Lecture #11: Introduction to Turing Machines

What To Do Before the Lecture

1. Watch the video for Lecture #11 —noting that it will probably be understandable if you play it at double speed. If you do not have time for this then look at the "Key Concepts" document that is found, immediately after the video for this lecture, on the course web site, instead.

2. ***Print*** and read through the rest of this document and — if you have time — try to solve the problems! These should help you to check that you understand how Turing machines process strings, and that you understand how to prove things about them.

3. The supplemental material include additional information about something that is introduced in the lecture video, but not explained in detail.

Problems To Be Solved

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider, once again, a Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

such that

$$Q = \{q_0, q_{\mathtt{a},1}, q_{\mathtt{a},2}, q_{\mathtt{b},1}, q_{\mathtt{b},2}, q_{\text{accept}}, q_{\text{reject}}\},$$

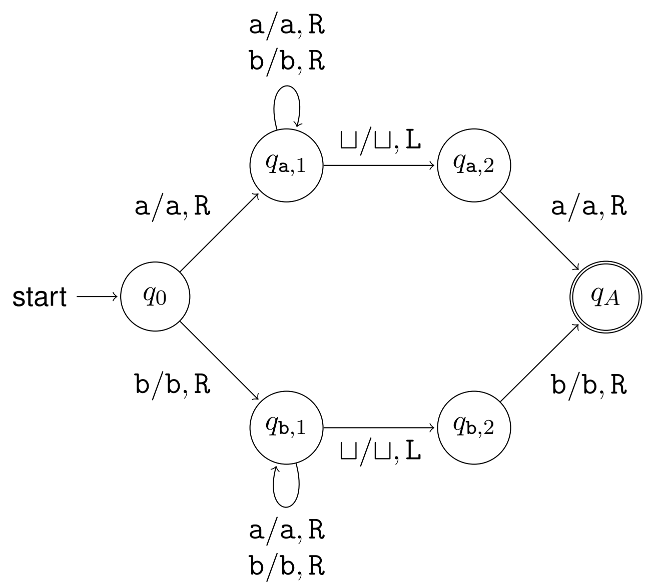
$\Gamma = \{\mathtt{a}, \mathtt{b}, \sqcup\}$, and the transition function, δ, is given by the following transition table:

	$\mathtt{a}$	$\mathtt{b}$	$\sqcup$
q_0	$(q_{\mathtt{a},1}, \mathtt{a}, \mathrm{R})$	$(q_{\mathtt{b},1}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$
$q_{\mathtt{a},1}$	$(q_{\mathtt{a},1}, \mathtt{a}, \mathrm{R})$	$(q_{\mathtt{a},1}, \mathtt{b}, \mathrm{R})$	$(q_{\mathtt{a},2}, \sqcup, \mathrm{L})$
$q_{\mathtt{a},2}$	$(q_{\text{accept}}, \mathtt{a}, \mathrm{R})$	$(q_{\text{reject}}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$
$q_{\mathtt{b},1}$	$q_{\mathtt{b},1}, \mathtt{a}, \mathrm{R})$	$q_{\mathtt{b},1}, \mathtt{b}, \mathrm{R})$	$(q_{\mathtt{b},2}, \sqcup, \mathrm{L})$
$q_{\mathtt{b},2}$	$q_{\text{reject}}, \mathtt{a}, \mathrm{R})$	$(q_{\text{accept}}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$

This Turing machine can also be described using the following picture. To keep the picture simple the accepting state is shown as "q_A" instead of q_{accept}, and transitions to the rejecting state, and this state, are left out — but, as the transition table indicates,

$$\delta(q, \sigma) = (q_{\text{reject}}, \sigma, \mathtt{R})$$

for $q = q_0$ and $\sigma = \sqcup$, for $q = \mathtt{a}, 2$ and either $\sigma = \mathtt{b}$ or $\sigma = \sqcup$, and for $q_{\mathtt{b},2}$ and either $\sigma = \mathtt{a}$ or $\sigma = \sqcup$.



The goal of this lecture presentation is to sketch a proof that this Turing machine ***decides*** the language

$$\begin{aligned} L &= \{\omega \in \{\mathtt{a}, \mathtt{b}\}^\star \mid |\omega| \geq 1 \text{ and } \omega \text{ begins, and ends, with the same letter}\} \\ &= \{\mathtt{a}, \mathtt{b}\} \cup \{\mathtt{a}\mu\mathtt{a} \mid \mu \in \{\mathtt{a}, \mathtt{b}\}^\star\} \cup \{\mathtt{b}\mu\mathtt{b} \mid \mu \in \{\mathtt{a}, \mathtt{b}\}^\star\}. \end{aligned}$$

1. Give a ***trace of execution*** to show that M ***rejects*** the empty string, λ:

   Trace of Execution:

2. Give a ***trace of execution*** to show that M ***accepts*** a.

Trace of Execution:

3. Give a ***trace of execution*** to show that M ***accepts*** b.

Trace of Execution:

Consider a string

$$\omega = \mathsf{a}\,\alpha_1\alpha_2\ldots\alpha_n \in \Sigma^\star$$

where $n \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma$ (so that $|\omega| = n+1$).

Consider the following:

Lemma 1. *For every integer i, if $0 \leq i \leq n$ then*

$$q_0\omega \vdash^\star \mathsf{a}\,\alpha_1\alpha_2\ldots\alpha_i\,q_{\mathsf{a},1}\,\alpha_{i+1}\alpha_{i+1}\ldots\alpha_n.$$

Note: This asserts that

$$q_0\omega \vdash^\star \mathsf{a}\,q_{\mathsf{a},1}\alpha_1\alpha_2\ldots\alpha_n$$

(because this is what is claimed for the case that $i=0$). It also asserts that

$$q_0\omega \vdash^\star \mathsf{a}\,\alpha_1\alpha_2\ldots,\alpha_n\,q_{\mathsf{a},1}$$

(because this is what is claimed for the case that $i=n$).

4. Prove Lemma 1.

Proof of Lemma:

Lemma 2. *Suppose that*

$$\omega = \mathtt{a}\,\mu\,\mathtt{a}$$

for a string $\mu \in \Sigma^{\star}$. Then M **accepts** ω.

5. Prove Lemma 2.

   Proof of Lemma:

Lemma 3. *Suppose that*

$$\omega = \mathtt{a}\,\mu\,\mathtt{b}$$

for a string $\mu \in \Sigma^\star$. Then M **rejects** *ω.*

6. Prove Lemma 3.

   Proof of Lemma:

Now consider a string

$$\omega = \mathtt{b}\,\alpha_1\alpha_2\ldots\alpha_n \in \Sigma^\star$$

where $n \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma$ (so that $|\omega| = n+1$).

7. State, and prove, another lemma, "Lemma 4", that resembles Lemma 1 — but that concerns the beginning of M's execution on a string that begins with "b" (like the above one) instead of a string that begins with "a".

   Lemma 4:

   Proof of Lemma 4:

Lemma 5. *Suppose that*

$$\omega = \mathtt{b}\,\mu\,\mathtt{b}$$

for a string $\mu \in \Sigma^\star$*. Then* M **accepts** ω*.*

8. Use Lemma 4 to prove Lemma 5.

   Proof

Lemma 6. *Suppose that*

$$\omega = \mathtt{b}\,\mu\,\mathtt{a}$$

for a string $\mu \in \Sigma^\star$*. Then* M **rejects** ω.

9. Use Lemma 4 to prove Lemma 6.

   Proof:

Theorem 7. *M decides the language L.*

1. Use the above results to prove Theorem 7.

   Proof: