CPSC 351 — Tutorial Exercise #14
Reintroduction to Discrete Probability Theory

These questions are intended to give you practice in using discrete probability theory to study problems in computer science.

Problems To Be Solved

A ***random walk*** is a basic concept in probability theory. It describes a path that consists of a sequence of "random" steps in some kind of structure or "mathematical space" — possibly a grid, or a directed or undirected graph (where one takes a step by choosing an edge to follow, to move from a current vertex in the graph to a next one). Applications of random walks to computer science, that have been described, include applications to study problems in graph analysis, artificial intelligence, optimization, and networking.

In this exercise one of the simplest situations (of interest): We will consider a situation in which the walker (that is, person who walks) is ***walking along a line*** and taking n steps (for some positive integer n) — starting at a central "origin" point, which we will call "position 0", and moving either ***left*** by one unit (from position i to position $i-1$, for some integer i) or ***right*** by one unit (from position i to position $i+1$, for some integer i) every time a step is taken.

In the experiment being defined, each ***outcome*** consists of a complete sequence of steps have been taken. This can be modelled by setting the ***sample space*** to be the set

$$\Omega_n = \{(\alpha_1, \alpha_1, \dots, \alpha_n) \mid \alpha_i \in \{\mathtt{L}, \mathtt{R}\} \text{ for } 1 \leq i \leq n\}. \tag{1}$$

An outcome

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega_n \tag{2}$$

represents an outcome in which the walker moves ***left*** during the i^{th} step if $\alpha_i = \mathtt{L}$, and the walker moves ***right*** during the i^{th} step if $\alpha_i = \mathtt{R}$, for every integer i such that $1 \leq i \leq n$.

Several functions from the sample space, Ω_n, to the set of real numbers, will be considered in this exercise. Suppose, here, that k is an integer such that $0 \leq k \leq n$.

- $\mathsf{LS}_k : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the *left* during the first k moves — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{LS}_k(\vec{\alpha})$ is the number, ℓ, such that exactly ℓ of the values $\alpha_1, \alpha_2, \dots, \alpha_k$ are $\mathtt{L}$— and so that $\mathsf{LS}_0(\vec{\alpha}) = 0$ for all $\vec{\alpha} \in \Omega_n$.
- $\mathsf{RS}_k : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the ***right*** during the first k moves are $\mathtt{R}$ — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{RS}_k(\vec{\alpha})$ is the number, r, such that exactly r of the values $\alpha_1, \alpha_2 \dots, \alpha_k$ — and so that $\mathsf{RS}_0(\vec{\alpha}) = 0$ for all $\vec{\alpha} \in \Omega_n$.
- $\mathsf{Pos}_k : \Omega_n \to \mathbb{N}$ represents the number of steps that the walker is to the right of the origin after the first k moves — so that this is a negative number if the walker is to the *left* of the origin after the first k moves. In particular for $\vec{\alpha} \in \Omega_n$ such that exactly ℓ of $\alpha_1, \alpha_2, \dots, \alpha_k$ are $\mathtt{L}$ and exactly r of $\alpha_1, \alpha_2, \dots, \alpha_k$ are $\mathtt{R}$, then

$$\mathsf{Pos}_k(\alpha) = r - \ell = \mathsf{RS}_k(\vec{\alpha}) - \mathsf{LS}_k(\vec{\alpha}).$$

  Furthermore, $\mathsf{Pos}_0(\vec{\alpha}) = 0$ — for all $\vec{\alpha} \in \Omega_n$.

1. Consider the functions LS_1, RS_1, and Pos_1.
   (a) State the values of $P_n(\mathsf{LS}_1 = i)$ and of $P_n(\mathsf{RS}_1 = i)$ for every integer i for every integer i such that $0 \leq i \leq 1$ — explaining why your answers are correct.
   (b) Use these probabilities to state the values of $P_n(\mathsf{Pos}_1 = i)$ for every integer i such that $-1 \leq i \leq 1$ — explaining why your answers are correct.
2. Suppose, now, that $n \geq 2$. Consider the functions LS_2, RS_2, and Pos_2.
   (a) State the values of $P_n(\mathsf{LS}_2 = i)$ and of $P_n(\mathsf{RS}_2 = i)$ for every integer i for every integer i such that $0 \leq i \leq 2$ — explaining why your answers are correct.
   (b) Use these probabilities to state the values of $P_n(\mathsf{Pos}_2 = i)$ for every integer i such that $-2 \leq i \leq 2$ — explaining why your answers are correct.

3. Suppose, next, that n and k are positive integers such that $n \geq k \geq 1$. Consider the functions LS_k, RS_k, and Pos_k.

   (a) State the values of $P_n(\mathsf{LS}_k = i)$ and of $P_n(\mathsf{RS}_k = i)$ for every integer i for every integer i such that $0 \leq i \leq k$ — explaining why your answers are correct.

   Note: Material from a prerequisite course about “Combinatorics” (or “Counting”) will probably be useful here.

   (b) Use these probabilities to state the values of $P_n(\mathsf{Pos}_k = i)$ for every integer i such that $-k \leq i \leq k$ — explaining why your answers are correct.

   Note: It might be helpful to consider the cases that $k + i$ is even, and that $k + i$ is odd, separately here.

   Note: If you do not see how to solve this problem then it might be helpful to start by solving this problem for the special cases $k = 3$ and $k = 4$ — and review material about ***permutations*** and ***combinations*** from CPSC 251 (or MATH 271). Can you see a pattern here? What material, from CPSC 251 (or MATH 271) is relevant?