Visualizing logical aspects of conceptual structures

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Abstract

Conceptual structures are studied in many different disciplines and represented in a variety of forms including natural language, formal logic, and graphical or spatial representations. This article analyzes the common foundations of such representations across all disciplines, and the relationships between the different forms of representation. Representations of the logical relations in conceptual structures are compared, including logical symbolism, Euler diagrams, semantic networks, conceptual grids and conceptual spaces. It is shown that these representations are formally equivalent and can be inter-translated algorithmically, but provide different and complementary visualizations such that the use of multiple representations may provide greater insight than any alone. It is shown that a wide range of significant conceptual structures can be represented and visualized in a substructural logic having only two logical relations, entailment and contrast/opposition, that is naturally represented in Euler diagrams and in semantic networks with two types of connecting arrow. It is shown that human everyday reasoning not involving definitions and rules but based on abduction over schemata representing traces of past experience can be modeled and visualized in these representations. The extension of the visual representation to include the constructs of a description logic and bridge from the substructural logic to structural mathematical logics is illustrated. It is concluded that interactive, computer-based visualization tools supporting a range of different representation schemes and inference based on a heterogeneous mix of representations can provide significant support for education and research across the many disciplines concerned with conceptual structures.

1 Introduction

Many disciplines, including linguistics, psychology, anthropology, sociology, philosophy and mathematics, study relationships between various forms of conceptual structure. The terminology varies across disciplines: concepts, constructs, categories, universals, properties, tropes, attributes, taxonomies, genera, determinables, determinants, predicates, propositions, statements, postulates, axioms, words, adjectives, adverbs, nouns, and so on; and there are disciplinary nuances in how each of these terms is intended to be used. However, there is much commonality across disciplines in the ways in which logical relationships between conceptual structures are modeled and represented symbolically, graphically, spatially, and so on.

The research reported in this article is concerned with the relationship between these various representations of conceptual structures, in particular: the formal links between the spatial, graphical and symbolic notations; and the relative perspicuity of the notations in conveying what is represented both naturally and correctly to specialists and non-specialists in the disciplines. It is shown that a strict algorithmic correspondence may be defined between several forms of representation such that each has the same logical power and is capable of representing the same range of conceptual structures.

It is shown that some inferences about the structures are more readily apparent in one representation than another, and using them together as complementary perspectives may be more insightful than using any alone. However, some inferences based on aspects of the notation that do not carry representational information may lead to invalid conclusions, emphasizing the need to convey precisely the valid usage of the notation to those interpreting it. The use of any notation requires skill in proper interpretation, and the ‘naturalness’ of a graphical or spatial representation while making some valid inferences more intuitive may also make some invalid inferences equally compelling.

This research parallels recent developments in the study of Euclid’s geometry where Hilbert (1902) and Tarski’s (Tarski and Givant, 1999) development of axiomatic formulations of the *Elements* was a major advance in the rigor of its proofs. It avoided false inferences derived from misleading figures (Maxwell, 1959) and led to widespread acceptance of the notion that “the ‘general triangle’ drawn on the page has no genuine role to play in the reasoning” (Tennant, 1986). However, it also reduced the comprehensibility of geometric proofs relative to those conveyed naturalistically through Euclid’s diagrams (Miller, 2007). In recent years there have been major advances in providing rigorous logical semantics for the roles of diagrams in geometric proofs (Miller, 2007; Avigad, Dean and Mumma, 2009), that axiomatize the diagrammatic aspects of Euclid’s proofs as an integral component of the axiomatization of his geometry.

The essence of these recent developments is that they specify a precise interpretation of legitimate diagrams such that logical inference based on that interpretation is sound, and inference that goes beyond it is not rigorous and likely to be in error. Since the interpretation is essentially a translation of the diagram into a logical formalism, there can be no claim that diagrammatic reasoning is supra-logical. Fricke (2003) has already noted the misleading nature of such a claim for Hyperproof (Barwise and Etchemendy, 1994) in that the reason why the visual reasoning in that system was more powerful than its symbolic reasoning was the impoverished representational capabilities of the latter. On the other hand, empirical evidence that the diagrammatic reasoning in Hyperproof was more readily accomplished than the equivalent symbolic reasoning is a reasonable outcome.

The integration of diagrammatic and symbolic representation/reasoning for geometry provides a role model for any visual language for logic: there should be a precise interpretation of the visual language in logical terms; and users need to be trained in the skills to follow that interpretation even if is naturalistic, since there may be other naturalistic inferences that are misleading.

The primary objective of the research reported in this article is to develop a framework for the study and development of visual languages for the logics of conceptual structures that satisfies these criteria and can be used to support studies of knowledge representation and inference from psychological, cultural, philosophical and computational perspectives. Some historical, philosophical, psychological and linguistic background will be discussed where it provides a context for discussing the role and value of different visual representations of the logic of conceptual structures. However, the intention is not to attempt to resolve issues in those disciplines, but rather to illustrate how the use of visual representation might clarify scholarly debate within and across them.

2 What are conceptual structures?

For much of what is discussed it would be appropriate to take the notion of a 'conceptual structure' as primitive, choose a generic term for such structures from those listed in the first paragraph, and focus on the possible logical relations between such structures and the visualization of these relations. However, there are major scholarly literatures associated with each of those terms and they all have technical connotations, often multiple variant ones, so that it would be better to adopt a more neutral term, but one with colloquial connotations that fit the general roles of conceptual structures in human activity.

The words 'term' as used in classical logic to denote either subject or predicate, and 'structure' as an abbreviation of 'conceptual structure' come to mind, but both have colloquial connotations useful to the discussion and it would be confusing to use them also as technical terms. Goodman and Elgin (1988) have addressed the problem by using 'label' as a neutral term with few connotations. We have used the word 'distinction' (Gaines and Shaw, 1984) as a generic term for a conceptual structure following Brown's (1969) notion of "making a distinction" and the connotations he ascribes to it: "a universe comes into being when a space is severed or taken apart...By tracing the way we represent such a severance, we can begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying linguistic, mathematical, physical and biological science, and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance."

However, making a distinction seems a somewhat passive word for an activity that may 'sever a space,' and we have moved to the notion of 'fitting a templet' as capturing the essence of the way in which conceptual structures are used, and not overloading terms that already have significant technical connotations. In this article the generic term 'templet' will be used for the conceptual structure imputed to underlie the process of making a distinction. It nicely accommodates all the various terminologies for conceptual structures listed at the beginning of Section 1, and has appropriate connotations.

Dictionary definitions illustrate how the term 'templet' captures significant roles played by conceptual structures in human activity. "A model or standard for making comparisons," emphasizes the role of a conceptual templet in enabling experiences to be *compared*. "A pattern or gauge used as a guide in making something accurately," captures the role of a psychological templet in *shaping* experience, that something is modified to fit the templet. Such modification supports a model of "action as the control of perception" as developed by Powers (1973). A conceptual structure is often imposed through an active process of changing the world, not just a passive process of gauging whether the world fits the associated templet.

The term *templet* was first used in this sense by the constructivist psychologist, George Kelly: "Man looks at his world through transparent patterns or templets which he creates and then attempts to fit over the realities of which the world is composed. The fit is not always very good. Yet without such patterns the world appears to be such an undifferentiated homogeneity that man is unable to make any sense out of it. Even a poor fit is more helpful to him than nothing at all." (Kelly, 1955, p.8-9). He subsumed the various uses made of conceptual structures under the notion of 'anticipation' drawing on Dewey's notion that anticipatory processes underlie all psychological processes: "Ability to anticipate future consequences and to respond to them as stimuli to present behavior may well define what is meant by a mind or by consciousness." (Dewey, 1917, p.28).

Kelly's axiomatic theoretical psychology formalizes Dewey's pragmatic instrumentalism which accepted Hume's (1888) argument there is no logical rationale for it to be possible to anticipate future events, and hence it is an empirical phenomenon that the world we live in exhibits patterns that sometimes enable future experience to be anticipated from past experience. As Dewey (1911) notes: "While there is no *a priori* assurance that any particular instance of continuity will recur, the mind endeavors to regulate future experience by postulating recurrence. So far as the anticipation is justified by future events, the notion is confirmed. So far as it fails to work the assured continuity is dropped or corrected."

The connotations of conceptual structures as templets span the range of meanings that Dewey and Kelly accommodate within the term *anticipation*: of *prediction* of what may happen; of *action* to make something happen; of *creative imagination* of what might happen or be made to happen; and of *preparation* for eventualities that may well never happen. The term *templet* will be taken as a generic primitive encompassing all other terms for conceptual structures with expectation that the specific issues discussed in the literature for various other terms can be represented within a logic of templets, and that significant similarities and differences will emerge.

3 The logic of templets

A major impediment to developing a logical framework for human psychological processes is that notions of what is a *logic* have been highly conditioned by the success of mathematical logic as developed by Boole, Frege, Hilbert, Russell, Tarski, Gödel, *et al* (Kneale and Kneale, 1962). We take for granted the logical constants and modes of inference of such logical systems and seek to impose them on all phenomena, in particular using notions of logical conjunction, disjunction, negation, definition and rules that may be inappropriate to human reasoning. Empirical psychological studies of human rationality (Shafir and LeBoeuf, 2002) then puzzle us because people do not use such notions 'correctly' and their reasoning processes appear 'irrational.' Hence, it is important to analyze the logic of conceptual structures using a minimalist logical framework that presupposes no more than is necessary to account for the phenomena of interest.

We will take a *templet* having a binary valuation of *fit* as a logical primitive, and leave as extra-logical issues what it is that a templet fits, such as 'experience,' 'phenomenon,' 'situation,' 'reality,' and so on, and how the fit is accomplished, whether a fit is possible, and so on. These extra-logical issues are important but it should be possible to analyze them through the definition of additional templets to represent them, essentially meta-templets—derived rather than primitive notions.

If we consider the possible logical relations between templets that indicate whether one templet will fit if the other one does, then there are two possibilities: *entailment,* if templet A fits then so will templet B; and *contrast/opposition,* if templet B fits then templet C will not. We can represent these symbolically by arrows of entailment and contrast/opposition:

$A \rightarrow B$ for A entails B, $B \nrightarrow C$ for B is in contrast/opposition to C

Entailment is a transitive, asymmetric binary relation between templets, and contrast/opposition is an intransitive, symmetric one. The two relations interact in that if both the relations above hold we can also derive A $\nrightarrow$ C. The symmetry of contrast/opposition would allow a non-directional symbol to be used to represent it, and the symbol "—" has sometimes been used for this purpose (Gaines, 1991), but the logical symmetry masks an underlying cognitive asymmetry that is reflected in linguistics through such distinctions as that between the *marked* and *unmarked* sides of a contrast (Lyons, 1968).

These two logical relations, of entailment and contrast/opposition, were studied in classical times as constraints upon the meaning of terms by Plato and Aristotle, and have played major roles in psychological, philosophical and linguistic studies ever since. For example, in linguistics, Humboldt, Saussure, Abel, Trier, *et al*, saw them as the fundamental relations between the meaning of words constituting a semantic 'field,' 'mosaic' or 'network', and argued that the meaning of a word was determined by its location in a network of other words related in meaning by entailment and contrast (Reuning, 1941; Nerlich and Clarke, 2000).

The relations are constraints that exist prior to the fitting of templets, committing us to asserting that if one templet fits then some related templets also fit and some others do not. They support memory and communication and, if we change them, we undermine both. They are not in themselves empirical but they facilitate empirical processes. If we use different relations from others in our community we are not asserting different facts but rather talking a different language from them. The relations do not tell us how to fit a templet but constrain what other templets we may fit when some have been fitted. Induction does not determine the relations but may enable us to infer them from the verbal behavior of others. Developmentally, the entailment and contrast relations between words are learnt before the capability to fit those words to experience (Soja, 1994).

Relations between templets are not necessarily encoded specifically but may be generated by considerations of the internal structure of templets, for example that "has red hair" entails "has hair" and contrasts with "has black hair". Factoring the logic of such internal constraints gives rise to other logical forms such as relational structures, but these are secondary to the fundamental relations of entailment and contrast/opposition. What we require of any particular model of such internal constraints is that it be able to derive the two basic relations from the specified internal structures.

There may also be meta-relations between templets where one templet is fitted to another or to a templet structure, for example, that a templet may be difficult to fit or that two templets have a particular relation. Such higher-order templets need to be distinguished from lower-order ones to avoid category errors, for example to say red is a color is not to say that red is colored but rather that if an object is red it is entailed that it is colored (Johansson, 1989, p.15). However, the logic of entailment and contrast/opposition relations between templets applies at every level without implying any interaction between levels.

The initial focus of this article will be on the various representations of the two basic relations of entailment and contrast/opposition between conceptual structures. Further sections will show how these inter-translate, providing different perspectives on the same conceptual structure, how much of human reasoning can be captured through these two relationships, and how mathematical logic emerges through extension of the relations and their representation.

4 Visualizing the logic of templets

Any pair of symbols that can represent a transitive, asymmetric binary relation and an intransitive, symmetric one that interacts with it as noted above, can be used to portray the relations between templets representing conceptual structures. If that portrayal can be naturalistic without conveying any relations other than those represented then it may support the comprehension of those structures. It is must also be able to represent the lack of either relation between templets, that is, conceptual structures where there are no mutual constraints.

One natural visualizable phenomenon with the desired properties is the spatial enclosure of closed curves in some manifold. The enclosure of one enclosure by another is a transitive, asymmetric relation, and the separation of a pair of enclosures is an intransitive, symmetric relation that interacts with it in the desired manner. Throughout the history of logical scholarship one-dimensional lines, separated in a second dimension for visual perspicuity, and two-dimensional closed curves, have both been used to represent the logical relations between conceptual structures (Greaves, 2002, p.115-121).

Figure 1 on the left shows four templets A, B, C and D represented by horizontal lines. The relation A → B is represented by the line for A being included in the line for B, and the relation B ↛ C represented by the separation of their lines. Non-inclusive overlap, such as that between the lines representing D and those for B, C and D, indicates that neither logical relation applies.

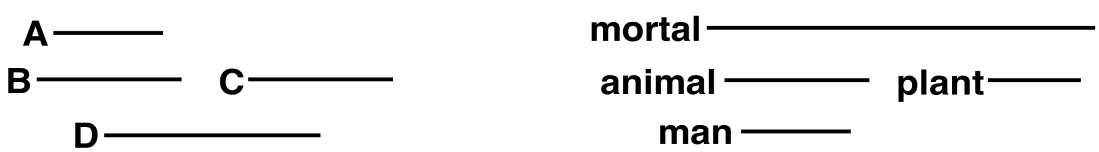

Fig. 1 Line diagrams for entailment and contrast/opposition relations and syllogisms

On the right some moods of syllogistic reasoning are represented in this way:

animal→ mortal together with *man→ animal* implies that *man→ mortal* — *Barbara*

animal ↛ plant together with *man→ animal* implies that *man ↛ plant* — *Celarent*

animal ↛ plant together with *man→ animal* implies that *plant ↛ man* — *Camestres*

Barbara and *Celarent* are the classical mnemonics for two moods of the first figure of Aristotle's syllogistic to which nearly all the other syllogisms may be reduced (Weidemann, 2004), and syllogisms in general and their extension to more than three terms can be represented by the two relations of entailment and contrast/opposition and their various representations. Engelbretsen (1992) has extended linear diagrams for syllogisms to include relationals, and used this in an exposition of Sommers' (1982) logic of terms for modeling reasoning in natural language (Englebretsen, 1996).

Two-dimensional enclosure representations are generally called 'Euler diagrams' following Euler's extensive use of them to explain syllogisms in his tutorial letters to a German Princess (Euler, Brewster and Griscom, 1840, Letters 102-105). Euler describes the general forms of the syllogism and then illustrates them using circles, remarking, "These four species of propositions may likewise be represented by figures, so as to exhibit their nature to the eye. This must be of great assistance towards comprehending more distinctly wherein the accuracy of a chain of reasoning consists." (Letter 102). Figure 2 shows the conceptual structures of Figure 1 represented by Euler circles.

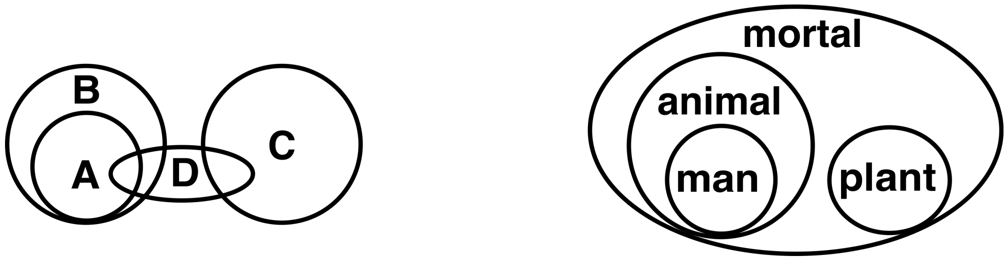

Fig. 2 Euler circles for entailment and contrast/opposition relations and syllogisms

Euler diagrams and their extensions have been widely studied formally and computationally as tools that can support the human reasoning process (Fish and Flower, 2005; Stapleton, 2005; Swoboda and Allwein, 2005; John, Fish, Howse and Taylor, 2006; Mineshima, Okada and Takemura, 2009), and have been used as computational interfaces to computer directory and library catalogue systems (De Chiara, Hammar and Scarano, 2005; Thièvre, Viaud and Verroust-Blondet, 2005).

Venn's (1881) book on symbolic logic presents a new form of diagram designed to support Boole's (1848) mathematical representation of the syllogistic. Chapter 1 illustrates syllogisms using Euler diagrams but Venn criticizes their utility in proving theorems where the relations between propositions have to be inferred rather than specified in advance. In Chapter 5 he presents a new form of diagrammatic representation that has come to be known by his name. Venn's notation is based on Euler diagrams where all the circles overlap with one another, which would mean in Euler's interpretation that there are no relations between the propositions represented, that is, when one templet is fitted it places no constraints upon whether any of the other templets fit. Venn then shades the areas of intersection between the circles, or more generally closed curves, to indicate that no other circle can be placed in that area. Venn and Euler diagrams thus have the same capability to represent relations between templets but do so in different ways: Venn by specifying the general situation and then shading it to develop a particular one; Euler by specifying the final situation immediately.

Figure 3 left shows Venn diagrams for two, three or four templets, and on the right the Venn diagrams for the relations $A \rightarrow B$ and $B \nrightarrow C$.

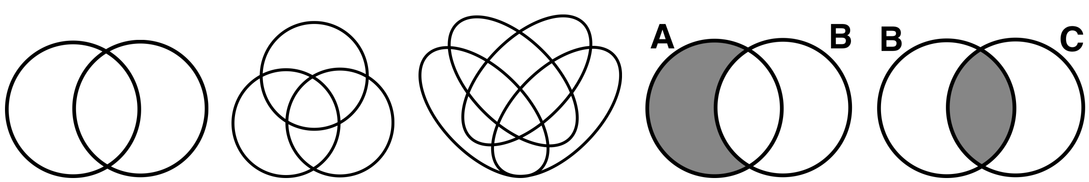

Fig. 3 Venn diagrams for 2, 3 and 4 templets, and for relations between 2 templets

The principles of representation of entailment by spatial inclusion and contrast/opposition by spatial separation remain the same as for Euler diagrams, but the creation of the shapes is through the shading of a generic Euler diagram. Figure 4 shows the Venn representations of the relations already shown in Figures 1 and 2.

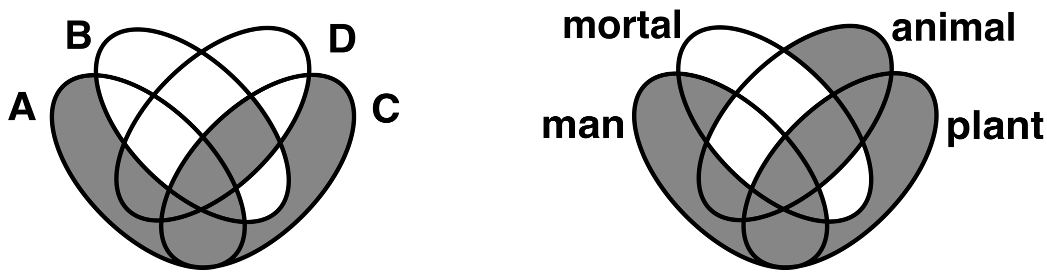

Fig. 4 Venn diagrams for entailment and contrast/opposition relations and syllogisms

Because they divide space into the 2^n possible intersections of n closed curves Venn diagrams become increasingly complex for n beyond four and are most familiar for n=3. They are often used to show how three binary properties divide possible exemplars into coherent sets, rather than for syllogistic inference *qua* Venn. In the general literature when Euler diagrams are used they are often termed Venn diagrams.

The two forms of arrow used in the linear symbolic representation of relations between templets generalize naturally to two-dimensional semantic network representations as shown by the representations of the examples of Figures 1, 2 and 4 on the left of Figure 5. On the right are shown two examples of interaction with a computer programmed to make deductions from the assertions that particular template fits to derive the fit of other templates. A vertical bar indicates that a template fits and a horizontal bar that it does not.

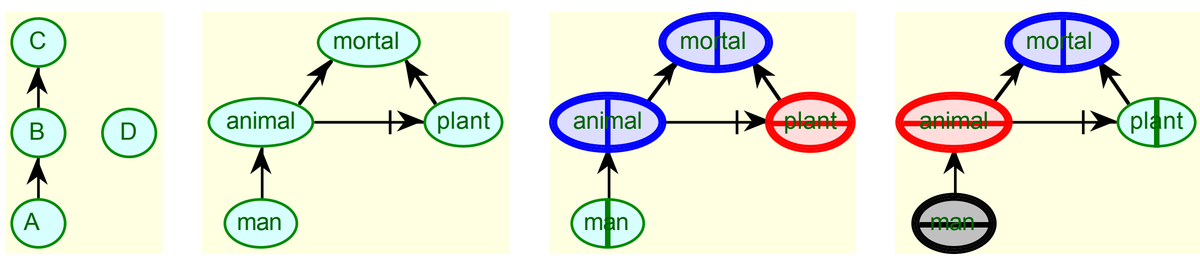

Fig. 5 Semantic networks for entailment and contrast/opposition relations and syllogisms

The leftmost of the two figures shows that when it is asserted that the templet 'man' fits it is inferred that 'animal' fits and hence that 'mortal' fits (syllogism *Barbara*) and that 'plant' does not (*Celarent*). The rightmost that when it is asserted that the templet 'plant' fits it is inferred that 'animal' does not fit and hence that 'man' does not fit (*Camestres*). The program indicates the source of the derivation by coloring the assertions green, the entailments blue, the oppositions red, and the inverse entailments black. The inverse entailments where a templet not fitting implies that none of its subordinates can fit is particularly significant because it can be seen as indicating the *relevance* of templets. If a superordinate templet does not fit then its subordinates are no longer relevant in the sense we should not waste effort in investigating their possible fit, and asking a question about them would seem strange since we should know they cannot fit.

This type of interactive user interface to an inference program can be implemented for any of the other graphic representations of conceptual structures shown in Figures 1 through 4. They are all logically equivalent to one another and diagrams in any of these forms inter-translate with diagrams in the other forms, linear logical symbolism and natural language.

5 Comparing different representations

The preceding section has described four graphic notations equivalent to the linear symbolic representation of relations between conceptual structures. This section considers how the representations differ in extra-logical features such as clarity of representation, and ease of understanding. Four major dimensions of comparison are apparent: do they scale up to represent complex conceptual structures; what naturalistic metaphors do they evoke when people attempt to understand them and use them to support reasoning processes; do they also evoke misleading interpretations; how do they extend to support more logical constants than entailment and contrast/opposition? This section discusses the first three questions, and Section 10 addresses the fourth. Examples in the remainder of the article illustrate aspects of all four dimensions.

Line and Venn diagrams are both very useful in presenting simple conceptual structures, but do not scale up to represent more complex ones. In particular, the representation of all possible combinations in Venn diagrams is inappropriate for complex conceptual structures because people manage the exponential increase in complexity by ensuring that a high proportion of combinations are irrelevant to any particular application. This corresponds to the majority of the 2^n intersections being grayed out, but usually with little scope for this to be crafted into a visual form that clarifies the conceptual structure.

Euler diagrams and semantic networks do both scale to complex conceptual structures often allowing issues associated with them to be clarified in a helpful way, and this article will focus on these two forms of graphic representation, extending them to conceptual spaces in Section 8. One major difference between the two representations has already been noted: Euler diagrams represent templets having no mutual constraints by partially overlapping circles, and semantic nets do so by separated circles with no interconnecting arrows; conversely, Euler diagrams represent contrasting templets by separated circles and semantic networks by a connecting arrow. Hence Euler diagrams, as illustrated in Section 7, become complex when two or more structures that have no mutual constraints need to be represented, for example, orthogonal dimensions such as taste and color.

Semantic networks represent such structures without difficulty as separate sub-networks, but have problems with large numbers of mutually contrasting structures where there needs to be a negated arrow between every structure and every other one. This is typically the situation for distinct enumerable templets such as those representing individuals, and semantic network systems usually adopt additional notational features to represent mutually contrasting structures, such as a rectangle around the unique identifier of an individual.

The two representations differ in the naturalistic metaphors that they evoke for the entailment and contrast/opposition relations and the logical inferences that can be drawn from them. Euler diagrams evoke the metaphor of spatial enclosure, with separated enclosures being distinct, and enclosures being enclosed in any enclosure of their enclosures. If we place an additional templet in the Euler diagram representing a conceptual structure it is immediately apparent what encloses it, what overlaps it, and what is separated from it. It is also apparent where we can place further templets relative to it that achieve certain relationships, and so on. Our natural grasp of spatial relationships through visual perception becomes a powerful logical tool.

Semantic networks evoke a flow metaphor, that the fit of one template flows outward through the arrows of entailment activating the fit of other templets, through the arrows of contrast/opposition inhibiting the fit of other templets, and that the lack of fit of a templet flows backwards through the arrows of entailment inhibiting the fit of other templets subordinate to it. The visual flow forward from a templet that fits through the outgoing arrows of entailment is very similar to the visual flow through layers of enclosure. The visual flow backward from a templet that does not fit through the incoming arrows of entailment to irrelevant templets is very similar to the visual flow through layers enclosed containing irrelevant templets. Flow through arrows of contrast/opposition is less perspicuous than visual separation, but in many common conceptual structures, such as taxonomies, it is highly localized so that only one such arrow is involved in any particular inference.

One potentially misleading aspect of spatial representations of the logical relations as graphical structures is that the spatial layout portrayed has not only a topological aspect that conveys the underlying logical relations, but also a metric aspect that is only loosely constrained by them and should not be used in inference. For example, the size and relative orientations of the Euler circles in Figure 2 are a matter of visual convenience and carry no logical import; it is only the containment, overlap and separation relations that convey the logical structure. Similarly, the relative positions of the nodes in the semantic networks of Figure 5 are irrelevant to the logical structure which is completely encoded in the connecting arrows. As will be shown in Sections 7 and 8, there are natural spatial relationships that arise from the logical structure, giving rise to the notion of a 'conceptual space' (Gärdenfors, 2000), and the way we lay out diagrams often reflects this. However, the spatial layout may well reflect nothing more than convenience in the way that it is drawn, and, in general, it can be misleading to read more than that into it.

The impact of the differences in the representation of relations between templets are discussed further and illustrated through examples in the following sections.

6 Visualizing taxonomic structures

Conceptual structures are human constructs which evolve to support human activity and their logical structure emerges through that process of evolution and reflects the nature of that process. For example, we may fit a templet in order to classify our current experience in order to compare it with past experience, or shape it to be like some past experience, in order to anticipate further aspects of the current experience. If the outcome is unsatisfactory we may change the templet we are using to a contrasting one, or may refine it further through contrasting sub-templets that entail it but make additional distinctions. This process leads naturally to the growth of taxonomic structures of templets classifying and shaping our experience. The contrasting sub-templets of 'mortal', 'animal' and 'plant' in the conceptual structure represented in various ways in the preceding section, illustrate the nature of this process.

De Morgan (1847) discusses the logic of such conceptual structures having two contrasting propositions both entailing the same proposition which constrains where they are applicable, and terms the constraining templet the 'universe' of the contrasting propositions. Boole (1848) adopted the term as the 'universe of discourse' in the exposition of his logic, and Kelly (1955) terms it the 'range of convenience' of a 'bipolar construct,' the fundamental building block of his psychology. The essence of such a construct is the way in which the templet "mortal" characterizes a significant similarity between sub-templets such as a man and a tree, and the contrast between templets "animal" and "plant" characterizes a significant difference. The line diagram of Figure 1, Euler diagram of Figure 2 and semantic network of Figure 5 all represent this structure of similarity and difference very clearly without the additional subordinate templet "man" in any way obscuring it. A Venn diagram with three circles also represents the triad clearly, but the representation of a fourth templet, "man," obscures it, illustrating the problems of scalability for Venn diagrams.

The growth of a taxonomic structure through the refinement of subordinate templets to represent a universe of discourse for further contrasts is illustrated in Figure 6. On the left is a conventional representation of a taxonomy of which the mortal-animal-plant construct is part, with the entailment relation represented by a line and its direction by the vertical ordering, and the contrast/opposition relation by the vertical alignment and horizontal separation of the contrast pairs. In the center is an Euler diagram representation of the taxonomy where the relations are represented through enclosure and separation, and on the right a semantic network where the relations are represented by two types of arrow.

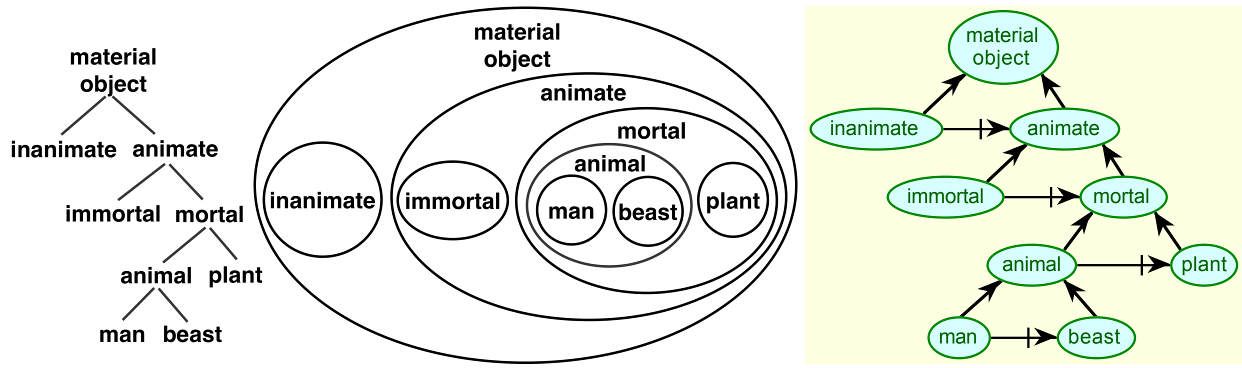

Fig. 6 Taxonomy with Euler diagram and semantic network representations

The linear symbolic representation of this taxonomy is:

$$\text{inanimate} \lor \text{animate} \rightarrow \text{material object}, \quad \text{inanimate} \nrightarrow \text{animate}$$
$$\text{immortal} \lor \text{mortal} \rightarrow \text{animate}, \quad \text{immortal} \nrightarrow \text{mortal}$$
$$\text{animal} \lor \text{plant} \rightarrow \text{mortal}, \quad \text{animal} \nrightarrow \text{plant}$$
$$\text{man} \lor \text{beast} \rightarrow \text{animal}, \quad \text{man} \nrightarrow \text{beast}$$

where the disjunctions on the left are convenient abbreviations for two separate entailments with no implication that there is a templet representing the disjunction. All four representations convey the same information and do so formally in that each can be translated algorithmically to the same linear symbolic form. The use of vertical ordering and horizontal separation in the conventional diagram works very well for taxonomic structures, and is still apparent in the other representations, but does not generalize well to non-taxonomic conceptual structures where the more explicit representation of the relations in the other diagrams enables layouts to be used that do not have to conform to spatial relationship conventions.

Taxonomies have a simple and elegant structure because they show only one family of contrasts refining their superordinate templet, and one entailment of each templet. This reflects the origins of taxonomies in the representation of biological species where evolutionary speciation splits one species into others that have much in common initially but diverge in their characteristics because members cannot inter-breed. More generally however, there may be several families of templets refining a common superordinate templet, and large numbers of entailments reflecting connotations of a templet. For example, the templet "mortal" might be refined through the contrast “short-lived—long-lived” rather than, or as well as, “animal—plant,” and the templets “animal” and “plant” both have many more entailments beyond “mortal.”

The biological schema has been extended to other forms of ‘natural kind’ by abstracting the criterion that ‘kinds’ have large numbers of entailments in common (Hacking, 1991). The “animal—plant” contrast differentiates kinds of “mortal” through two templets each of which has many more significant entailments than does “mortal”, whereas the contrasting templets generated by “short-lived—long-lived” does not. It is usual to term “short-lived” a ‘property’ and “animal” a ‘kind’ to mark this distinction between contrasts that are logically the same in structure but have different impacts in the way they partition a conceptual structure. However, there is no absolute basis for the distinction since what are significant entailments depends on what use we are making of the conceptual structure, and communities with different objectives develop different taxonomies (Medin, Lynch, Coley and Atran, 1997).

There is an analogy between chemical structures as bonds between atoms and conceptual structures as bonds between templets. Section 4 introduced the basic notions of templets and the logical relations that bond them, and illustrated a fundamental construct of contrasting templets entailing a common superordinate templet, similar in status to that of a benzene ring. Section 6 showed how more complex structures are formed through bonding multiple instances of this basic construct to form taxonomies, just as benzene rings bond to form complex organic molecules. Just as in chemistry, the graphic presentations make these structures, their relationships, their construction and their properties, more perspicuous than a natural language description or linear symbolic representation.

There are several other generic ‘molecular’ conceptual structures that are also common components of more complex conceptual structures, and the following sections discuss some of the major ones.

7 Deriving and visualizing the spatial structure of conceptual dimensions

The basic construct of contrasting templets entailing a common superordinate templet constraining their universe of discourse has been illustrated for a pair of contrasts but generalizes to ‘contrast sets’ (Frake, 1969) where there are several mutually contrasting templets refining a common superordinate. Johnson (1921) introduced the terminology ‘determinable’ for the superordinate templet, and ‘determinants’ for its subordinate contrast set, and this is widely used in the philosophical literature. The taxonomy illustrated in the previous section shows that a given templet can be both a determinant for a templet it entails and a determinable for those which entail it.

Johnson introduced this terminology as part of an analysis of conceptual structures which he argued did not fit Aristotle’s notion of *genus* and *differentia*, for example that the concept red was subordinate to color but had no entailments to differentiate it, for example, from green—something is attributed the property of red without there being any specifiable reason. This is arguably a specious distinction if one accepts as entailments of red, “evoking the perception I have learned to term ‘red’,” “being termed by others whose perception I trust as ‘red’,” and so on. However, other parts of Johnson’s analysis are significant, in particular that there are often additional relationships between determinants other than contrast, such as ordering and graduation along a scale, a well-known phenomenon linguistically (Cruse, 1986) and psychologically (Rosch and Lloyd, 1978).

The ordering and grading of templets in a contrast set is not an immediately apparent consequence of the logical relations of entailment and contrast/opposition between them, and there is a temptation to assume that it must be introduced as the extra-logical imposition of a metric space (Denby, 2001), or by the use of a multi-valued logic (Fourali, 2009) such as Łukasiewizc infinitely-valued logic (Rescher, 1969), nowadays often known as 'fuzzy logic' (Gaines, 1976) following Zadeh's (1965) terminology. However, Gaines (2009) shows that order relations between templets emerge naturally from the logical relations between templets and can be derived algorithmically.

Figure 7 show an Euler diagram for the conceptual structure that Gaines (2009, Fig.10) uses as an example of a logical structure equivalent to a rating scale.

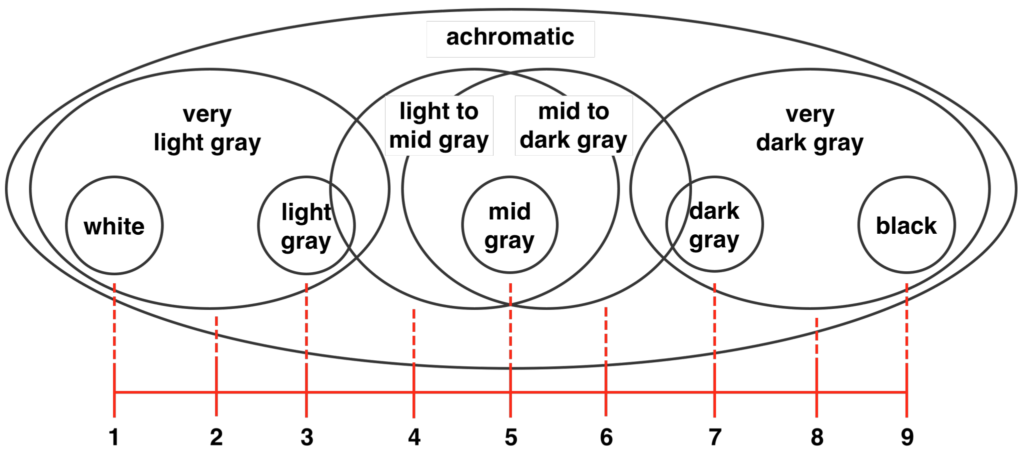

Fig. 7 Euler diagram for a graded conceptual structure with nine points on a scale

Figure 8 shows the same structure as a semantic network with five-, seven- and nine-point scales indicating the graded structure.

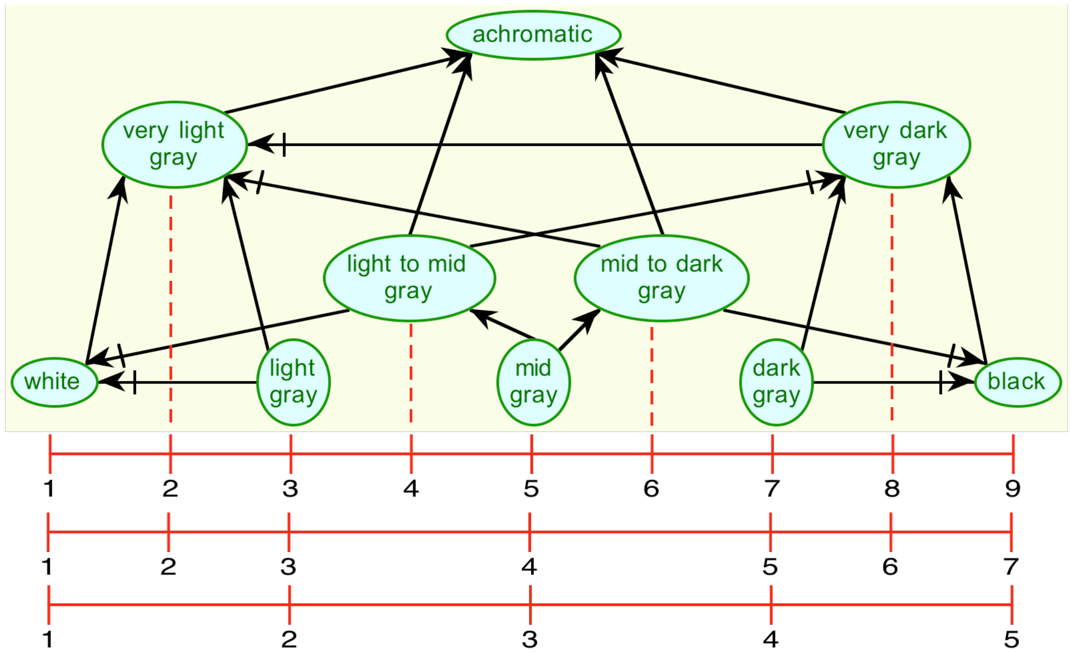

Fig. 8 Semantic network for a graded conceptual structure

The linear symbolic representation of this structure is:

very light gray ∨ light to mid gray ∨ mid to dark gray ∨ very dark gray → achromatic

very dark gray ↛ very light gray

white ∨ light gray → very light gray, light gray ↛ white

black ∨ dark gray → very dark gray, dark gray ↛ black

light to mid gray ↛ white ∨ very dark gray

mid to dark gray ↛ black ∨ very light gray

mid gray→ light to mid gray ∧ mid to dark gray

Gaines derives the scalar relation between the templets by associating each templet with a feature vector of the other templets that fit or, in complemented form, do not fit when the primary templet fits. The matrix whose columns are the feature vectors for each templet characterizes the network of relations in the conceptual structure intensionally in terms of its templets and is sufficient to regenerate that network. Kelly (1955, p.270) terms this matrix a 'conceptual grid' and his 'repertory grid' elicitation technique, that is widely used to elicit conceptual structures, generates such a matrix. Gaines notes that the cardinality of the symmetric difference between the feature vectors is a distance measure establishing a metric that characterizes the scalar relationship between templets. This metric is not imposed from without but generated algorithmically from the conceptual structure's internal relations.

Figure 9 shows the conceptual grid generated algorithmically from the conceptual structures represented in Figures 7 and 8. The first column indicates that if it is true that "achromatic" fits then there are no further implications about what other templets fit; the second that if it is true that "white" fits" then it also true that "achromatic" and "very light gray" fits, and false that any of the other templets fit; and so on for each templet.

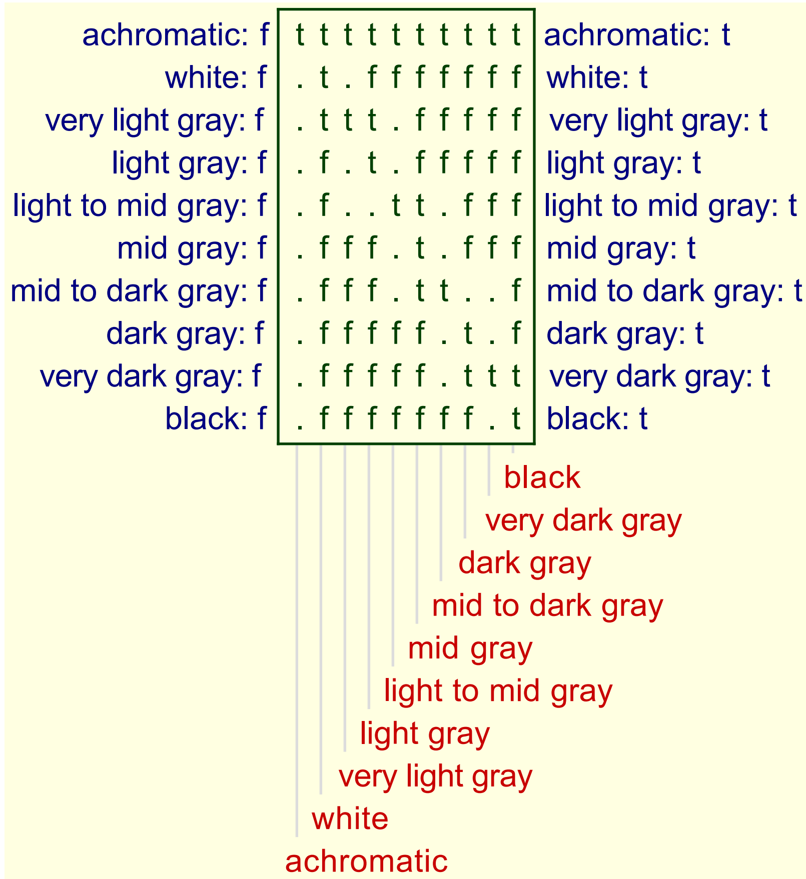

Fig. 9 Conceptual grid of feature vectors

Figure 10 shows the inter-templet distances derived as the cardinality of the symmetric difference between the set of features in the feature vectors for the four templets that Gaines takes as salient, "white," "very light gray," "very dark gray" and "black." For example, the achromatic templet entails none of the features and the white templet entails four of them, 'white: t,' 'very light gray: t,' 'very dark gray: f,' 'black: f,' so that the cardinality of the symmetric difference is 4.

	1	2	3	4	5	6	7	8	9	10	
1:	0	4	3	4	3	4	3	4	3	4	: achromatic
2:	4	0	1	2	3	4	5	6	7	8	: white
3:	3	1	0	1	2	3	4	5	6	7	: very light gray
4:	4	2	1	0	1	2	3	4	5	6	: light gray
5:	3	3	2	1	0	1	2	3	4	5	: light to mid gray
6:	4	4	3	2	1	0	1	2	3	4	: mid gray
7:	3	5	4	3	2	1	0	1	2	3	: mid to dark gray
8:	4	6	5	4	3	2	1	0	1	2	: dark gray
9:	3	7	6	5	4	3	2	1	0	1	: very dark gray
10:	4	8	7	6	5	4	3	2	1	0	: black

Fig.10 Inter-templet distances derived from conceptual grid based on salient attributes

This distance matrix can be analyzed for its underlying spatial structure through multi-dimensional scaling (Davison, 1983), a technique that Gardenförs (2000, Sect.1.7) uses to derive conceptual spaces from psychological data. Figure 11 shows a plot of first two principal components of such an analysis of the data of Figure 9.

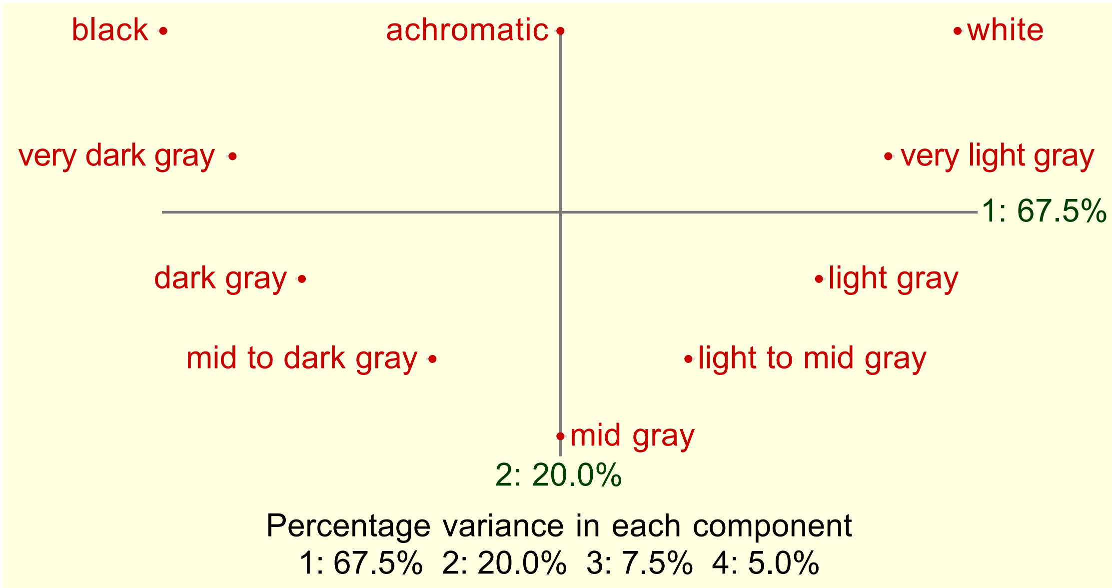

Figure 11 Multi-dimensional scaling of the inter-templet distances

The expected ordering of templets along a linear scale is apparent horizontally, and there is smaller but significant vertical component representing the similarity of the extremities of the scale in that they are both distant from the center of the scale.

This derivation shows that the order and similarity properties that Johnson (1921) discusses for determinables, and the metric space that Denby (2001) introduces to model them, need not be imposed *ad hoc* but can be derived from the logical relations between the determinants. It also provides an insight into the notion of 'salient.' Analysis of the distance matrix based on all the features in Figures 7 and 8 produces a similar model to that of Figure 11. A salient set of the features may be defined as a minimal subset of features that produces the same spatial structure of ordered dimensions as analysis of the full set. There may be more than one such subset, or 'basis.'

In a recent paper on counseling techniques for overcoming conflict arising from polarized thinking Fourali (2009) models 'shades of gray' through an Euler diagram of overlapping concepts illustrated in Figure 12, and suggests that this be modeled through fuzzy logic.

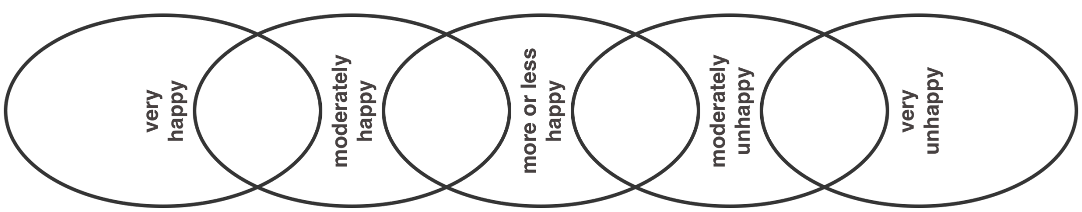

Fig. 12 Euler diagram: "semantic ambiguities between levels of happiness" (Fourali, 2009)

This conceptual structure is interesting because it is proposed to model the scalar properties of contrast sets entirely through the contrast relations. It also exemplifies Körner's notion in his debate with Searle on the nature of determinables (Körner and Searle, 1959) that it is the overlap due to 'inexactitude' of concepts in a contrast set that makes the set coherent.

The equivalent semantic network is shown in Figure 13.

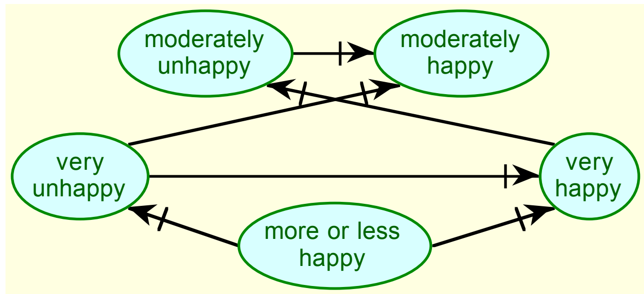

Fig. 13 Semantic network of semantic ambiguities between levels of happiness

The linear symbolic representation of this structure is:

moderately unhappy ↛ moderately happy

very unhappy ↛ very happy ∨ moderately happy

very happy ↛ moderately unhappy

more or less happy ↛ very unhappy ∨ very happy

Figure 14 shows the conceptual grid generated by the conceptual structures of Figures 12 and 13.

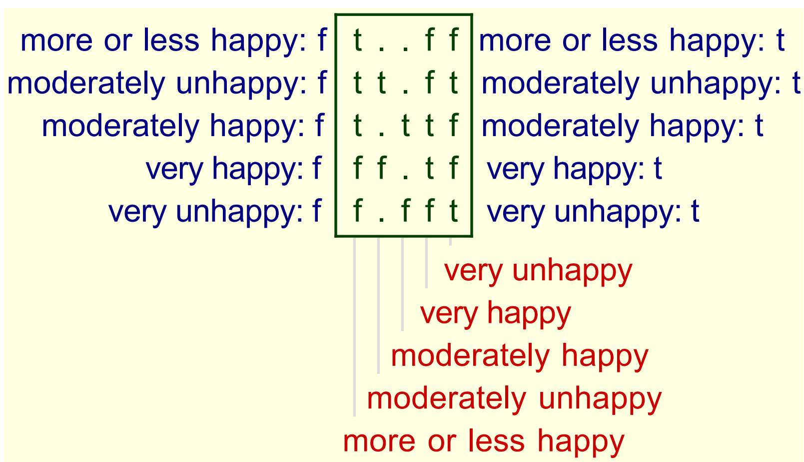

Fig. 14 Conceptual grid for happiness

Figure 15 shows the spatial structure resulting from multi-dimensional scaling of the distance matrix generated from Figure 14 and it can be seen that it verifies Körner and Fourali's hypothesis that chained overlap does imply an underlying scale.

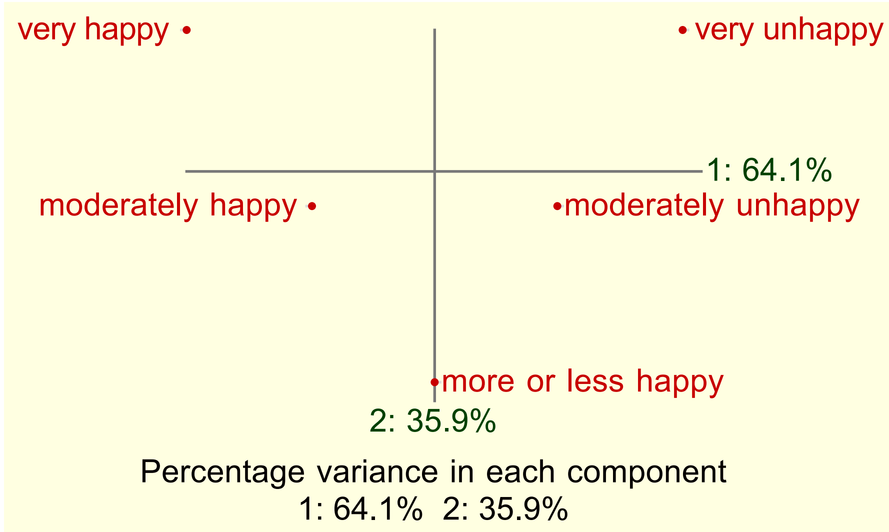

Fig. 15 Multi-dimensional scaling of the inter-templet distances for happiness

It is notable that a rating scale has emerged only from the standard logical relation of contrast/opposition, and that no non-standard logic has been imposed. In addition a second multi-valued logical dimension has emerged. While the horizontal axis in Figure 15 may be seen as representing a 'degree of membership' to happiness, the vertical dimension may be seen as one mediating between moderation and extremism in happiness. This second dimension is an important one psychologically because the preferred templet whose fit is homeostatically maintained is often that between the extremes, for example, in Berlyne's (1960) model of arousal, Csikszentmihalyi's (1990) of optimal experience, and Gaines' (1997) of the optimal error rate to maximize learning.

8 Deriving and visualizing conceptual spaces

The derivation of scalar dimension from the relations between templets representing the determinants of a determinable raises the question of whether a similar analysis of several independent systems of determinables and determinants will generate a multi-dimensional space. Kelly (1955, Ch.6) saw his conceptual grid as a tool for representing the "geometry of psychological space" (Shaw and Gaines, 1992), and Gardenförs (2000) has developed a theory of conceptual spaces providing a "geometry of thought." Is the spatial allusion only a metaphor, does the metric need to be imposed or is it latent in the logical relations forming a conceptual structure, and, conversely, are those relations also latent in the geometry of a psychological space?

The techniques described in the previous section for extracting the latent spatial dimensions of a conceptual structure may be used with any arbitrary structure, and in particular ones with several independent sub-structures that one might expect to correspond to different dimensions. For example, Figure 16 is an Euler diagram derived from that of Figure 12 by adding a second chain of overlapping templets for healthiness. Since they are independent of the first chain they overlap every member of it, and all the templets have been elongated and one set rotated to portray this.

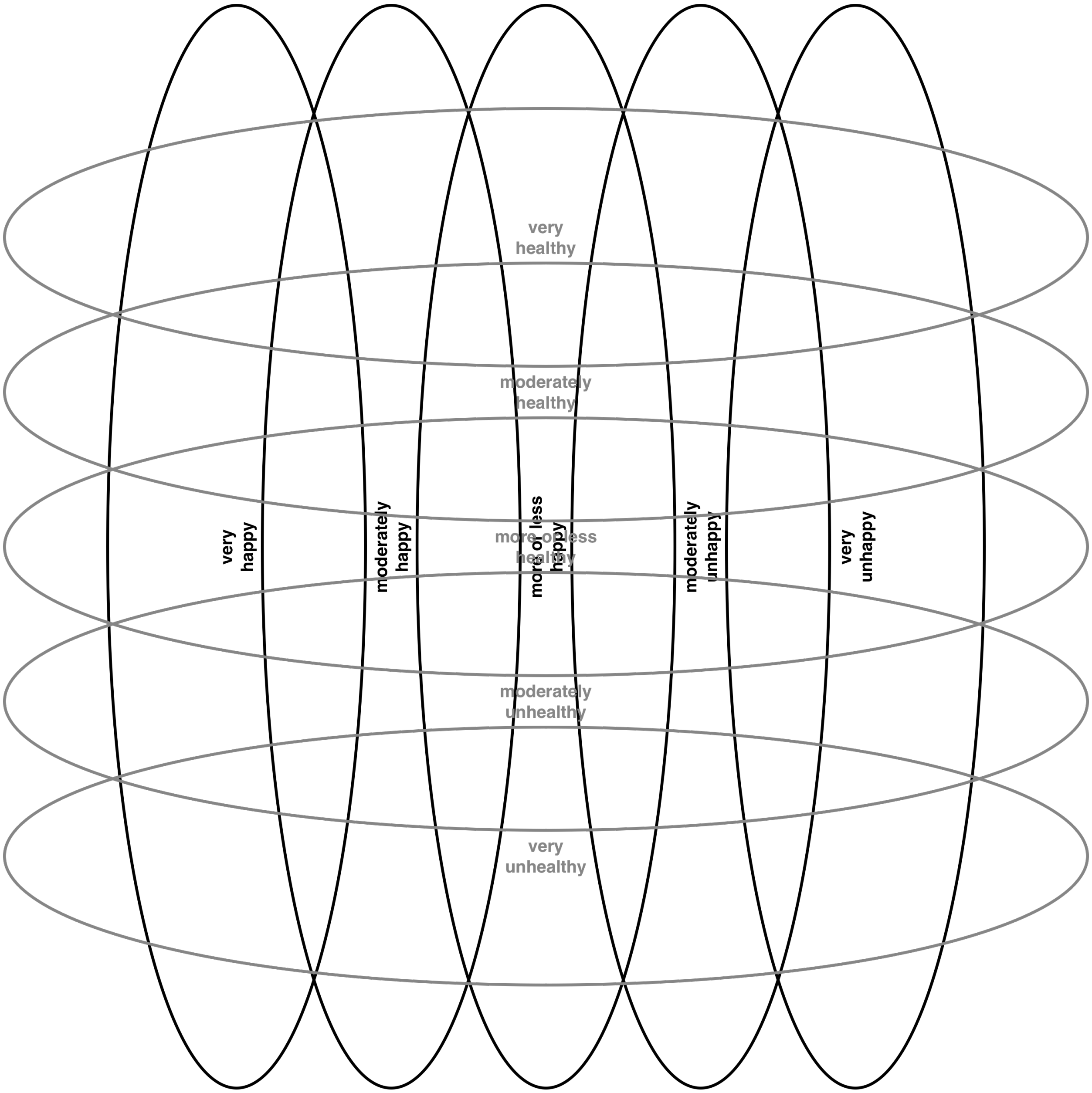

Fig. 16 Euler diagram for two independent dimensions

Figure 17 show the equivalent semantic network.

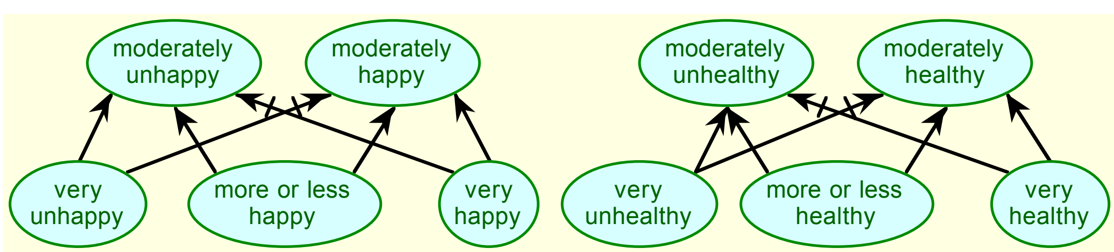

Fig. 17 Semantic network for two independent dimensions

Figure 17 shows the latent structure derived from the conceptual matrix of feature vectors for each templet generated from their logical relations with the entire set of templets. The first two components are precisely the expected orthogonal five-point scales. The curvature noted in Figure 14 representing moderation-extremism dimensions is represented in the remaining components and does not appear in the two-dimensional plot.

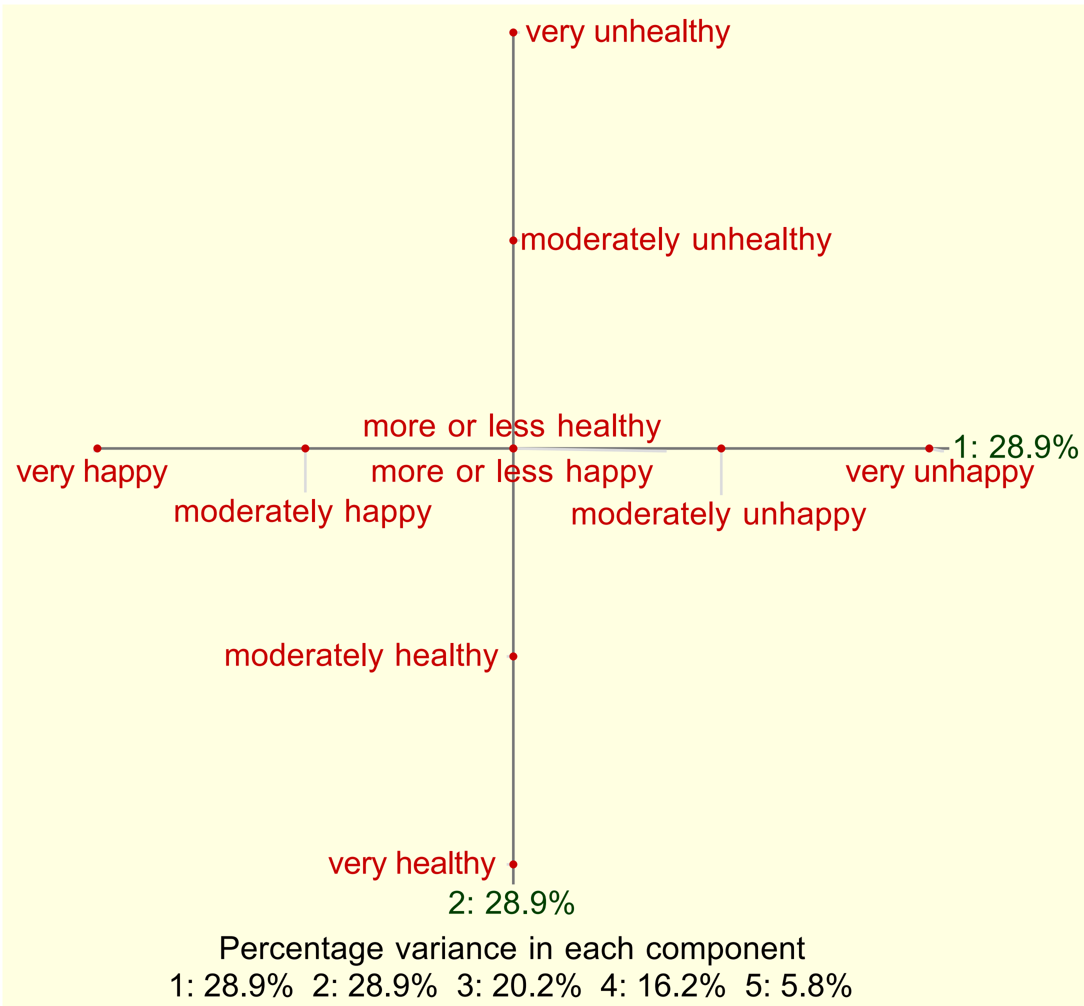

Fig. 18 Scaling of the joint inter-templet distances for happiness and health

Figure 19 is a three-dimensional plot of the conceptual space latent in a semantic network consisting of that shown in Figure 17 together with a third network of the same type where the nodes are labeled for pleasantness rather than happiness or health. It can be seen that the three major components generate a spatial structure with three five-point orthogonal scales as axes.

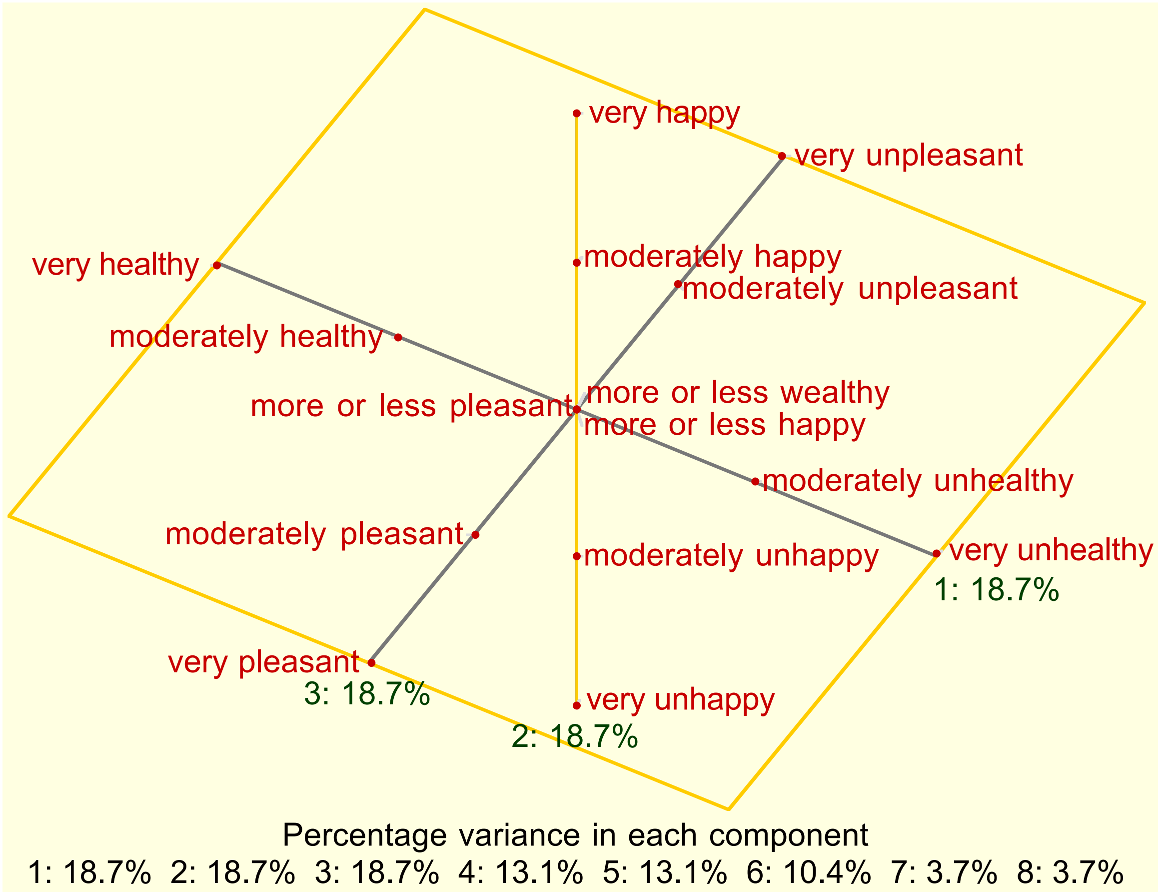

Fig. 19 Scaling of the joint inter-templet distances for happiness, health and wealth

An Euler diagram for three independent structures cannot be drawn with simple shapes in a plane and, even if drawing one with suitable closed curves is possible with a computational tool (Flower and Howse, 2002), it would be so convoluted as to be meaningless to human perception. The natural extension of Figure 16 to three dimensions is to use three chains of overlapping ellipsoids to represent templets such that every ellipsoid in each chain overlaps every ellipsoid in the other two chains and no ellipsoid contains any other.

The use of circles or ellipses in Euler diagrams is a common convention, but any closed curve serves the same purpose and Charles Dodgson (aka Lewis Carroll) used rectangles for Venn diagrams (Abeles, 2007). If this is done in Figure 16 then the partitioning of the two-dimensional space is similar to the normal way of doing so on graph paper, but the adjacent rectangles have a slight overlap rather than abutting. Similar considerations apply to an extended Euler diagram using rectangular prisms instead of ellipsoids.

These considerations of multi-dimensional Euler diagrams also address the issue of how conceptual space partitions naturally into regions representing specific concepts, and whether the logical structure of such concepts can be recovered from their spatial relationships. Gardenförs (2000, Section 3.9) suggests that Voronoi tessellations provide the convex partitions of conceptual space predicated by prototype theory (Rosch, 1983). Such partitions can be treated as overlapping closed curves in an Euler diagram if the adjacent partitions are taken to have a slight overlap. The examples described in this section show that the logical structure of the Euler diagram will generate a conceptual matrix and associated a distance matrix such that multi-dimensional scaling will reconstruct the connectivity of the original tessellation of the space. The metric used to portray this connectivity is somewhat arbitrary since the distance measure used in constructing the spaces from the logical relations is not unique, for example, any Minkowski metric might be used (Shaw, 1980, pp.155-161). As Kelly (1969, p.105) notes 'psychological space' is a topological structure characterized by its connectivity rather by a metric.

The procedure for deriving conceptual spaces from logical relations also takes into account the similar effect of other logical relations. For example, the Euler diagram of Figure 7 uses a mixture of entailments represented by enclosure, contrasts represented by separation, and independence represented by overlap and its the latent geometry is that of a nine-point scale. Thus, similar spatial structures may arise from different networks of logical relations. The general principle is that the topology of the space can be represented through the connectivity of neighboring regions and the lack of connectivity of distant regions. The presence of a templet enclosing other templets indicates a connection between those templets, as does overlap between those templets. The interpretation of these connections through Euler diagrams provides a formal derivation of the logical structure of the conceptual space.

For complex, multi-dimensional spaces a semantic network will be the most convenient form of representation, but the logically equivalent generalized Euler diagram provides an intuitive bridge to the geometry of the logically equivalent semantic space, even if both are so high in dimension as to be beyond human visualization.

9 Visualizing anticipation as abduction over prototypes in conceptual space

The logical relations between templets discussed in the previous sections are internal constraints of the systems of meaning that we impose on phenomena, requiring that when it is asserted that one or more templets fit some phenomenon it is implicit that some other templets also fit or do not fit the phenomenon. The act of fitting the templets ascribes meaning but it is normative, not anticipatory. Our assertions describe the phenomenon but additional logical processes are necessary to go beyond the description we have fitted to anticipate the fit of additional templets that do not form part of our description.

There are many possible mechanisms for anticipation, but a foundational one is the comparison of stored descriptions of past phenomena, schemata or prototypes, with that of the current phenomena to anticipate other templets that might fit it because we have fitted them to similar phenomena in the past. As Dewey (1910, p.174-175) phrases it, "To be able to use the past to judge and infer the new and unknown implies that, although the past thing has gone, its meaning abides in such a way as to be applicable in determining the character of the new." Bartlett (1932) termed the traces of templets that we had fitted to past phenomena 'schemata' to emphasize that memories were not recordings of phenomena but traces of the abstract structures we had fitted to phenomena. Rosch (1983) termed them 'prototypes' in order to emphasize their role in facilitating comparison with new phenomena.

Goodman and Elgin (1988, p.8) emphasize that the system of templets and associated schemata that we use determines the similarities we will find and hence the anticipations that may arise: "A system integrates an expression into a network of labels that organizes, sorts or classifies items in terms of diversity to be recognized. It thus reflects or establishes likenesses; and systems that describe the matter differently may share a realm." Kelly (1955) emphasizes that the system of templets that we chose to fit to a phenomenon is our choice, constrained only by our willingness to accept the other templets that our meaning constraints require us fit as a consequence. Schütz (1943) notes that these requirements include a wide range of social constraints: an end's relationship with other ends; the consequences and side-effects of achieving an end; the means appropriate to the end; the interaction of such means with other ends and means; the accessibility of those means; the construal that others might place on the actions; its interaction with their own planned actions; and so on. Kelly (1955) coined the term "constructive alternativism" for the availability to choice of different systems of templets to provide alternative meanings for what we choose to construe as the 'same' phenomenon. In a philosophy of science context, Giere (2006) terms the recognition of the availability of such choice 'perspectivism.'

This process of choosing a system of templets to fit a phenomenon and comparing it with traces of past phenomena in order to anticipate that other templets fitted in the past might also fit now can be modeled within the framework of templets and relations already developed. The additional step required is that of the selection of the traces from which we might chose a possible anticipatory fit.

Figure 20 illustrates the general templet structure involved. At the center is a templet that we might choose to fit to a new phenomenon, to put it in 'perspective,' to embed it in a 'conceptual space,' to 'frame' it. This templet entails a number of universes of discourse, templets superordinate to a network of templets that, as already noted, Kelly terms the 'range of conveniences' of a 'construct,' Johnson the 'determinable' of a set of 'determinates,' and Gardenförs the 'dimensions' of a 'conceptual space.' It is entailed by templets that Bartlett terms 'schemata' and Rosch 'prototypes, which encode traces of past phenomena characterized by the determinates of the determinables above that had previously been fitted to them.

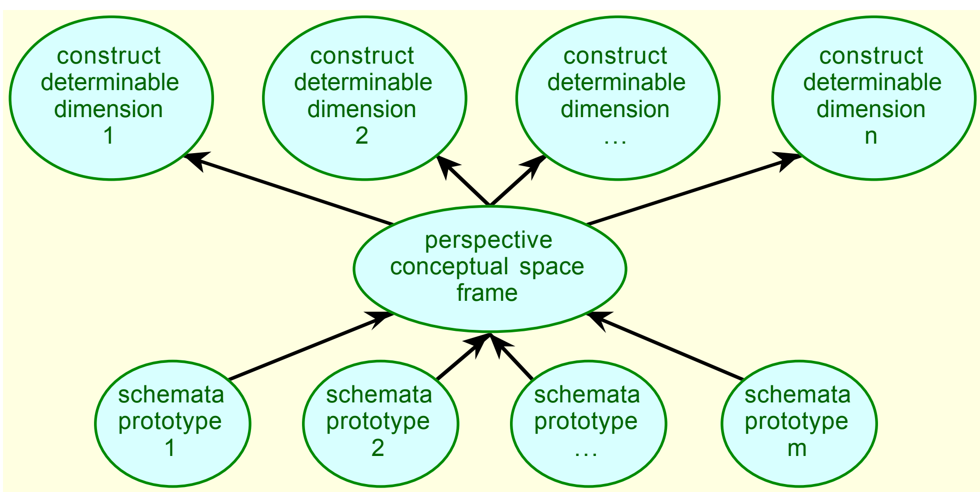

Fig. 20 General templet structure for anticipation through abductive inference

Goffman (1974) discussed the choosing of the central templet as an act of 'framing' and that term has entered the artificial intelligence literature through Minsky's (1974) use of it to model computer vision, the linguistics literature through Fillmore's (1985) use it to model the semantics of natural language, and the cognitive psychology literature through Barsalou's (1992) analysis of the role of frames in human cognition. In the library science literature Ranganathan (1964) terms this structure a 'faceted taxonomy.'

The process of selecting one or more schema from the stored schemata as being compatible with and extending the schema we have fitted to a new phenomenon is one that Peirce (1931) termed 'abduction.' The inference form is to choose a schema that entails the same templets as the ones fitted to the target phenomenon to be anticipated, and thus 'explains' it, while also entailing additional templets which can be hypothesized to fit the target as well.

The contrast/opposition relations play an important role in ruling out schemata that could not fit the target phenomenon without contradiction. Andersen (2000) notes that contrast sets are central to Kuhn's account of family resemblance, noting that "it diverges from Wittgenstein's account by including dissimilarity between instances of contrasting concepts on a par with similarity among instances of a single concept." The entailment relation supports generalization creating a volume in conceptual space encompassing the traces of many relevant schemata, and the contrast/opposition relations supports bounding these volumes so that only the phenomena they anticipate correctly fall within them.

Gaines (2009) uses a simple test case from the knowledge acquisition literature to illustrate anticipatory inference using semantic networks with additional notation to represent all the logical constructs of a *description logic* (Baader, Calvanese, McGuinness, Nardi and Patel-Schneider, 2003), essentially a subset of first-order logic (FOL) capable of representing definitions and rules. The problem is one of prescribing hard or soft contact lenses for a client using the values of four attributes (Cendrowska, 1987). The principle is that a patient whose tear production is normal will be prescribed a hard lens if astigmatic and a soft lens if not. However, there is an exception to the soft prescription if the patient is presbyopic and myopic, and to the hard if hypermetrope and old. Figure 21 shows the twenty four possible cases.

none	reduced	not astigmatic	myope	young	none case 1
soft	normal	not astigmatic	myope	young	soft case 1
none	reduced	astigmatic	myope	young	none case 2
hard	normal	astigmatic	myope	young	hard case 1
none	reduced	not astigmatic	hypermetrope	young	none case 3
soft	normal	not astigmatic	hypermetrope	young	soft case 2
none	reduced	astigmatic	hypermetrope	young	none case 4
hard	normal	astigmatic	hypermetrope	young	hard case 2
none	reduced	not astigmatic	myope	pre-presbyopic	none case 5
soft	normal	not astigmatic	myope	pre-presbyopic	soft case 3
none	reduced	astigmatic	myope	pre-presbyopic	none case 6
hard	normal	astigmatic	myope	pre-presbyopic	hard case 3
none	reduced	not astigmatic	hypermetrope	pre-presbyopic	none case 7
soft	normal	not astigmatic	hypermetrope	pre-presbyopic	soft case 4
none	reduced	astigmatic	hypermetrope	pre-presbyopic	none case 8
none	normal	astigmatic	hypermetrope	pre-presbyopic	none case 9
none	reduced	not astigmatic	myope	presbyopic	none case 10
none	normal	not astigmatic	myope	presbyopic	none case 11
none	reduced	astigmatic	myope	presbyopic	none case 12
hard	normal	astigmatic	myope	presbyopic	hard case 4
none	reduced	not astigmatic	hypermetrope	presbyopic	none case 13
soft	normal	not astigmatic	hypermetrope	presbyopic	soft case 5
none	reduced	astigmatic	hypermetrope	presbyopic	none case 14
none	normal	astigmatic	hypermetrope	presbyopic	none case 15
lens	tear production	astigmatism	myopia	presbyopia	

Fig. 21 Stereotypical cases for a simple expert system problem

Figure 22 shows an instance of the general templet structure for anticipation through abductive inference shown in Figure 20 that solves the contact lens problem. The four determinables at the upper left are those whose determinates are necessary to characterize the client and allow the appropriate determinant of the determinable, "prescription," at the upper right to be inferred. The "contact lens client" frame in the center entails the determinables, making their determinates relevant to the problem. The seven schemata at the bottom are sufficient to solve the problem. The conceptual grid of test cases at the middle right holds the data of Figure 21 and is used to check the solution.

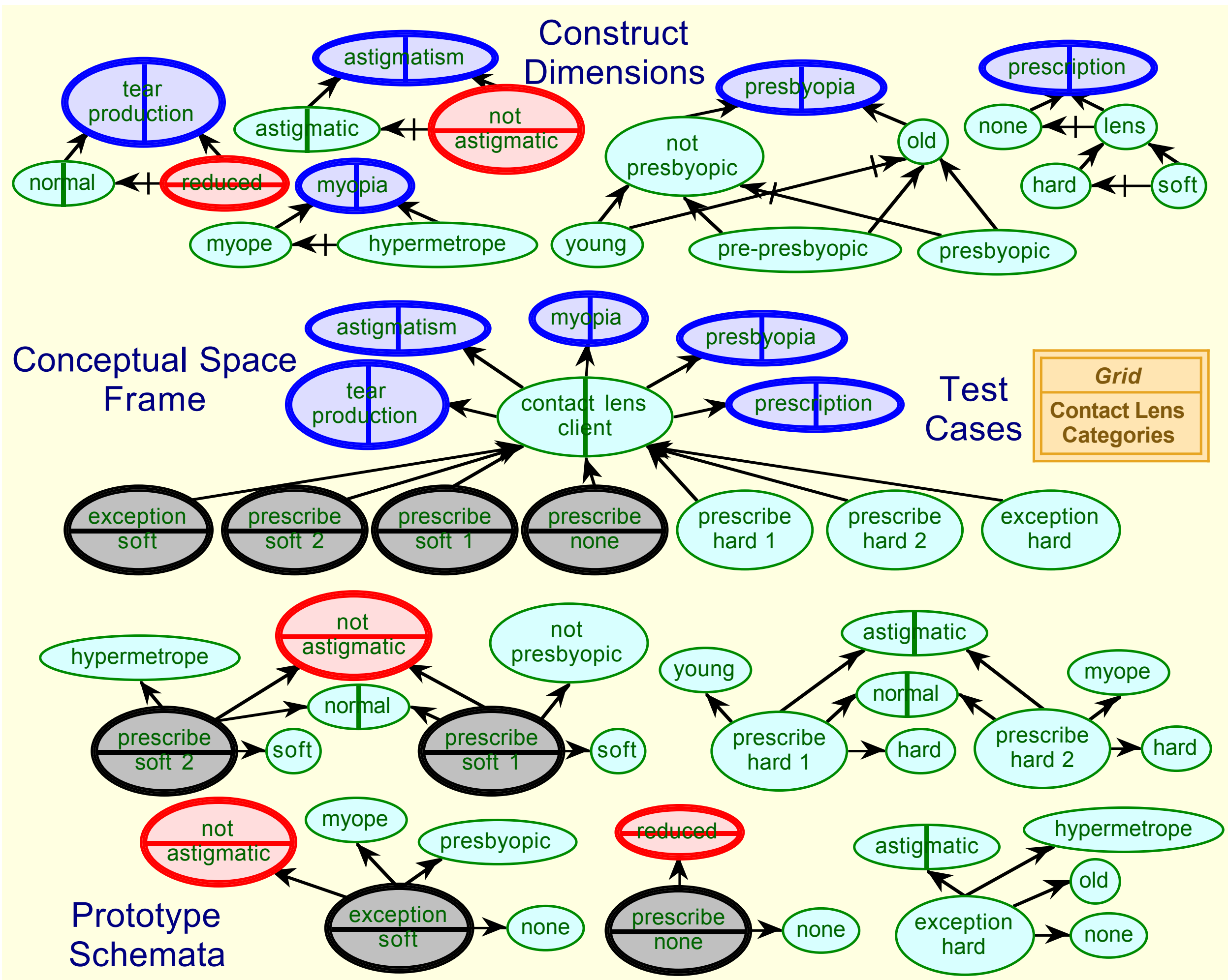

Fig. 22 Prototypes in a conceptual space solving a simple expert system problem

Figure 22 shows a stage in a solution for a client with the characteristics of "none case 9" where the templets "contact lens client," tear production is "normal" and astigmatism is "astigmatic" have been asserted to fit. The logical relations between the templets have already classified the soft prescription and reduced tear production schemata as irrelevant, leaving the hard prescription and exception schemata as possible. Asserting that the templet "hypermetrope" fits will exclude the "prescribe hard 2" templet as irrelevant. Asserting "pre-presbyopic" will also exclude "prescribe hard 1," leaving only "exception hard" to be inferred abductively as the best, and only remaining, fit, which then entails that the prescription sub-templet "none" fits.

The abductive inference process can be treated as deduction by introducing the metalogical constraint that one of the schemata must fit, and abductive reasoning in general has solid logical foundations (Gabbay and Woods, 2005; Aliseda, 2006). It is noteworthy that the logic of entailment and contrast/opposition between templets is sufficient to support the anticipatory process with the introduction of only one metalogical abductive constraint, and that this constraint applies only to the schemata—a hypothetical assertion that at least one of them fits.

This brings the psychology and the logic nicely together in the defeasible assumption that some trace of a past phenomenon must fit the target one. If the ensuing anticipation turns out to be misleading then the adjustment necessary is clear and localized, to add a schema for a more appropriate anticipation and change those schemata that led to the misleading outcome so that they do not fit when the new schema does. This adjustment may be possible with the existing construct dimensions or may involve adding an additional one to discriminate the schemata. The overall process exemplifies Mill's (1875, Book 3 Ch.8) "joint method of agreement and difference" for the inductive development of models of phenomena.

Figure 22 is just one example of a very large number of instances of the generic structure of Figure 20 that can solve the contact lens problem. There are several possible structures for the non-binary constructs, "presbyopia and "prescription." There is a very large number of possible sets and logical arrangements of schemata, even if one does not take into account those with preferences over schemata. The set of twenty four cases in Figure 21 will obviously serve as schemata, as will subsets of those together with more skeletal schemata, schemata with additional irrelevant construct dimensions, and so on.

This illustrates some of the problems of eliciting conceptual models for purposes such as "knowledge engineering" (Hayes-Roth, Waterman and Lenat, 1983). The conceptual structures in use may be highly idiosyncratic and vary greatly between individuals of similar competence. Even the simple problem represented in Figure 22 can be solved through a very large range of conceptual structures. Conceptual structures involving more dimensions, with their associated range of possible structures and the combinatorial explosion of equivalent sets of schemata, can become very complex and difficult to compare.

These issues are difficult to convey textually or through logical symbolism but can be made very apparent through graphic representation of the logical structures, and exploratory interaction with them to see the effect of the logical constraints as templets are fitted to particular situations.

This and the preceding sections have presented a number of visualization tools for representation and inference in a system of logic based on only two connectives, those of entailment and contrast/opposition, and shown how many aspects of the conceptual structures people use in meaning creation, representation, communication and reasoning can be modeled with this minimalist logical system and illustrated through the associated visualization tools.

Section 3 noted that a background in mathematical logic tends to lead to a tacit preconception that other logical constructs such conjunction, disjunction, definition and rules, are an automatic and essential feature of any logical system, but that these notions from mathematical logic may be misleading if applied to modeling many aspects of human reasoning. However, they play a major role in reasoning in mathematics and the exact sciences, and the following section outlines how the visualization tools and techniques can be extended to facilitate understanding of such formal reasoning.

10 Visualizing conjunction, disjunction, negation, definitions, rules, and individuals

Logical constants such as disjunction, conjunction and negation have not been defined or used in any of the examples in previous sections. They have a natural interpretation as procedures for forming new templets from existing ones, but such templets are not necessarily needed or created in everyday human reasoning and, indeed may be problematic to create. However, their role in allowing primitive templets to be combined in various ways to define non-primitive templets derived from the primitives is foundational for mathematics and the sciences that depend on it. The issues involved can be illustrated in terms of the visualization methods already discussed, and it is instructive to do so since it throws light on the nature and role of the logical constants, and the differences between mathematical logic and natural human reasoning.

Figure 23 left shows an Euler diagram for the formation of a new templet representing the logical conjunction of A and B, C ≡ A∧B, as the spatial intersection of the templets A and B. The new templet is the grey region and its label is white, with the convention that the white label names the gray region. On the right is shown an Euler diagram for the formation of a new templet representing the logical disjunction of A and B, D ≡ A∨B, as the spatial union of the templets A and B, again with the convention that the white label names the entire gray region.

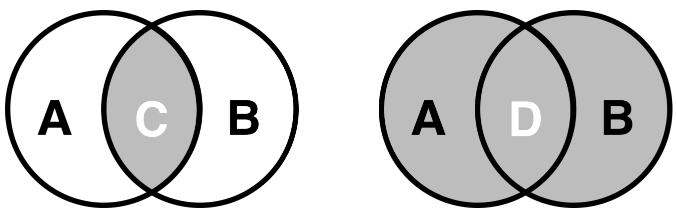

Fig. 23 Euler diagrams for the formation of conjunction and disjunction templets

Figure 24 shows the essential features of conjunction and disjunction. On the left the templet E has been placed inside the templet C and it can be see that it entails not only C but also A and B. On the right the templet F has been placed to enclose the templet D and it can be seen that it is entailed not only by D but also by A or B.

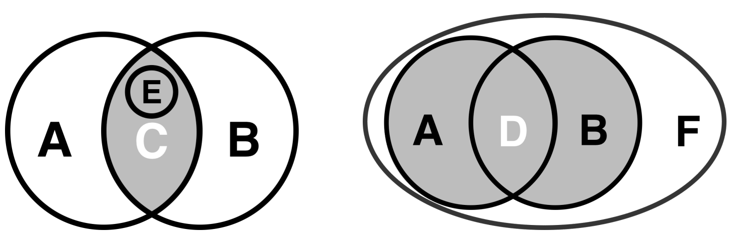

Fig. 24 Euler diagrams for inferences from conjunction and disjunction templets

Figure 25 shows the same conceptual structures as Figure 24 as a semantic network:

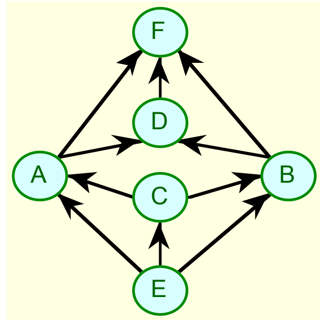

Fig. 25 Semantic network for inferences from conjunction and disjunction templets

For both representations, we have:

$$E \rightarrow C \text{ implies that } E \rightarrow A \text{ and } E \rightarrow B$$
$$D \rightarrow F \text{ implies that } A \rightarrow F \text{ and } B \rightarrow F$$

which are the logical definitions C as A∧B and D as A∨B.

Now consider the situation in which the templets C and D are not defined, or not definable within the conceptual system represented. If we place templets E and F as shown, we still have:

$$E \rightarrow A \text{ and } E \rightarrow B, \quad A \rightarrow F \text{ and } B \rightarrow F$$

and we might write this in abbreviated form as:

$$E \rightarrow A\wedge B, A\vee B \rightarrow F$$

without there being any implication that A∧B or A∨B correspond to templets in the conceptual structures of which A, B, E and F are part.

A system of conceptual structures that lacks templets corresponding to the normal logical constants is said to have a 'substructural' logic, a term based on Gentzen's use of the word 'structural' in his generic sequent calculus for any logical system (Došen, 1993). If, following Tarski's (1956) minimalist axiomatization of what it is to be a 'logic,' we characterize a logic entirely in terms of its consequence operator, then the operations associated with the other logical constants can be defined in terms of the consequence operator but the conceptual structures they generate will not necessarily exist within the conceptual system being modeled.

The definitions of conjunction and disjunction in this way can be visualized in terms of the diagrams of Figures 24 and 25. The templet E is a proto-conjunction of the templets A and B since it entails both of them. Any templet with this property that is maximal in the sense that it is entailed by any other templet having the property is a standard structural conjunction (Koslow, 1992). In Figure 24 left it is apparent that if templet E expands to the maximum size that will fill the shaded space it satisfies this condition; templet C if existed would have this property. Similarly in Figure 25 if the proto-conjunction E is maximal in entailing no other proto-conjunction it would have this property.

The templet F is a proto-disjunction of the templets A and B since it is entailed by both of them. Any templet with this property that is minimal in the sense that it entails any other templet having the property is a standard structural disjunction (Koslow, 1992). In Figure 24 right it is apparent that if templet F contracts to the minimum size that will enclose the shaded space it satisfies this condition; templet D if existed will have this property. Similarly in Figure 25 if the proto-disjunction F is minimal in being entailed by no other proto-disjunction it will have this property.

Similar diagrams and definitions can be given for absolute negation and complementation, that is negation relative to an entailed templet. Thus, the role of the standard logical constants can be illustrated and explained through the visualization techniques already described. In particular, it is apparent that templets for them may not exist and yet much human reasoning can still be modeled as shown in previous sections. For Venn, the use of the shaded areas in Figure 22 to visualize the definitions of the logical constants on which Boole's new mathematical logic was founded was a significant tutorial tool. However, the assumption as that logic became generally adopted, that such constructions corresponded to universally available concepts in human thought, can be misleading.

This is important for the human sciences where definitions and rules have been found to be inappropriate models for much human activity. It was natural early attempts to model human reasoning to assume that all concepts were similar to those in mathematics, in that they were defined through necessary and sufficient conditions. However, Waismann (1945) in a symposium on whether scientific concepts should be defined through their verifiability conditions argued that all such concepts were essentially 'open' in being "always corrigible or emendable." Weitz (1977) analyzes the openness of human concepts in depth in his book on the *opening mind* and his analysis has come to be accepted in many disciplines. Conversely, mathematical concepts are not open and can be uniquely characterized as having no connotations other than their definitions (Tharp, 1989). The open nature of human concepts has been confirmed empirically in developmental psychology (Smith and Medin, 1981; Keil, 1989), anthropology (Rosch and Lloyd, 1978) and scientific practice (Nersessian, 2008).

A templet implementing a structural conjunction represents a *definition* in that the satisfaction of its entailments is a sufficient condition to require it be fitted. If only a subset of the entailments are sufficient then the templet represents a *rule* such that when those entailments are satisfied the templet has to be fitted and its remaining entailments imply that other templets will also be fitted. The omission of definitions and rules in models of human activity is equivalent to postulating that human reasoning is based on a substructural logic where the normal logical constants are not freely available. However, the logical constants have become tacitly accepted as freely available constructions in any rational system, and one needs persuasive visualization techniques to demonstrate that they are very strong constructions that are not necessary for much that we associate with human rationality.

It is possible to extend the semantic network representation to support all the logical constants including structural conjunction and disjunction, definitions, rules, relations, cardinality constraints and individuals (Gaines, 2009). Logical systems for such extended semantic networks have become called *description logics*, and have been studied in depth as tractable sub-sets of first-order logic suitable for computer implementation and used to support the semantic web and other practical applications of formally defined ontologies (Baader *et al*., 2003). Figure 26 shows an extended semantic network representing the conceptual structure of family relationships *Handbook* (Baader *et al*., 2003, p.52) in which the heavy arrows indicate necessary and sufficient, definitional entailments and contrast/oppositions, the unboxed text relations, the boxed text cardinality constraints and the double-ringed templet one defined disjunctively. There are three primitive templets, "person," "female" and "woman," and the remaining nine are defined in terms of these. The impact of the definitional structure in reducing the number of primitive, "open" conceptual structures is apparent, and minimal primitives is highly appropriate to mathematics and, perhaps, to legalistic definitions of family relationships, but less so for the more open conceptual structures of everyday life.

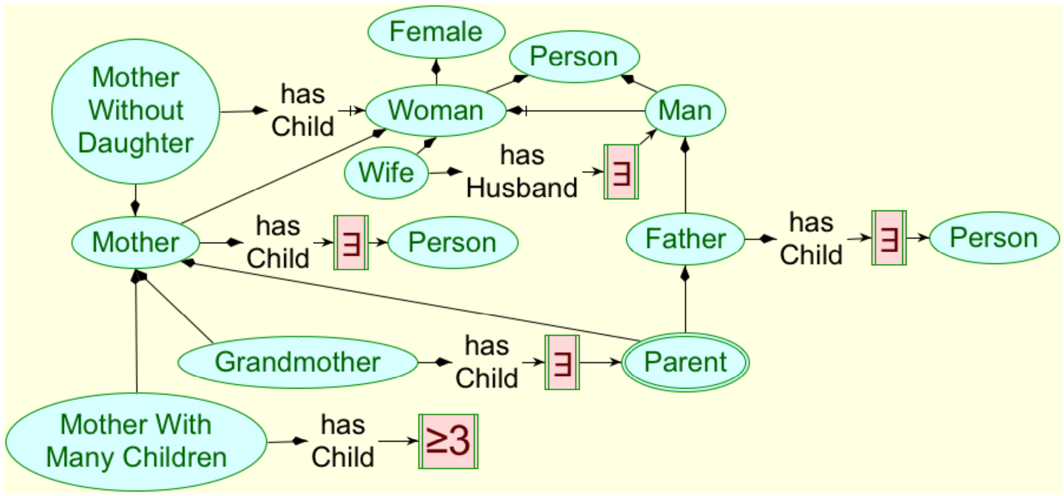

Fig. 26 Defined conceptual structure of family relationships (Gaines, 2009)

The semantic networks studied in this article are a subset of such extended networks restricted to non-definitional entailment and contrast/opposition relations. Given how much can be achieved without the extensions and the difficulties of developing large-scale systems emulating human reasoning with systems based on definitions and rules, it is an interesting open question whether there may be weaker substructural extensions that are better suited to such emulation.

11 Conclusions

Conceptual structures are foundational to many disciplines concerned with human reasoning and their logical structures and applications are significant in education, research, scholarly studies and industrial applications. However, the properties of such structures, their logical relationships, the inferences that may be drawn from them, are not easy to comprehend, particularly given that the logic underlying much human activity is known to be non-standard, involving neither definitions nor rules and hence also not involving the structural logical constants that are normally taken for granted.

A number of techniques for visualizing conceptual structures have been developed in order to support their study, including modeling and simulating them, and communicating about the research issues and their proposed resolutions. This article describes seven different representations of conceptual structures, linear symbolic logic, line diagrams, Euler diagrams, Venn diagrams, semantic networks, conceptual grids and conceptual spaces. These are shown to be rigorous and formally equivalent representations such that one can translate between them algorithmically and use them in a heterogeneous mix as alternative perspectives if appropriate.

The relative merits of these representation schemes for different types of conceptual structure have been discussed and illustrated, and has been shown that each has features that make them useful for some forms of exposition and analysis. In particular, the power of Euler diagrams and semantic networks, both representing only two types of logical relation, entailment and contrast/opposition, has been demonstrated through examples of the representation of significant conceptual structures such as taxonomies, determinables and determinants, frames and schemata.

It has been shown that the spatial structures of ordered scales and multi-dimensional spaces arises naturally and algorithmically from the logical relations within conceptual structures treated as features of their components. Conversely, it has been shown that the logical relations between components of a conceptual structure represented in a conceptual space may be derived from the spatial relations between the components regarded as a multi-dimensional generalization of an Euler diagram.

The relations between the substructural logic underlying everyday human reasoning and the structural constructs of mathematical logic have been illustrated using the visualization techniques presented. The converse issue of bridging from the representations of the substructural logic by extending the visualization techniques has also been addressed and links to research on visual languages for description logics.

In conclusion, logical and psychological studies of the entire range of human reasoning from everyday life through to mathematics and the exact sciences is now at advanced stage across many disciplines and it is reasonable to hope that we are approaching a synthesis of value to all disciplines. Computer graphics, human-computer interaction, and theorem-proving technologies are also all at an advanced stage where it should be possible to support the education and research effort required to support the scholarly community in achieving this synthesis and benefiting from it in many domains. Hopefully, this article and the others in this special issue will contribute to these objectives.

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