Analysing Analogy

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Analogical reasoning is one of the most important techniques used by people, yet it has proved very difficult to represent the process in formal terms amenable to operational implementation in computer algorithms. In this paper we analyse the concept of an analogy and its application to reasoning processes. An analogy is seen to be a partial correspondence between two systems. In representing it in map-theoretic terms a third system naturally arises which may be injected into each of the others to represent the partial correspondence. This system may be called an 'analogy' system capturing the particular analogy under consideration. We then go on to consider multiple analogies between two systems and their inter-relationships and show that these form a semi-lattice with a truth-system as the minimal element. Because analogies are between systems with structure they have to capture the transformations that define the structure rather than just the elements of the systems. Hence the systems themselves are most simply represented as categories and the mappings as faithful functors between them. In this paper we give not only the formal theory but a number of systemic and programming examples to illustrate our analysis.

INTRODUCTION

Analogical reasoning is a powerful technique for problem solving in human thinking yet considered "weak" logically because it is non-deductive. There have been attempts to define its role in science, notably the studies of Hesse (1970) and Leatherdale (1974). However, in the study of formal reasoning using logic and mathematics it has been in one sense deliberately expurgated from the formal structure even though it plays a major part in the creative thought that lies behind that structure; yet in another sense analogy is the very essence of logical and mathematical formalism.

This is a multi-faceted paradox which throws light on the nature of analogical reasoning itself. From Plato through Aristotle to Aquinas analogical reasoning has been a major topic of enquiry at the centre of attempts by philosophers to comprehend and generalize the processes of inference in human reasoning (Burrell 1973). In the modern philosophy of science it has been neglected or treated as a peculiar tool to be used as scaffolding but carefully dismantled and all vestiges removed when the edifice created is made public.

This removal of analogical reasoning from the legitimation of inference may be seen as a major distinguishing feature in the rise of positivism. At the end of the last century Frege in his <u>Grundlagen der Arithmetik</u> refined mathematical reasoning into a purely logical axiomatic system. Husserl was initially critical of Frege's work but later accepted the framework of it in developing his own formal approach in his <u>Philosophie der Arithmetik</u> which led to his system of phenomenology. Between them Frege with formal logic and Husserl with formal phenomenology provided the ontological and epistemological foundations for modern positivist science and its formalization in such works as Carnap's <u>Der Logische Aufbau der Welt</u>.

The common factor underlying the positivist movement has been anti-psychologism in removing all aspects of science peculiar to human reasoning as processing in the human mind. This is not to say that acts of creative thought, individual genius and intuition, and so on, are denied, but that it this the procedures thereafter for explicating and legitimating in a public, verifiable form the results of these acts and so on which have become to be regarded as the proper subject for the philosophy of science. In retrospect one suspects that this emphasis, although presented as fundamental, is instead just technological. Tools existed to formalize the ontology of science based on mathematics and to formalize the epistemology of the processes of confirmation and falsification. Tools did not exist until recently with which one could begin to formalize the processes of innovation and creation. Thus it is timely to re-appraise the foundations of science and examine the possibility of formalizing some of the processes which come before the more routine activities already studied.

We noted above that there was a paradox in the eradication of the notion of reasoning by analogy in the axiomatic approach to science. This stems from the fact that by basing science on mathematics and logic the use of analogy became implicit in all aspects of scientific activity. These ultimate formal abstractions have only a remote relation with the actual world of the scientist and he is taught to use analogical reasoning to map every 'real-world' experience into an analogical experience in the 'world' of mathematics. In creating the analogy the scientist is allowed to neglect features of the real world:

> "The explicatum is to be similar to the explicandum in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however, close similarity is not required, and considerable differences are perimtted." (Carnap 1962 p.7)

Thus the scientist is mapping part of the real world "explicandum" into part of the formal "explicatum" system. It is this type of partial-partial mapping which we shall formally characterize in this paper as an analogy.

When one criticizes the positivist movement in science one must not discount its actual achievements. The progress in science and technology of the last 100 years owes much to the drive to present the subject matter in formal, simple and universal terms. In doing this we have generated tools for analysis rather than design. We cannot show where the explicata, the theories, the laws, and so on, come from. We can show only what to do with them when they are available: how to evaluate them; how to manipulate them; and how to use them. We rely on the unformalized activities of people to generate the new ideas that will then be subject to the formal scientific process.

This lack of formal foundations for major areas of scientific activity has mattered little in the past because these areas were peculiarly human and could be left outside the technological infrastructure of science. Now that the computer has become a tool operating at the level of man's mind, an engine for the exploration of Popper's (1968) World 3 of knowledge (Gaines 1979), it has become important to understand and formalize as wide a domain as possible of the scientific process so that man and machine may tackle it in a symbiotic relationship.

We have noted elsewhere (Shaw & Gaines 1979) that the interactive computer can provide means for:

(1) Modelling of data within a given framework (confirmation);

(2) Indication of search strategy for data most likely to cause a change of model (falsification);

(3) Indication of the effect of actions on the state of the model (simulation);

(4) Indication of actions most likely to lead to desired model or state of model (decision);

(5) Indication of presuppositions underlying the above four processes (paradigm);

and have emphasized the importance of adding to this list a further process:

(6) Indication of a change in presuppositions that would improve applications (1) through (5) (paradigm shift).

The first four programs are straightforward applications of computer technology and the fifth is covered by programs which elicit construct structures (Shaw 1980). Recently we have described computer programs (Shaw & Gaines 1980) that act as truly dialectical partners in a conversational process of critical discussion that encourages the paradigm shifts of (6). Currently such programs embody a simple model of analogical processes and this paper represents a further step in the development of more refined models.

WHAT IS AN ANALOGY?

The following formulation of analogy derives from a an attempt to put software engineering on a sound footing by formalizing concepts of virtual machines and structured programming (Gaines 1975). The notion of analogy used in that work was first put forward in another paper which gave a formal model for what we mean by analog computing (Gaines 1968). In this paper we have extracted the theory from the application to computing in order to present it as a basis for general-purpose tools.

What is an analogy? When we speak of there being one between two situations, things, systems, and so on, we mean that there is some correspondence between them, some similarity. This similarity is not an identity, that the two are exactly alike, because we would then say they are the same rather than similar (note that the term "the same" is often used colloquially to mean analogous - identities do not exist out in the real world). Neither is the similarity usually such that one situation subsumes another so that we can say "forgetting some aspects of it, this situation is the same as the other". In general for an analogy to be established we have to forget aspects of both of the analogous situations in order to see the similarity that remains. Thus, as we noted previously, the formal basis for analysing analogy is some theory of partial correspondences.

Before establishing a formal theory of analogy it is worth examining some concrete examples. If we just have two arbitrary sets, X and Y, then any partial correspondence between non-overlapping sub-sets of them might be thought of as an analogy as illustrated in Figure 1. There are two obvious difficulties with such an extremely weak concept of analogy. Firstly, that we are usually concerned with the preservation of structure in examining analogies. Figure 2 shows how an analogy might arise between two automata which puts the states into partial correspondence such that regardless of the input sequence when one automata is in a particular state the other one is in a corresponding state. This analogy preserves the connectivity structure of the automata under inputs in some way.

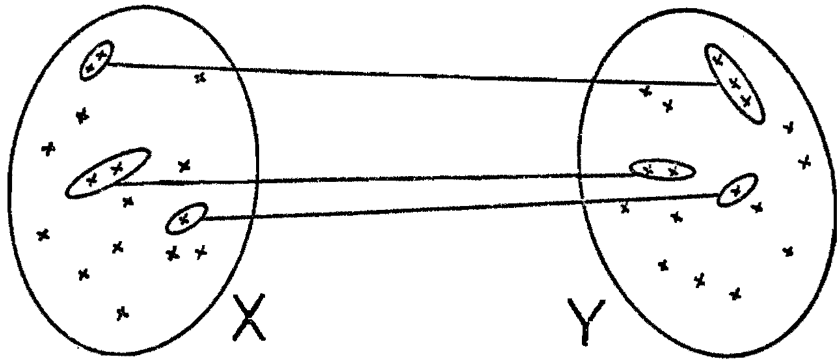

Figure 1 An analogy between sets is a correspondence between non-overlapping sub-sets

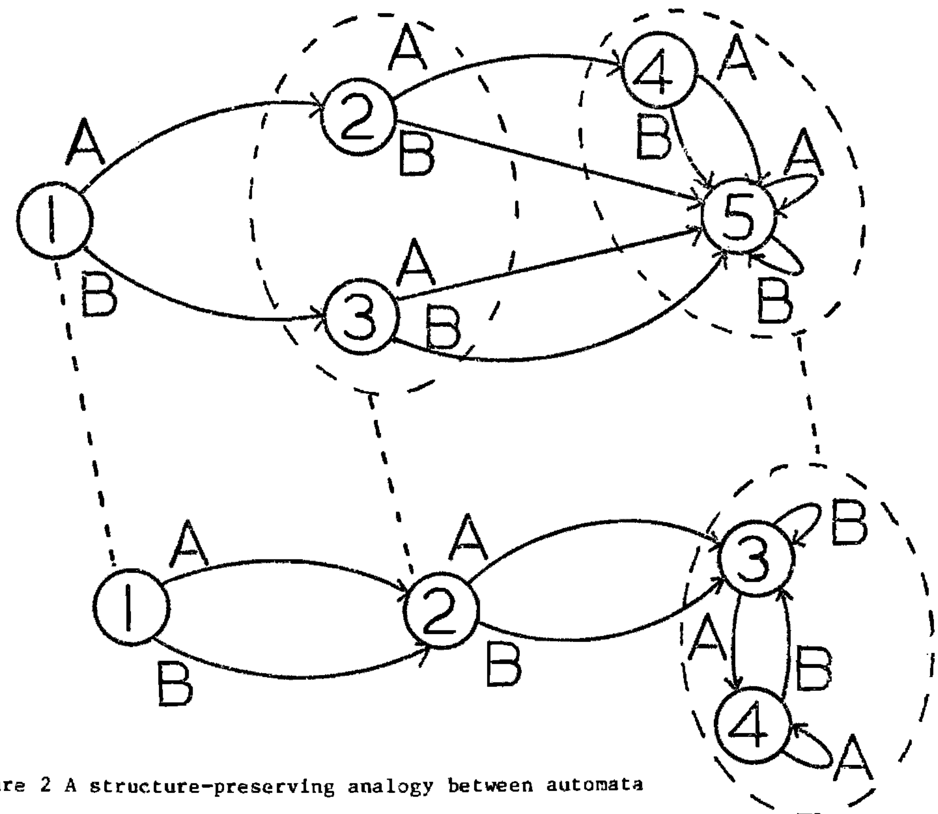

Figure 2 A structure-preserving analogy between automata

Figure 3 shows a similar phenomenon for the analysis of an analogy between two BASIC programs, both of which extract the roots of a quadratic equation. The input and output behaviours of the programs are very similar but what goes on inside is quite different for the two. The intermediate variables used and calculations made need have no detailed correspondence. In fact in this example both calculations do reach a similar step so there is one point of internal correspondence as noted in Figure 3. It is also worth pointing out in relation to this example that there is an implied correpondence not shown between the arithmetic systems in use by both programs.

The second problem with the weak form of analogy depicted in Figure 1 is that, if we place no further constraints on the notion of analogy, then the partial correspondences can be trivial or meaningless. How can we express what is meant by "meaningful", "significant", "relevant", and so on, analogies. Not only do we expect to preserve structure under an analogy but we also expect to that particular structure which for us is essential to the situations between which an analogy is being discussed. We will develop a theory of analogy that takes into account both these requirements.

```
1 INPUT A, B, C
2 LET X=B*B-4*A*C
3 IF X<0 THEN 10
4 LET X=SQR(X)
5 PRINT (X+B)/(2*A),(X-B)/(2*A)
10 PRINT "NO REAL ROOTS"
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```
1 INPUT A, B, C
2 LET Y=B*B
3 LET Y=Y-4*A*C
4 LET X1=SQR(Y)/(2*A)
5 LET X2=-B/(2*A)
6 PRINT X2+X1,X2-X1
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Figure 3 An analogy between BASIC programs for roots of quadratic

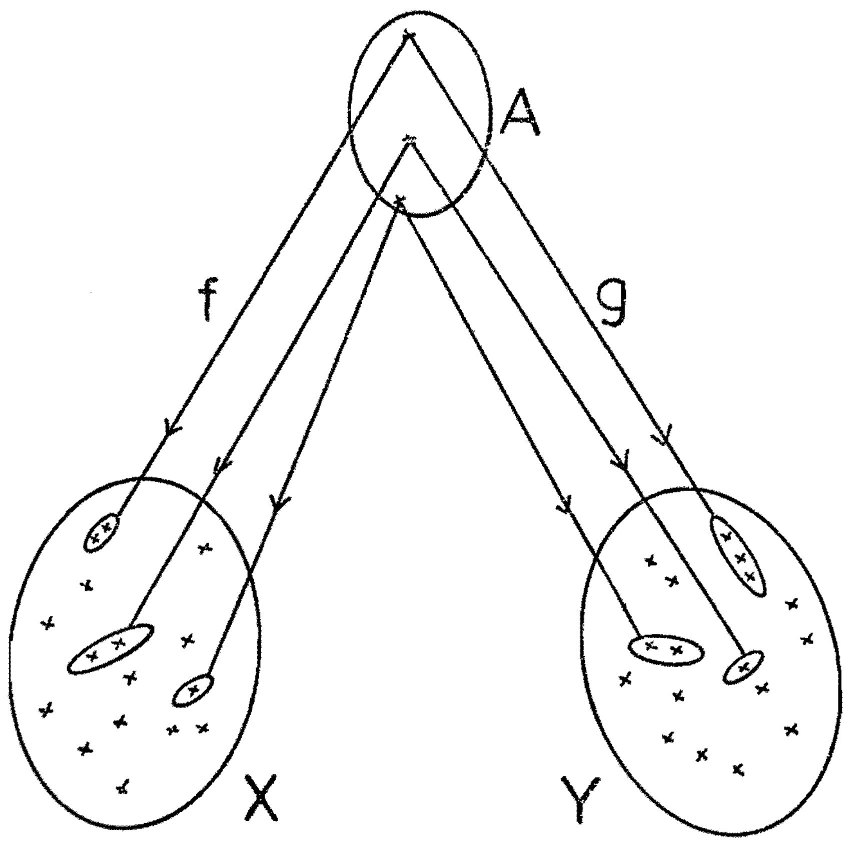

Figure 4 Representing the correspondence by mappings introduces an "analogy set"

It is convenient to continue with our set-theoretic example of Figure 1 initially and develop the argument further in relation to this. The correspondence shown in this figure is not a mapping but it may be represented by a pair of mappings, f and g, from an additional set, A, as shown in Figure 4. The additional set is introduced solely to carry the mappings and we call it the <u>analogy set</u>. This construction generates a definite entity which somehow <u>is</u> the analogy and this seems to correspond to the way in which we thing of an analogy as something which exists in its own right.

Now consider a further analogy which <u>extends</u> that given by A: Figure 5 shows a second analogy set, A', preserving the analogy of A, but extending it to other sub-sets in X and Y. The preservation is such that there exists a mapping, a, from A into A' which factors the maps, f and g, from A into X and Y through the maps, f' and g', from A' into X and Y, respectively. That is we have a commutative diagram such that:

$$f = f'a, \quad g = g'a \qquad (1)$$

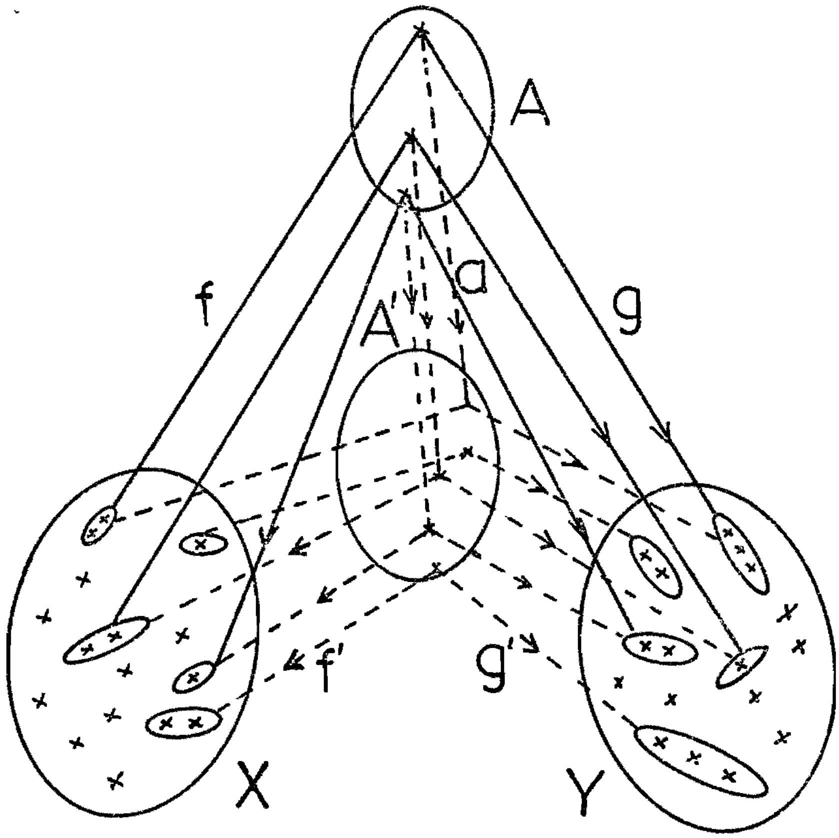

Figure 5 Extension of analogy gives a commutative diagram (f=f'a,g=g'a)

Thus the analogy represented by the maps from A' is an extension of that represented by the maps from A in that it preserves all the correspondences of A and adds additional ones. Clearly there could be a number of different extensions of the analogy A which might, or might not, themselves be extensions of one another. We shall note in the more general formulation, and it is obvious in the set-theoretic case, that the extensions to an analogy form a semi-lattice ordered by the factor mappings of which the original analogy is the supremum.

If we now return to our discussion of the "relevance" or "significance" of an analogy then we shall assume that this is always accounted for in terms of a element in the semi-lattice of analogies which forms a supremum for all those analogies we are prepared to consider relevant or significant. This element gives a correspondence between the two sets which preserves the minimal amount of structure in both of them that we require for a non-trivial analogy: in a previous paper (Gaines 1975) it was termed a "truth" element for the analogy and corresponds to what we regard as the essential truth underlying the correspondence - for example, in the analogy between a scientific theory and the real world.

The truth-automaton for the example of Figure 2 is shown in Figure 6: it is a simple three-state automaton that prserves the structure represented by the analogy. The truth-program for the example of Figure 3 is shown in Figure 7: it is a simple program that preserves the essential property of the two programs in extracting the roots of the quadratic. Note that it does not preserve the full analogy shown in Figure 3 since the intermediate correspondence is inessential. In a previous paper (Gaines 1968) it is argued that it is the existence of such intermediate correspondences that gives us the abstract notion of an analog computer. The advantage of such computers is that when we want to modify the computation we find it easier to do so because of the rich correspondence to the problem being solved - there is more than just a behavioral analogy.

In the next section we generalize the theory of analogy developed above to include arbitrary structural correspondences.

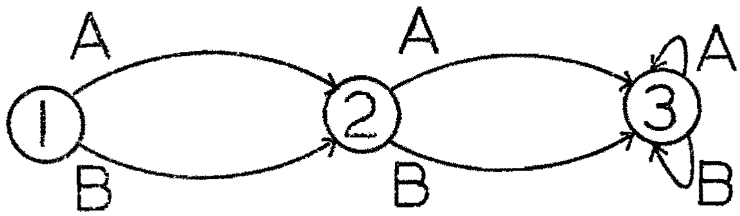

Figure 6 "Truth-automaton" for example of Figure 2

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1 INPUT A, B, C
2 PRINT -(B+SQR(B*B-4*A*C))/(2*A),-(B-SQR(B*B-4*A*C)/(2*A)
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Figure 7 "Truth-program" for example of Figure 3

A CATEGORY-THEORETIC FORMULATION OF ANALOGY

To take into account the structure-preserving properties required of an analogy we have to go beyond our set-theoretic example. If we had attempted to formulate the concept of an analogy relation a decade ago we would have been forced to frame it in terms of particular algebraic or topological structures. There would be a theory of analogies between sets, between automata, between programs, between topologies, and so on. A category-theoretic framework for a theory of analogy avoids these problems. A category can be highly specific, e.g. a single discrete set, or highly general, e.g. a class of algebras, and it can express constraints upon both objects and mappings.

How may we compare two categories for an analogy between them? The notion of an isomorphism, or any kind of morphism, between the categories is not useful because in general we expect each to have a structure not reflected in the other. As we have already noted, an analogy is a partial correspondence. It is a simple matter to extend the set-constructions we have already developed and introduce a "correspondence" category that maps into each of the categories between which we are analysing an analogy. To ensure that the mappings from the correspondence category are non-trivial we require them to preserve the structure in each category, that is to preserve commutative diagrams in each category. Such a structure-preserving mapping is termed a "faithful" functor between the categories (MacLane 1971).

The weakness that we noted in relation to set-theoretic correspondences still applies to the more general case and we need to introduce the notion of a "truth" element again in order to ensure that a correspondence is meaningful. A "truth" category for an analogy may be thought of as a category with the minimal structure sufficient to express the essence of what we want to preserve in the categories between which there is an analogy. The equivalent of Figure 5 is now the commutative diagram of Figure 8 in which the mappings are now faithful functors but the equations in (1) above still hold. Note that we have termed one of the categories between which the analogy holds a model. This is a deliberately suggestive terminology since we feel, as discussed in section 1, that models are usually not just abstractions from the modelled system but also have characteristics of their own which do not derive from the modelled system. Figure 8 gives a more accurate representation of the modelling process than does a diagram in which a model is shown mapping directly into the modelled system, or vice versa.

There can clearly be many analogy categories for a given category/truth/model triple (CTM). The direction and faithfulness of the functors ensures that the analogy categories are all "smaller" than both the category and its model. Figure 9 shows a set of four analogies each of which has necessarily a triple of arrows to the CTM triple. However, there may also be faithful functors between the analogies themselves and these define an important relation on the set of possible analogies. Because the existence of faithful functors is reflexive, asymmetric and transitive the relation induced by them is a partial order. Because least upper bounds, if they exist are unique, and greatest lower bounds always exist (the truth category is a universal lower bound) and are unique, the order corresponds to that of a lower semi-lattice.

This semi-lattice structure is very important in representing various other features of our usual analyses of analogy. It gives a rigorous basis for the concept that one structure is more analogous to another than a third. It ensures that

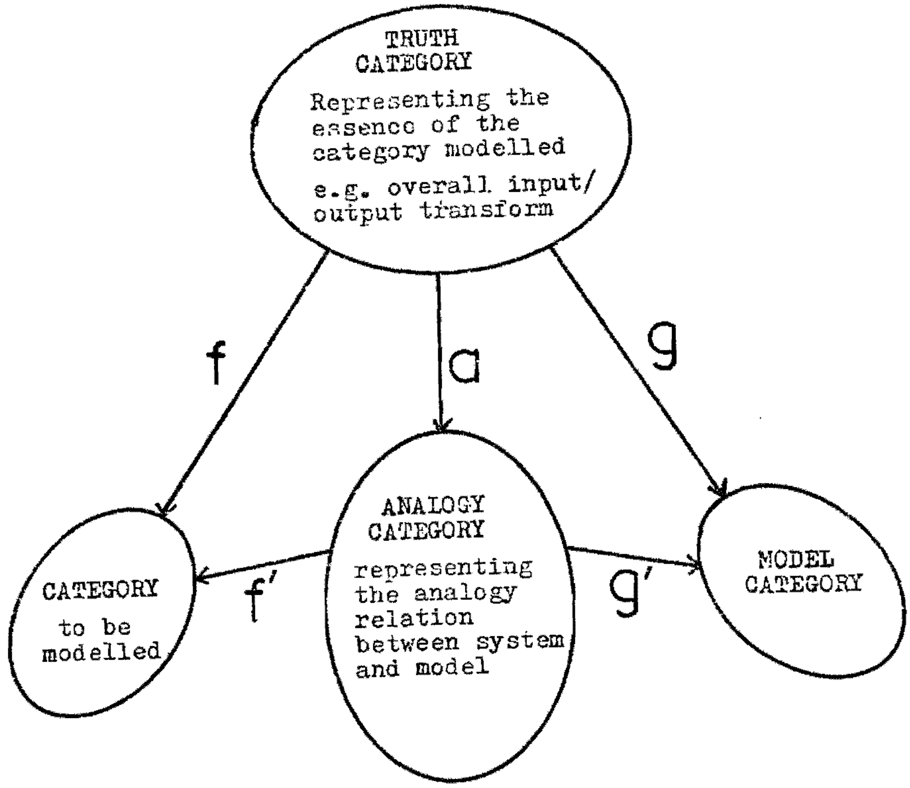

Figure 8 Category-theoretic representation of general analogy by faithful functors

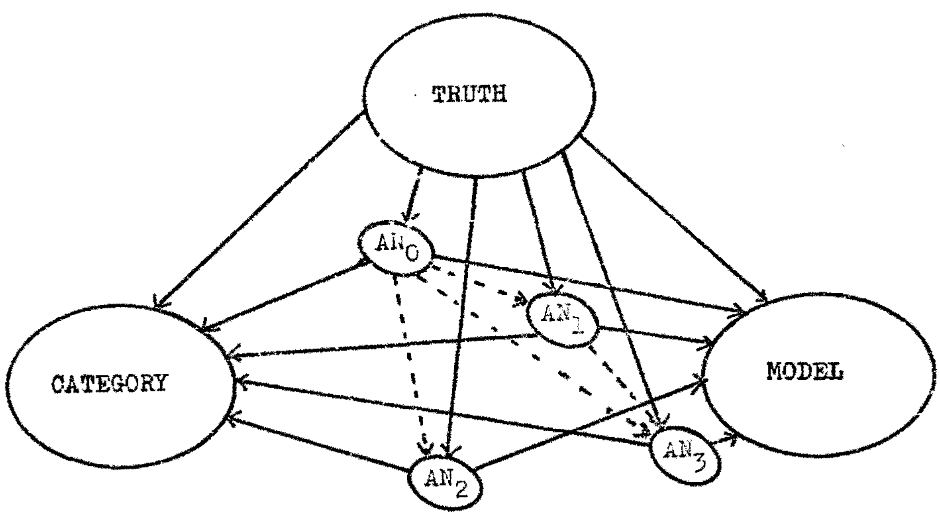

Figure 9 Semi-lattice of analogies (⟶ necessary, --→ possible)

two analogies between the same systems cannot compared directly then there is a unique common analogy (their greatest lower bound) that expresses their maximum mutual content. The semi-lattice ordering of analogies seems to correspond to what we mean by one analogy being "more comprehensive", "closer", or "more detailed", than another.

The role of the truth category can be seen as that of a constraint ensuring the relevance of an analogy. The non-existence of a maximal element making the semi-lattice into a lattice corresponds to the possibility of forming different and incomparable analogies between two systems. Two analogies being incomparable corresponds to people having different "points of view": you may form an analogy which helps you and I may form a very different one one that helps me, but providing they are both adequate for the task in hand (have the truth, at least, in common) the theory presented here does not judge between them.

It is important to note that, given a CTM triple, it is possible to compute the complete semi-lattice of possible analogies between them. Clearly it is also feasible to select out only the maximal elements to provide the set of the largest, incomparable analogies.

CONCLUSIONS

We have shown how the notion of an analogy between two systems can be formalized as a partial correspondence between two categories. We have shown that this partial correspondence is naturally represented by the introduction of an analogy category from which their is a faithful functor into each of the corresponding categories. We have introduced the notion of a truth-category as corresponding to what we mean by a relevant analogy. We have shown how all the possible analogies between two systems form a semi-lattice with the truth-category as the minimal element and with the order relation corresponding to the comprehensiveness of the analogy. In conclusion we would claim that the formal approach adopted here is adequate to provide complete foundations for the analysis of analogy.

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