

THE UNIVERSITY OF CALGARY

FACULTY OF SCIENCE

FINAL EXAMINATION

Computer Science 351 — Practice Examination

DATE: December, 2025

TIME: 2 hrs.

NAME: _____

Academic integrity is the foundation of the development and acquisition of knowledge and is based on values of honesty, trust, responsibility, and respect. We expect members of our community to act with integrity. Research integrity, ethics, and principles of conduct are key to academic integrity. Members of our campus community are required to abide by our institutional code of conduct and promote academic integrity in upholding the University of Calgary's reputation of excellence.

Aids Allowed: One double-sided letter-sized page of notes. No other aids are allowed.

Instructions:

- Answer ALL questions.
- Answer questions in the space provided. Continue answers on the blank pages at the end of the test if you need more room.
- This test is out of 30.

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1. Consider **Turing machines** and their languages.

- (a) (2 marks) Say what it means for a Turing machine to **recognize** a language $L \subseteq \Sigma^*$.

What it Means for a Turing Machine to Recognize a Language:

- (b) (2 marks) Say what it means for a Turing machine to **decide** a language $L \subseteq \Sigma^*$.

What it Means for a Turing Machine to Decide a Language:

- (c) (3 marks) Say whether the following claim is **true** or **false** and sketch a **brief** proof of your answer.

Claim: Let $L_1, L_2 \subseteq \Sigma^*$ for an alphabet Σ . If L_1 and L_2 are both **decidable** then $L_1 \cup L_2$ is **decidable** as well.

Is This Claim True?:

Brief Proof of Your Answer:

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- (d) (3 marks) Say whether the following claim is **true** or **false** and sketch a **brief** proof of your answer.

Claim: Let $L_1, L_2 \subseteq \Sigma^*$ for an alphabet Σ . If L_1 and L_2 are both **recognizable** then $L_1 \cup L_2$ is **recognizable** as well.

Is This Claim True?:

Brief Proof of Your Answer:

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2. (10 marks) Suppose $L \subseteq \Sigma^*$ (for some alphabet Σ). Let

$$L^{(2)} = \{\mu\mu \mid \mu \in L\}.$$

For example, if $\Sigma = \{a, b, c\}$ and $L = \{a, bc\}$ then $L^{(2)} = \{aa, bcbc\}$.

Use a **many-one reduction** to prove that if L is undecidable then $L^{(2)}$ is undecidable as well.

Note: If you cannot find a many-one reduction that can be used to prove this then for part marks, you should **say so**. You should then give the **definition** of a **many-one reduction**, describe how to use these to prove that languages are undecidable, and give as much detail as you can about the properties that a many-one reduction should have in order to prove the result that is given **in this problem** — so that your answer does not look like something you copied from a cheat sheet.

Solution:

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3. Let n be a positive integer. Suppose that two basketball players are taking turns, trying to throw a ball into a basket, with each trying to do this n times. (The “first player” goes first.) Each can either “Hit” (that is, have the basketball go through the “Hoop”) or “Miss” each time, so that this can be modelled using a **sample space**

$$\Omega = \{(\alpha_1, \alpha_2, \dots, \alpha_{2n}) \mid \alpha_i \in \{H, M\} \text{ for } 1 \leq i \leq 2n\}$$

whose size is 2^{2n} .

The first player is better at this than the second: Each time the first player tries to throw the ball into the basket, the ball goes through the **hoop** with probability $\frac{3}{4}$ and it **misses** with probability $\frac{1}{4}$. On the other hand, each time the second player tries to throw the ball into the basket, the ball goes through the **hoop** with probability $\frac{1}{4}$ and it **misses** with probability $\frac{3}{4}$.

This can be modelled using a **probability distribution** $P : \Omega \rightarrow \mathbb{R}$ such that, if $0 \leq k, \ell \leq n$, and $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$ such that exactly k of $\alpha_1, \alpha_3, \dots, \alpha_{2n-1}$ are “H”, and exactly ℓ of $\alpha_2, \alpha_4, \dots, \alpha_{2n}$ are “H”, then

$$P(\vec{\alpha}) = \left(\frac{3}{4}\right)^k \times \left(\frac{1}{4}\right)^{n-k} \times \left(\frac{1}{4}\right)^\ell \times \left(\frac{3}{4}\right)^{n-\ell} = \frac{3^{n+k-\ell}}{4^{2n}}.$$

For $1 \leq i \leq n$, let $FP_i : \Omega \rightarrow \{0, 1\}$ be the **random variable** whose value is 1 if the first player hits the basket the i^{th} time they try to, and whose value is 0 otherwise: For $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$,

$$FP_i(\vec{\alpha}) = \begin{cases} 1 & \text{if } \alpha_{2i-1} = H, \\ 0 & \text{if } \alpha_{2i-1} = M. \end{cases}$$

It follows by the above that $P(FP_i = 1) = \frac{3}{4}$, if $1 \leq i \leq n$.

(1 mark)

- (a) Give the definition for the **expected value of a random variable, with respect to a probability distribution**.

Definition of “Expected Value”:

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(1 mark)

- (b) Give the expected value of FP_i with respect to the above probability distribution P (for $1 \leq i \leq n$).

Expected Value of FP_i :

For $1 \leq i \leq n$, let $SP_i : \Omega \rightarrow \{0, 1\}$ be the random variable whose value is 1 if the *second* player hits the basket the i^{th} time they try to, and whose value is 0 otherwise:
For $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$,

$$SP_i = \begin{cases} 1 & \text{if } \alpha_{2i} = H, \\ 0 & \text{if } \alpha_{2i} = M. \end{cases}$$

(1 mark)

- (c) Give the expected value of SP_i with respect to the above probability distribution P (for $1 \leq i \leq n$).

Expected Value of SP_i :

It can be shown that (for this probability distribution) the random variables

$$FP_1, FP_2, \dots, FP_n, SP_1, SP_2, \dots, SP_n$$

are *pairwise independent*. **You may use this fact, without proving it, when answering the rest of this question.**

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Let $W : \Omega \rightarrow \mathbb{N}$ be the number of times the ball goes through the hoop — so that W is a random variable such that

$$W = FP_1 + FP_2 + \cdots + FP_n + SP_1 + SP_2 + \cdots + SP_n.$$

(2 marks)

- (d) Give the expected value of W and explain, briefly, why your answer is correct.

Expected Value of W :

Why Your Answer is Correct:

(3 marks)

- (e) Compute the **variances** of FP_i (for $1 \leq i \leq n$), SP_i (for $1 \leq i \leq n$), and W — showing enough work so that a marker can understand how your answers were obtained.

Variance of FP_i :

Variance of SP_i :

Variance of W :

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(2 marks)

- (f) Give the best (that is, smallest) upper bound that you can for the probability that $W \geq \frac{3}{2}n$, when n is large — giving enough work so that a marker can understand how your answer was obtained.

Please make sure to name any results from lectures that you used when obtaining this bound.

Bound for Probability that $W \geq \frac{3}{2}n$:

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