

Computer Science 351

Regular Operations and Regular Expressions

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Lecture #7

Goal for Today

Goals for Today:

- Introduce the ***regular operations***.
- Introduce ***closure properties*** and their uses.
- Introduce ***regular expressions***.

The “Union” of Two Languages

Definition: Let Σ be an alphabet and let $A, B \subseteq \Sigma^*$. The **union** of the languages A and B is the language

$$A \cup B = \{\omega \in \Sigma^* \mid \omega \in A \text{ or } \omega \in B \text{ (or both)}\}.$$

- **Example:** Suppose that $\Sigma = \{a, b, c\}$, $A = \{a\}$, and $B = \{bb, c\}$. Then

$$A \cup B = B \cup A = \{a, bb, c\}.$$

The “Concatenation” of Two Languages

Definition: Let Σ be an alphabet and let $A, B \subseteq \Sigma^*$. The **concatenation** of the languages A and B is the language

$$A \circ B = \{\omega_1 \cdot \omega_2 \mid \omega_1 \in A \text{ and } \omega_2 \in B\}.$$

- **Example:** Suppose that $\Sigma = \{a, b, c\}$, $A = \{a\}$, and $B = \{bb, c\}$. Then

$$A \circ B = \{abb, ac\} \quad \text{and} \quad B \circ A = \{bba, ca\}.$$

The Kleene Star of a Language

Definition: Let Σ be an alphabet and let $A \subseteq \Sigma^*$. The **Kleene star** of the language A is the language

$$A^* = \{\omega_1 \cdot \omega_2 \dots \omega_k \mid k \geq 0 \text{ and } \omega_i \in A \text{ for } 1 \leq i \leq k\}$$

This language is also, sometimes called the **Kleene closure** of A — or the **star** of A .

- **Example:** Suppose that $\Sigma = \{a, b, c\}$, $A = \{a\}$, and $B = \{bb, c\}$. Then

$$\begin{aligned} A^* &= \{\lambda, a, aa, aaa, aaaa, \dots\} \\ &= \{\omega \in \Sigma^* \mid \omega \text{ only includes } a\text{'s}\}. \end{aligned}$$

The Kleene Star of a Language

On the other hand, B^* includes the following strings (along with lots more):

- λ (obtained by setting $k = 0$)
- bb (obtained by setting $k = 1$ and $x_1 = bb$)
- c (obtained by setting $k = 1$ and $x_1 = c$)
- $bbbb$ (obtained by setting $k = 2$ and $x_1 = x_2 = bb$)
- bbc (obtained by setting $k = 2$, $x_1 = bb$, and $x_2 = c$)
- ccb (obtained by setting $k = 2$, $x_1 = c$, and $x_2 = bb$)
- cc (obtained by setting $k = 2$ and $x_1 = x_2 = c$)

Regular Operations

- Each of ***union***, ***concatenation*** and ***Kleene star*** can be thought of as ***operations*** on the set of languages over an alphabet Σ .

Definition: For any alphabet Σ , the set of operations

- union,
- concatenation, and
- Kleene star

are the ***regular operations over Σ*** .

Closure Properties

Theorem 1. Let Σ be an alphabet, and let $A, B \subseteq \Sigma^*$.

- (a) If A and B are regular languages then $A \cup B$ is a regular language, as well.
- (b) If A and B are regular languages, then $A \circ B$ is a regular language, as well.
- (c) If A is a regular language then A^* is a regular language as well.

The proof of this makes use of a technical result that is given on the next slide.

Closure Properties

Lemma 2. Let Σ be an alphabet, and let $L \subseteq \Sigma^*$. Then L is a regular language if and only if L is the language $L(M)$ of some nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ which satisfies the following properties.

- (a) There are no transitions into q_0 , at all. That is, $q_0 \notin \delta(q, \sigma)$ for any state $q \in Q$ or any symbol $\sigma \in \Sigma_\lambda$, so that the only string $\omega \in \Sigma^*$ such that $q_0 \in \delta^*(q_0, \omega)$ is the empty string, $\omega = \lambda$.
- (b) M has exactly one accepting state, q_F , and there are no transitions out of this state. That is, $F = \{q_F\}$ and $\delta(q_F, \sigma) = \emptyset$ for every symbol $\sigma \in \Sigma_\lambda$.

Closure Properties

To prove part (a) of Theorem 1, let $A, B \subseteq \Sigma^*$ such that A and B are both regular languages.

- Then there exist nondeterministic finite automata

$$M_1 = (Q_1, \Sigma, \delta_1, q_{1,0}, F_1) \text{ and } M_2 = (Q_2, \Sigma, \delta_2, q_{2,0}, F_2)$$

that each satisfy the additional properties in Lemma 1 (so that $F_1 = \{q_{1,F}\}$ for a state $q_{1,F} \in Q_1$ and $F_2 = \{q_{2,F}\}$ for a state $q_{2,F} \in Q_2$) such that $L(M_1) = A$ and $L(M_2) = B$.

- Renaming states, as needed, we may assume that $q_0 \notin Q_1 \cup Q_2$ and that $Q_1 \cap Q_2 = \emptyset$.

Closure Properties

Consider an NFA

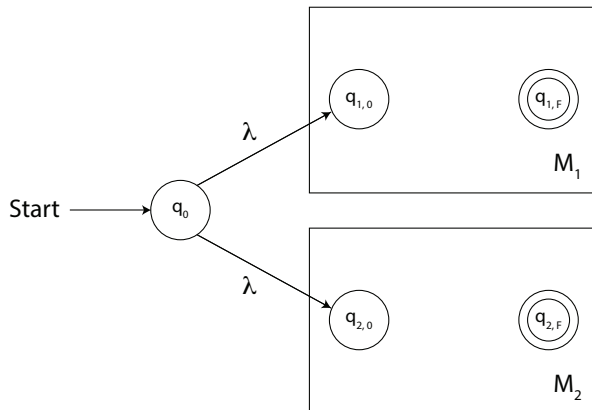
$$M = (Q, \Sigma, \delta, q_0, F)$$

where $Q = \{q_0\} \cup Q_1 \cup Q_2$, $F = F_1 \cup F_2$, and where $\delta : Q \times \Sigma \rightarrow Q$ such that, for $q \in Q$ and $\sigma = \Sigma_\lambda$,

$$\delta(q, \sigma) = \begin{cases} \{q_{1,0}, q_{2,0}\} & \text{if } q = q_0 \text{ and } \sigma = \lambda, \\ \emptyset & \text{if } q = q_0 \text{ and } \sigma \in \Sigma, \\ \delta_1(q, \sigma) & \text{if } q \in Q_1, \\ \delta_2(q, \sigma) & \text{if } q \in Q_2. \end{cases}$$

Then M is as shown on the next slide.

Closure Properties



Closure Properties

- It is easily shown, by induction on the length of ω , that if $\omega \in \Sigma^*$ then

$$\delta^*(q_0, \omega) = \begin{cases} \delta_1^*(q_{1,0}, \omega) \cup \delta_2^*(q_{2,0}, \omega) \cup \{q_0\} & \text{if } \omega = \lambda, \\ \delta_1^*(q_{1,0}, \omega) \cup \delta_2^*(q_{2,0}, \omega) & \text{if } \omega \neq \lambda. \end{cases}$$

- Since $F = F_1 \cup F_2$, this can be used to show that $L(M) = L(M_1) \cup L(M_2) = A \cup B$, as required to show that $A \cup B$ is also a regular language.

Closure Properties

- Similar constructions can be used to show that if $A, B \subseteq \Sigma^*$ are regular languages then $A \circ B$ is a regular language, and that if $A \subseteq \Sigma^*$ is a regular language then so is A^* .
- A supplemental document includes the rest of the proofs of Lemma 2 and Theorem 1.

Closure Properties

Definition: A **closure property** for a set S of languages over an alphabet Σ , is a property stating — for an *operation* on languages over Σ — that if the operation is applied to languages that all belong to the set S , then the result is a language that belongs to the set S , as well.

- Parts (a), (b), and (c) of Theorem #1 are examples of closure properties for the set of regular languages over an alphabet Σ .
- This gives us ***another way to prove that a language is regular***: Show that it is the union, concatenation or star of other regular language(s).

Regular Expressions and Their Languages

- Let Σ be an alphabet that *does not* include any of the symbols

$$\lambda, \emptyset, \Sigma, (,), \cup, \circ, ^*$$

and let

$$\Sigma_{\text{regexp}} = \Sigma \cup \{\lambda, \emptyset, \text{"}\Sigma\text{"}, (,), \cup, \circ, ^*\}$$

so that Σ_{regexp} includes a copy of the *symbol*, $\text{"}\Sigma\text{"}$, that we are also using as the name of the language we are starting with.

- A **regular expression over the alphabet Σ** is a kind of string of symbols, in Σ_{regexp}^* , as defined by the following list of seven rules.
- The **language of a regular expression, over the alphabet Σ** , is a subset of Σ^* that is defined as follows.

Regular Expressions and Their Languages

1. For every **symbol** $\sigma \in \Sigma$, the **string** σ , with length one in Σ_{regex}^* is a regular expression over Σ .

The **language**, $L(\sigma)$, of the regular expression σ , is the **set** $\{\sigma\}$.

2. The **string** λ , with length one in Σ_{regex}^* , is a regular expression over Σ .

The **language**, $L(\lambda)$, of the regular expression λ , is the **set** $\{\lambda\}$.

Regular Expressions and Their Languages

3. The **string** $\emptyset$, with length one in Σ_{regex}^* is a regular expression over Σ .

The **language**, $L(\emptyset)$, of the regular expression $\emptyset$, is the **set** $\emptyset$.

4. The **string** Σ , with length one in Σ_{regex}^* , is a regular expression over (the alphabet) Σ .

The **language**, $L(\Sigma)$, of the regular expression Σ , is the **finite set** Σ .

Regular Expressions and Their Languages

5. If $R_1 \in \Sigma_{\text{regexp}}^*$ and $R_2 \in \Sigma_{\text{regexp}}^*$ are **regular expressions over Σ** then the **string**

$$(R_1 \cup R_2) \tag{1}$$

(of symbols in Σ_{regexp}) is a regular expression over Σ .

The **language** $L(R)$, of the regular expression R at line (1), is the **set**

$$L(R_1) \cup L(R_2).$$

Regular Expressions and Their Languages

6. If $R_1 \in \Sigma_{\text{regex}}^*$ and $R_2 \in \Sigma_{\text{regex}}^*$ are **regular expressions over Σ** then the **string**

$$(R_1 \circ R_2) \tag{2}$$

(of symbols in Σ_{regex}) is a regular expression over Σ .

The **language** $L(R)$, of the regular expression R at line (2), is the **set**

$$L(R_1) \circ L(R_2).$$

Regular Expressions and Their Languages

7. If $R_1 \in \Sigma_{\text{regexp}}^*$ is a **regular expression over Σ** then the **string**

$$(R_1)^* \quad (3)$$

(of symbols in Σ_{regexp}) is a regular expression over Σ .

The **language** $L(R)$, of the regular expression R at line (3), is the **set**

$$L(R_1)^*$$

Regular Expressions and Their Languages

Theorem 2: Let Σ be an alphabet that does not include any of the symbols “ λ ” “ $\emptyset$ ”, “ Σ ”, “(”, “)”, “ $\cup$ ”, “ $\circ$ ”, or “ $*$ ” and let $L \subseteq \Sigma^*$. Then L is a **regular language** if and only if there exists a regular expression R , over the alphabet Σ , such that $L = L(R)$.

- There is a **constructive** proof of this claim. That is, the proof describes processes to convert between deterministic finite automata with alphabet Σ and regular expressions over Σ (in both directions, preserving the language being described).
- A supplemental document for this lecture gives more information about this.

Regular Expressions and Their Languages

Why Study, and Work With, Regular Expression?

- More effective use of text editors, word processors, database systems, web scrapers...
- Introduction to issues and methods in system software development
- Practice using “inductive definitions” to solve problems

Some of this is considered in the next lecture.