CPSC 351 — Tutorial Exercise #15
Random Variables and Expectation

These questions are intended to give you practice in using discrete probability theory to study problems in computer science, involving random variables and their expected values.

Problems To Be Solved

As in the previous exercise, let us consider the problem of making a ***random walk along a line*** and taking n steps, for some positive integer n.

Once again, each ***outcome*** consists of a complete set of steps that have been taken. This can be modelled by setting the ***sample space*** to be the set

$$\Omega_n = \{(\alpha_1, \alpha_1, \dots, \alpha_n) \mid \alpha_i \in \{\mathtt{L}, \mathtt{R}\} \text{ for } 1 \leq i \leq n\}. \tag{1}$$

An outcome

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega_n \tag{2}$$

represents an outcome in which the walker moves ***left*** during the i^{th} step if $\alpha_i = \mathtt{L}$, and the walker moves ***right*** during the i^{th} step if $\alpha_i = \mathtt{R}$, for every integer i such that $1 \leq i \leq n$.

The previous exercise introduced several functions from the sample space, Ω_n, to the set of real numbers, that is, ***random variables*** (for this sample space):

- $\mathsf{LS}_k : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the *left* during the first k moves — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{LS}_k(\vec{\alpha})$ is the number, ℓ, such that exactly ℓ of the values $\alpha_1, \alpha_2, \dots, \alpha_k$ are $\mathtt{L}$— and so that $\mathsf{LS}_0(\vec{\alpha}) = 0$ for all $\vec{\alpha} \in \Omega_n$.
- $\mathsf{RS}_k : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the ***right*** during the first k moves are $\mathtt{R}$ — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{RS}_k(\vec{\alpha})$ is the number, r, such that exactly r of the values $\alpha_1, \alpha_2 \dots, \alpha_k$ — and so that $\mathsf{RS}_0(\vec{\alpha}) = 0$ for all $\vec{\alpha} \in \Omega_n$.

- $\text{Pos}_k : \Omega_n \to \mathbb{N}$ represents the number of steps that the walker is to the right of the origin after the first k moves — so that this is a negative number if the walker is to the *left* of the origin after the first k moves. In particular for $\vec{\alpha} \in \Omega_n$ such that exactly ℓ of $\alpha_1, \alpha_2, \ldots, \alpha_k$ are $\mathtt{L}$ and exactly r of $\alpha_1, \alpha_2, \ldots, \alpha_k$ are $\mathtt{R}$, then

$$\text{Pos}_k(\alpha) = r - \ell = \text{RS}_k(\vec{\alpha}) - \text{LS}_k(\vec{\alpha}).$$

  Furthermore, $\text{Pos}_0(\vec{\alpha}) = 0$ — for all $\vec{\alpha} \in \Omega_n$.

The following random variables will also be considered.

- $\text{Dis}_k : \Omega_n \to \mathbb{N}$ represents the *distance* of the walker from the origin after the first k moves — so that, for all $\vec{\alpha} \in \Omega_n$, $\text{Dis}_k(\vec{\alpha}) = |\text{Pos}_k(\vec{\alpha}|$.
- $\text{SqDis}_k : \Omega_n \to \mathbb{N}$ represents that *square* of the distance from the walker to the origin after the first k moves — so that, by definition, $\text{SqDis}_k(\vec{\alpha}) = \text{Dis}_k(\vec{\alpha})^2$ for all $\vec{\alpha} \in \Omega_n$.

1. Let $P : \Omega_n \to \mathbb{R}$ be a ***probability distribution*** for the sample space Ω_n. Let k be an integer such that $0 \leq k \leq n$.

   (a) Consider the ***constant function*** $k : \Omega_n \to \mathbb{R}$ such that $k(\vec{\alpha})$ is equal to the *number* k for all $\vec{\omega} \in \Omega_n$.
   Show that the expected value $\text{E}[k]$ of this constant function, with respect to the probability distribution P, is also equal to the *number* k.

   (b) Suppose the expected value, $\text{E}[\text{LS}_k]$, of the random variable LS_k with respect to this probability distribution, is a real number $\beta \in \mathbb{R}$.
   Use the above information to say as much as you can about the expected values $\text{E}[\text{RS}_k]$ and $\text{E}[\text{Pos}_k]$ of the random variables RS_k and Pos_k, respectively — with respect to the same probability distribution P.

As in the previous exercise, let us consider the expected values of these random variables with respect to the ***uniform probability distribution***, for various choices of the positive integer n[1]. In particular, for every positive integer n, let $P_n : \Omega_n \to \mathbb{R}$ such that, for all $\vec{\alpha} \in \Omega_n$,

$$P_n(\vec{\alpha}) = \tfrac{1}{|\Omega_n|} = 2^{-n}. \tag{3}$$

[1] By doing so, we are investigating a well-known problem that is sometimes called *the Drunkard's Walk problem*.

Consider the following values — which are each defined for every positive integer n, and for every integer k such that $0 \leq k \leq n$.

- *ls*$_k$ is the expected value of the random variable LS$_k$ with respect to the probability distribution P_n.
- *rs*$_k$ is the expected value of the random variable RS$_k$ with respect to the probability distribution P_n.
- *pos*$_k$ is the expected value of the random variable Pos$_k$ with respect to the probability distribution P_n.
- *dist*$_k$ is the expected value of the random variable Dis$_k$ with respect to the probability distribution P_n.
- *sqdist*$_k$ is the expected value of the random variable SqDis$_k$ with respect to the probability distribution P_n.

2. Let us consider these values when the number, k, is small.

   (a) Confirm that *ls*$_1$ $=$ *rs*$_1$ $= \frac{1}{2}$, *pos*$_1$ $= 0$, and *dist*$_1$ $=$ *sqdist*$_1$ $= 1$.

   (b) Suppose that $n \geq 2$. Confirm that *ls*$_2$ $=$ *rs*$_2$ $= 1$, *pos*$_2$ $= 0$, *dist*$_2$ $= 1$, and *sqdist*$_2$ $= 2$.

   (c) Suppose that $n \geq 3$. Compute *ls*$_3$, *rs*$_3$, *pos*$_3$, *dist*$_3$ and *sqdist*$_3$.

   (d) Suppose that $n \geq 4$. Compute *ls*$_4$, *rs*$_4$, *pos*$_4$, *dist*$_4$, and *sqdist*$_4$.

   If you completed this problem correctly then you have discovered that — when $n \geq 4$ — *dist*$_3$ $=$ *dist*$_4$ $= \frac{3}{2}$, *sqdist*$_3$ $= 3$, and *sqdist*$_4$ $= 4$.

3. Compute *ls*$_k$, *rs*$_k$ and *pos*$_k$ as functions of k (for all n and for all integers k such that $0 \leq k \leq n$) and explain why your answer is correct.

   Note: It might be helpful to start by observing that, for all integers k and i such that $0 \leq k \leq n$ and $1 \leq i \leq k$,

   $$\mathsf{LS}_k = \mathsf{LS}_{k,1} + \mathsf{LS}_{k,2} + \cdots + \mathsf{LS}_{k,k}$$

   where LS$_{k,i}$ is an ***indicator random variable*** such that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), LS$_k i(\vec{\alpha}) = 1$ if $\alpha_i = \mathtt{L}$ and LS$_k(\vec{\alpha}) = 0$ if $\alpha_i = \mathtt{R}$.

One might also wonder how *dist*$_k$ and *sqdist*$_k$ are related to k. Solving the following problem is a useful step along the way to answering these questions.

4. Let Ω be a sample space, with a probability distribution $\mathsf{P} : \Omega \to \mathbb{R}$, and, for a positive integer m, let $S_1, S_2, \ldots, S_m$ be events (that is, subsets of Ω) such that the following properties are satisfied:

   (a) $S_1 \cup S_2 \cup \cdots \cup S_m = \Omega$.

   (b) The events $S_1, S_2, \ldots, S_m$ are *pairwise disjoint*. That is, for all integers i and j such that $1 \le i, j \le m$ and $i \ne j$, $S_i \cap S_j = \emptyset$.

   (c) $\mathsf{P}(S_i) > 0$ for every integer i such that $1 \le i \le m$.

   Prove that, for every random variable $X : \Omega \to \mathbb{R}$,

   $$\mathsf{E}[X] = \sum_{i=1}^{m} \mathsf{E}[X \mid S_i] \times \mathsf{P}(S_i).$$

   Note that this generalizes Theorem #1 in the notes for Lecture #20.