so - however they are not the only assumptions and I have shown[21,57], and will further develop in the following sections, that alternative assumptions make probability logic truth-functional as a fuzzy logic in terms of definition (c) of section 2.1;

(b) "the allowable operations on distributions do not include minimum" (ref.18 p.340) - again this is a common impression for the discontinuous min/max operation of fuzzy logic seem strikingly different from our normal manipulation of probabilities - however, as noted above, this is not so;

(c) "it would be nice to combine probability theory with symbolic logic. But we do not seem to know how to do this". (ref. 19 p.109) - this, when there are the studies of Gaifmann[58], Scott and Kraus[59], Adams[60], Fenstad[61], and Rescher[62], together with the, perhaps less well-known, studies of Danielsson[63] and Miura[64] on probabilistic models of modal logics or that of Lukasiewicz on the logical foundations of probability theory, first published in 1913, but only recently available in English[65].

However, there is a strong element of truth in Lee's statement (c) because, despite these results, logicians in general do seem to have neglected probability logic to the same extent that they have given "vagueness the go-by"[66]. In his analysis of the emergence of the concept of probability Hacking[67] identifies it with the emergence of modern science when "theological views of divine foreknowledge were being reinforced by the amazing success of mechanistic models .... the specific mode of determinism is essential to the formation of concepts of change and probability". Yet in this peculiar symbiosis between determinism and chance it is clearly determinism that has the upper hand in the philosophy of science and its logic (ref. 68 pp.316), so much so that Suppes[69] has felt it necessary to attack what he calls the "new theology of science" that holds such tenets as "knowledge must be grounded in certainty".

Even in classical multivalued logic studies the rejection of probabilistic interpretations has been definite and intentional. Salomaa (ref.43 p.120) describes attempts to identify the truth-value of a proposition with a probability as "futile" quoting Mazurkiewicz as quoted by Zawirski (ref.69 p.516) on the grounds that probability logics are non-truth-functional. He concludes "there is no use in combining these two things" (multivalued and probability logics). This anti-probability ethos is so strong that it allows Sanford (ref.12 p.32) to reject probability logics as models for vagueness by "disproving" what is a tautology of probability logic by an error in arithmetic.

The critical comments of this section have not been made for their own sake but rather to identify a set of assumptions which are so prevalent in the literature as to command unquestioning acceptance, yet which are not valid. There is a common foundation to probability logic and the many "fuzzy logics" in sense (c) of section 2.2. The strength of this foundation is the very non-truth-functionality of probability logic that has been quoted as a disadvantage. Many properties may be proved of a general "probability logic" that continue to hold when it is made truth-functional in a variety of ways, one of which, for example gives Lukasiewicz L_1 All the important properties of the implication function, for example, may be derived in the basic non-truth-functional logic and the same valuation of implication holds in Rescher's probability logic and Lukasiewicz L_1. There are also alternative implication functions, all with the basic required semantics discussed in section 2.3 corresponding for example to conditional probability, that in their turn lead to alternative MVLs.

That is the technical background - an integrated mathematical structure for the logics of uncertainty. There is also a common semantic interpretation that links fuzzy logic with logics of subjective[48,49,71] and qualitative probability[72,73] and belief systems[74], a not unreasonable linkage if fuzzy logic is itself to be a foundation for models of human verbal reasoning and hence also of communication amongst people.

3.1 Connectives for a Basic MVL

In considering non-truth-functional logics it is necessary to make a clear distinction between the (algebraic) structure of the propositional calculus for conjunctively and disjunctively combining proposition and the truth values ascribed to the propositions themselves. There are many possible starting points but, for the purposes of this outline, it is convenient to assume a lattice of propositions, $L(X, F, T, \vee, \wedge)$, generated by a set of elements, X, under two (idempotent, commutative) monoid operations, $\vee$ (disjunction), $\wedge$ (conjunction), with maximum element, T, and minimum element, F, i.e. L satisfies:

$$\forall \; x \in L, \quad x \vee x = x \wedge x = x \tag{1}$$

$$\forall \; x, y \in L, \quad x \vee y = y \vee x, \; x \wedge y = y \wedge x \tag{2}$$

$$\forall \; x, y, z \in L, \quad x \vee (y \vee z) = (x \quad y) \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z \tag{3}$$

$$\forall \; x, y \in L, \quad x \wedge (x \vee y) = x, \; x \vee (x \wedge y) = x \tag{4}$$

$$\forall \; x \in L, \quad x \vee F = x, \; x \wedge F = F, \; x \vee T = T, \; x \wedge T = X \tag{5}$$

the idempotent, commutative, associative, and adsorption postulates, together with a definition of the minimal and maximal elements (ref.75 p.18). The usual order relation is also defined:

$$\forall \; x, y \in L, \quad x \leqslant y \iff \exists \; z \in L : y = x \wedge z \tag{6}$$

It is possible to make a case for weaker structures (e.g. dropping idempotency[75]) but, for present purposes, this will be taken as an unreasonably wide generalization of our concepts of conjunction and disjunction. Now suppose that every element of L is assigned a "truth-value" (for different applications different terminologies might be more appropriate, "probability", "degree of knowledge", "level of belief", etc.) in the closed interval, $[0,1]$ by a continuous, order-preserving function $p: L \to [0,1]$, with the constraints:

$$p(F) = 0, \; p(T) = 1 \tag{7}$$

the equivalence relation on L, $\equiv$, defined by:

$$x \equiv y \iff p(x \vee y) = p(x \wedge y) \tag{8}$$

is a congruence on L. Both these constraints ensure compatibility between the lattice and the truth valuation upon it. Note that by the order preservation we already have:

$$p(x \wedge y) \leqslant \mathrm{Min}(p(x), p(y)) \leqslant \mathrm{Max}(p(x), p(y)) \leqslant p(x \vee y) \tag{9}$$

To complete the definition of an MVL one needs values for equivalence, $\equiv$, implication, $\supset$, and negation $\sim$, of propositions. I shall leave open for the moment the question of whether these concepts are represented not only metalinguistically but also by lattice elements, and regard $p(x \equiv y)$, $p(x \supset y)$ and $p(\sim x)$ as being a notational convenience. The value of equivalence must obviously satisfy the constraint:

$$\forall \; x, y \in L, \quad x \equiv y \iff p(x \equiv y) = 1 \tag{10}$$

Implication may be defined in terms of equivalence by noting that (corresponding to the requirements of section 2.3) we require:

$$\forall \; x, y \in L, \quad x \leqslant y \iff p(x \supset y) = 1 \tag{11}$$

but that for x, y satisfying this we have: $x \vee y = y$, $x \wedge y = x$. Thus it is natural to define implication as the degree to which these equivalences hold:

$$\mathrm{Imp}_A \quad p(x \supset y) = p((x \vee y) \equiv y) \tag{12}$$

or

$$\mathrm{Imp}_B \quad p(x \supset y) = p((x \wedge y) \equiv x) \tag{13}$$

both of which satisfy (11).

Negation may then be defined in terms of implication in the normal manner (ref.77 p.50) as:

$$p(\sim x) = p(x \equiv F) = p(x \supset F) \tag{14}$$

We have immediately:

$$p(x) = 0 \iff p(\sim x) = 1 \tag{15}$$

I shall call an MVL with connectives satisfying the constraints so far a basic MVL. It is a weak, non-truth-functional structure of high generality. However, it is worth noting that if truth values are constrained to a binary set (0 and 1 by (7)) the basic MVL reduces to PC. This is a counter example to Lee's[19] remark that, "by rejecting the evaluation procedure of fuzzy logic, one would simultaneously reject that of two-valued logic" - long before the logic has been specialized to a particular set of truth-functional connectives it reduces to PC in the binary case.

3.2 Metrics and a Basic Probability Logic

Equn. (10) is only a constraint on the value of equivalence not a complete definition. There are two very natural definitions arising from (8), (9) and (10):

$$\mathrm{Equiv}_1 \quad p(x \equiv y) = 1 - p(x \vee y) + p(x \wedge y) \tag{16}$$

$$\mathrm{Equiv}_2 \quad p(x \equiv y) = p(x \wedge y)/p(x \vee y) \tag{17}$$

(with the convention that 0/0 = 1). Note that in both cases:

$$p(F \equiv T) = 0 \tag{18}$$

If these two definitions are coupled with the two alternatives for implication (16 and 17), we obtain:

$$\mathrm{Imp}_{A1} \quad p(x \supset y) = 1 - p(x \vee y) + p(y) \tag{19}$$

$$\mathrm{Imp}_{B1} \quad p(x \supset y) = 1 - p(x) + p(x \wedge y) \tag{20}$$

$$\mathrm{Imp}_{A2} \quad p(x \supset y) = p(y)/p(x \vee y) \tag{21}$$

$$\mathrm{Imp}_{B2} \quad p(x \supset y) = p(x \wedge y)/p(x) \tag{22}$$

Negation has two forms:

$$\mathrm{Neg}_1 \quad p(\sim x) = p(x \equiv F) = 1 - p(x) \tag{23}$$

$$\mathrm{Neg}_2 \quad p(\sim x) = p(x \equiv F) = \begin{cases} 0, & p(x) \neq 0 \\ 1, & p(x) = 0 \end{cases} \tag{24}$$

Two definitions that are already reminiscent of two families of MVLs[26].

There is a further important constraint that may be placed upon p. It can be obtained by requiring Imp_{A1} and Imp_{A2} to be identical, so that:

$$p(x \vee y) + p(x \wedge y) = p(x) + p(y) \tag{25}$$

i.e. p is a valuation on L (ref.75 p.74). This is the usual requirement upon a probability over a language and I shall call a basic MVL satisfying it a basic probability logic (BPL). Most classical MVLs, such as Lukasiewicz L_1, are also BPLs. Birkoff[75] proves many properties of an order-preserving valuation: notably that the congruence requirement (8) is automatically satisfied, L must be modular, and

$$d(x,y) = p(x \vee y) - p(x \wedge y) \quad (26)$$

is a metric on L.

It may be shown that $1 - p(x \equiv y)$ is a metric on L for both our definitions $Equiv_1$ and $Equiv_2$, and indeed many properties of a basic MVL may be derived by commencing with a postulated metric on L rather than our function p. The function p may then be defined through:

$$p(x \equiv y) = 1 - d(x,y) \quad (27)$$

$$p(x) = p(x \equiv T) = 1 - d(x, T) \quad (28)$$

Implication may be defined in such a metric logic by noting that x _ y only when the triangle inequality:

$$d(x,y) + d(y, T) \geq d(x, T) \quad (29)$$

becomes an equality.

Commencing with a metric on an arbitrary space, designating a point as T and deriving a basic MVL from the metric properties of the space is an alternative approach to that taken here. It is a particularly attractive approach for applications to pattern clustering[78,79] and taxonomy[80] where the metric has a natural meaning and the designation of T corresponds to the centre of a cluster or taxon.

3.3 Truth-Functionality in MVLs

The constraints upon the function p and its extension to other connectives in a basic MVL are not sufficient to make the logic truth-functional, (TF), i.e. $p(x \vee y)$, $p(x \wedge y)$, $p(x \equiv y)$, etc., cannot be defined in terms of p(x) and p(y). However, there are constraints upon the truth values of these connectives - no logic is completely non-truth-functional. Also various other constraints such as (12) or (13), (14), (16), or (17), and (25), give inter-definability of some connectives. This makes it convenient to define a weak form of truth-functionality:

Def_A: A basic MVL is weakly truth-functional to extent n if at least n of the 5 basic connectives are not definable in terms of the remainder. Thus a basic MVL is weakly TF to extent 3 (values of conjunction, disjunction and equivalence undefined). A basic probability logic is weakly TF to extent 2 since (25) gives interdefinability of the values of conjunction and disjunction. In both cases we normally reduce the extent by 1 by using either (16) or (17) to define equivalence in terms of conjunction and disjunction.

A stronger form of truth-functionality is:

Def_B: A basic MVL is truth-functional if the values of all binary connectives involving expressions s and t are equationally defined in terms of p(s) and p(t) where s and t are expressions having no generating element in common.

The lattice definitions enable one to decompose an arbitrary expression into components that have no elements in common so that the equational definitions may be used to compute truth values for arbitrary expressions. However, the equations themselves may not carry over to connectives between expressions with elements in common (e.g. the probabilistic "independence" assumption, $p(x \wedge y) = p(x)p(y)$ does not carry over to $p(x \wedge x)$).

This leads to a stronger definition:

Def_C: A basic MVL is strongly truth-functional if arbitrary binary connectives involving expressions s and t are equationally defined in terms of p(s) and p(t).

We can now give some interesting results:

(i) Strong TF and Min/Max Connectives A basic MVL is strongly TF $\Longleftrightarrow$ the outer inequalities of equn.(9) are equalities, i.e.:

$$p(x \wedge y) = Min(p(x),p(y)) \quad (30)$$

$$p(x \vee y) = Max(p(x),p(y)) \quad (31)$$

This is essentially the result of ref.15 but is is seen in a new light when it is realized that "fuzzy logic" (in sense (c) of section 2.3) is the only MVL satisfying a strong truth-functionality requirement. Essentially it means that, for a compound proposition, we do not have to remember its structure but only its truth-value. This seems an unrealistically strong requirement but it does uniquely distinguish "fuzzy logic" with its Max/Min connectives. It also shows why an emphasis on truth-functionality[43] leads to the universality of these connectives in classical MVLs[26].

(ii) Strong TF and Necessary Mutual Implication A strongly TF MVL has the theorem:

$$\forall\ x,y,\ \ p((x \supset y) \vee (y \supset x)) = 1 \quad (32)$$

This follows from the unit value necessarily ascribed to one of the two implications in a logic with Max/Min connectives and the constraints on equivalence and implication of (11) and (12) or (13). Equn. (32) is effectively the axiom which it is necessary to add to those of IPC to get Gödel's infinitely-valued logic[26] Any MVL with Min/Max connectives and a "reasonable" definition of implication will have (32) as a thesis and hence cannot be characteristic for IPC.

The following results apply to basic probability logics:

(iii) LEM and Probability Logic A basic probability logic with Equiv and the law of the excluded middle (LEM) is Rescher's probability logic[26].

The LEM gives us

$$p(\sim x \vee x) = 1 \quad (33)$$

and this together with (25) implies the law of contradiction:

$$p(\sim x \wedge x) = 0 \quad (34)$$

The LEM together with $Equiv_1$ and (25) give:

$$p(x \supset y) = p(\sim x) + p(x \wedge y) = p(\sim x \vee y) \quad (35)$$

i.e. the standard implication of both PC and PL. The other connectives also correspond.

(iv) Strong TF in Probability Logic and L_1 A strongly truth-functional basic probability logic with $Equiv_1$ is precisely Lukasiewicz's infinitely-valued logic, L_1.

$Equiv_1$ gives us:

$$p(x \supset y) = 1 - p(x) + p(x \wedge y)$$

$$= 1 - p(x) + \mathrm{Min}(p(x), p(y))$$

$$= \mathrm{Min}\,(1, 1 - p(x) + p(y)) \quad (36)$$

i.e. the implication of L_1 - the other connectives also follow.

The close relationship of PL and L_1 has been noted previously[21,57,76] and results (iii) and (iv) made it very clear - the equivalence, implication and negation are identical, and PL is derived by adding LEM whereas L_1 is derived by assuming strong truth-functionality or Min/Max connectives or necessary mutual implication as in equn.(32). L_1 may be seen to arise when a basic probability logic is made strongly truth-functional. Note that it is not possible to make Rescher's probability logic strongly TF. The following results are concerned with the various logics that arise when a BPL is made TF as in Def_B rather than strongly TF.

(v) Statistical Independence The assumption that:

$$p(x \wedge y) = p(x)p(y) \quad (37)$$

for propositions with no common element in a BPL, together with Equiv_1 and LEM, gives a probability logic of assumed statistical independence between the generating propositions, e.g. such as arises in the next state calculation for probabilistic automata.

(vi) Mutual Exclusion The assumption that:

$$p(x \wedge y) = 0 \quad (38)$$

for propositions with no common element in a BPL, together with Equiv_1 and LEM, gives a probability logic of assumed mutual exclusion between the generating propositions, e.g. such as arises when states are grouped together in a probabilistic automaton.

(vii) A Fuzzy Logic with LEM The assumption that:

$$p(x \vee y) = \mathrm{Max}\,(p(x), p(y)) \quad (39)$$

for propositions with no common element in a BPL, together with Equiv_1 and LEM, gives a fuzzy logic of assumed necessary implication between the generating propositions, i.e. a logic based on a generating set forming a chain. Such a logic with Max/Min connectives but LEM also has been proposed by Sanford[12] in his studies of "borderline" logics and is that obtained when one makes Rescher's PL truth-functional with Max/Min connectives[21].

The alternative forms of implication, Imp_{A2} and Imp_{B2}, are of interest in the context of results (iii) through (vi). Imp_{B2} is related to conditional probability:

$$p(x \supset y) = p(x \wedge y)/p(x) = p(y|x) \quad (40)$$

and gives a logic of conditional probability. Making logics with either of these implications strongly truth-functional gives a logic whose connectives are those of Gödel's logic except for implication which has Goguen's[18] value:

$$p(x \supset y) = \mathrm{Min}(1, p(y)/p(x)) \quad (41)$$

rather than Godel's $p(y)$ if $p(x) > p(y)$. It is not clear currently how the axiomatic forms of these logics differ.

3.4 Quantification and Modalities in MVLs

The treatment of quantification within the basic MVL framework developed so far is straightforward, and as usual, offers far richer possibilities as suggested by Mostowski[81] and developed by Rescher[82,26]. Quantifiers are essentially arithmetic predicates applied globally over L to the truth-values of elements satisfying specified constraints - they depend on a specific function, p over L.

Modalities have a similar global nature but are introduced as predicates over a family of admissible p's related by a binary relation of "reachability", i.e. a "possible worlds" semantics[83,84]. Again the availability of arithmetic predicates allows a richer set of modal connectives to be defined. The conventional alethic modalities have natural extended definitions - in a "world" p over L the value of $\Box x$ and $\Diamond x$ is:

$$\Box\, x = \mathrm{Inf}\quad p'(x)\ ,\ p' \text{ reachable from } p \quad (42)$$

$$\Diamond\, x = \mathrm{Sup}\quad p'(x)\ ,\ p' \text{ reachable from } p \quad (43)$$

In logics where Neg_1 of equn. (23) holds these are interdefinable in the usual way.

In a BPL with LEM and 1 designated it may be shown that "reachability" being an equivalence leads to a logic characteristic of S5[62]. In the study of stochastic automata "reachability" in the sense of a "possible future state distribution" is transitive but not necessarily reflexive unless a stationary cycle is reached. Since the designation of 1 in a BPL with LEM gives a logic characteristic of PC these considerations reduce to those analysed by Prior[85] in his studies of temporal logic. Similar considerations apply to the BPLs without LEM which, with 1 designated are closely related to IPC and hence to S4 under various "translations"[86].

Clearly this whole area of infinitely-valued MVLs with quantifiers and modalities in relation to axiomatic studies of standard and non-standard logics is worth detailed examination, if only because of its computational implications. By manipulating the valuation rather than the axioms we reduce symbolic algebra to arithmetic, making computation easier. This is already done for S5 where engineers typically use a probability logic to represent non-deterministic behaviour even though the exact values of the "probabilities" have no significance other than being non-zero, non-unity[87,88]

Also of interest are the non-linear arithmetic functions that may be used in defining modal operators. A "borderline logic"[12] may be developed using power functions such as:

$$D_N\, x = p(x)^N \quad (44)$$

where D_N is a "determinacy" operator expressing the extent to which p(x) is near unity. A "borderline" case is one that is neither near unity nor near zero so that the degree to which x is borderline in a given world is:

$$B_N\, x = (\sim D\ \ x) \wedge (\sim D \sim x) \quad (45)$$

Thus the truth value of x being necessarily borderline in p is:

$$B_N\, x = \underset{p' \text{ reachable}}{\mathrm{Inf}} \quad (\sim D\ x) \wedge (\sim D \sim x) \quad (46)$$

The relationship of D_2 to Zadeh's[24] definition of "very" is interesting and again would repay further exploration.

It is important to emphasize that many of the results of this section may be treated as properties of the underlying lattice structure and developed algebraically rather than numerically. This shows up particularly clearly in the elegant studies of subresiduated lattices reported recently by Epstein and Horn[89]. Particularly in the area of quantification and modalities it seems worthwhile to fully develop relationships between fuzzy logics, probability logics and classical infinitely-valued MVL's so as to fully exploit the wide range of concepts, techniques and results available in the literature.[90]

4 Semantics, Summary and Conclusions

In this paper I have been concerned to draw together all the logics of uncertainty. Doing so is not only of technical interest but also opens up the rich range of semantics that have been associated with classical modal logics, standard and non-standard propositional and predicate calculi, multi-valued and probability logics, together with the semantics of system theory and vague reasoning introduced by Zadeh.

There is one particular semantics that I have found very useful in illuminating the differences between PL and L_1 as noted in results (iii) and (iv). Consider a population each member of which can respond to certain questions with a binary, yes or no, reply. The forms of question will involve evaluating a statement about a proposition that belongs to the generating set, X, of a lattice, L, as defined in section 3.1. For example "is this proposition, x, true or false, reasonable or unreasonable, believed or not believed,"etc. The value of p(x) is defined to be the proportion of the population replying "yes" to the question. A compound proposition in L is given a valuation in terms of the proportion of the population who say "yes" to both x and y for propositions of the form $x \wedge y$, and so on for other cases.

One may now return to the initial discussion of section 3 and note that a degree of membership of .2 to the fuzzy set of "tall men" now means that 20% of a certain population would accept the membership. If the "population" is one of measurements of height of a person then this is a physical "noise" model of probability. If the "population" is one of people then this is a social acceptance model of linguistic usage, a reasonable model of Zadeh's "fuzzy reasoning" based on human linguistic behaviour. If the "population" is one of "neurons" then this is a model of individual decision-making. If we allow metalinguistic statements about the value of p(x) to be made by members of the population then this us a model of "subjective probability" or "belief", and so on.

Consider now the additional constraints that must be placed upon the behaviour of the population to correspond to results (iii) and (iv). Rescher's probability logic is obtained if someone who says "yes" to x must say "no" to ~x. Lukasiewicz's L_1 is obtained if members of the population each evaluated the evidence for x in the same way but applied differing thresholds of acceptance. The member with the lowest threshold would then always respond with "yes" when any other member did, and so on up the scale of thresholds, thus giving the required relation of implication between propositions of equn.(32). This model, although unusual, has its intuitive attractions, e.g. Reason[91] has shown that the threshold applied by human beings in coming to a binary decision on an essentially analog variable seems to be associated with personality factors and a trait of the individual. If so, human populations would tend to show more a fuzzy, than a stochastic, logic in their decision making.

Similarly populations showing the "statistical independence" of (v) or the "mutual exclusion" of (vi) may be defined. However, rather than argue the case for one type of population or another, one can now envisage that logics based on a real population will be of mixed type and hence it is more interesting to insert the concepts and talk in terms of a "fuzzy", "probabilistic", "independence", "exclusion", etc. relationship between propositions. Such relationships are mainly of interest to the extent that they are necessary, and thus fit naturally into the framework of arithmetic modal predicates discussed in section 3.4. For example, if the degree of fuzziness of x relative to y in p is defined by:

$$F(x,y) = 1 - p(x \vee y) + \mathrm{Max}(p(x), p(y)) \qquad (46)$$

then F(x,y) expresses the extent to which there is a necessary fuzzy connection between the two propositions.

In conclusion, there is no one logical system that stands out clearly as the logic of uncertainty. Many applied studies have commenced with a specific logic and found it necessary to modify it to match the required semantics. There is technical scope for an explosive proliferation of MVLs, particularly with respect to modal operators. This proliferation is to be welcomed rather than contained - the real-world, particularly that involving human agents, sustains a far greater diversity of patterns of reasoning than allowed for in classical logic. However, to utilise and make sense of these diverse application studies the common foundations of the many logics of uncertainty need to be firmly established. This paper has ranged far and wide and certainly does not yet present a totally coherent and complete view of the logics of uncertainty. If it at least clarifies some aspects of the literature of these logics, demonstrates the close relationships between them, and indicates the wealth of both technical and semantic interchange possible when the logics are viewed as a whole, then it will have served its purpose.

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