CPSC 351 — Tutorial Exercise #9

Discrete Probability for Computer Science — Part One

These questions are intended to give you practice in using discrete probability theory to study problems in computer science.

Problems To Be Solved

A ***random walk*** is a basic concept in probability theory. It describes a path that consists of a sequence of "random" steps in some kind of structure or "mathematical space" — possibly a grid, or a directed or undirected graph (where one takes a step by choosing an edge to follow, to move from a current vertex in the graph to a next one). Applications of random walks to computer science, that have been described, include applications to study problems in graph analysis, artificial intelligence, optimization, and networking.

In this exercise one of the simplest situations (of interest): We will consider a situation in which the walker (that is, person who walks) is ***walking along a line*** and taking n steps (for some positive integer n) — starting at a central "origin" point, which we will call "position 0", and moving either ***left*** by one unit (from position i to position $i-1$, for some integer i) or ***right*** by one unit (from position i to position $i+1$, for some integer i) every time a step is taken.

In the experiment being defined, each ***outcome*** consists of a complete sequence of steps have been taken. This can be modelled by setting the ***sample space*** to be the set

$$\Omega_n = \{(\alpha_1, \alpha_1, \dots, \alpha_n) \mid \alpha_i \in \{\mathtt{L}, \mathtt{R}\} \text{ for } 1 \leq i \leq n\}. \tag{1}$$

An outcome

$$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega_n \tag{2}$$

represents an outcome in which the walker moves ***left*** during the i^{th} step if $\alpha_i = \mathtt{L}$, and the walker moves ***right*** during the i^{th} step if $\alpha_i = \mathtt{R}$, for every integer i such that $1 \leq i \leq n$.

Several ***random variables*** might be of interest to someone who is interested in this:

- $\mathsf{LS}_n : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the *left* — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{LS}_n(\vec{\alpha})$ is the number, ℓ, such that exactly ℓ of the values $\alpha_1, \alpha_2, \dots, \alpha_n$ are $\mathtt{L}$.
- $\mathsf{RS}_n : \Omega_n \to \mathbb{N}$ represents the number of times that the walker steps to the ***right*** — so that, for $\vec{\alpha} \in \Omega_n$ as shown at line (2), $\mathsf{RS}_n(\vec{\alpha})$ is the number, r, such that exactly r of the values $\alpha_1, \alpha_2 \dots, \alpha_n$ are $\mathtt{R}$.
- $\mathsf{Pos}_n : \Omega_n \to \mathbb{N}$ represents the number of steps that the walker is to the right of the origin after completing this walk — so that this is a negative number if the walker is to the *left* of the origin at the end of the walk. In particular for $\vec{\alpha} \in \Omega_n$ such that exactly ℓ of $\alpha_1, \alpha_2, \dots, \alpha_n$ are $\mathtt{L}$ and exactly r of $\alpha_1, \alpha_2, \dots, \alpha_n$ are $\mathtt{R}$, then $\mathsf{Pos}_n(\alpha) = r - \ell$.
- $\mathsf{Dis}_n : \Omega_n \to \mathbb{N}$ represents the *distance* of the walker from the origin at the end of the walk — so that, for all $\vec{\alpha} \in \Omega_n$, $\mathsf{Dis}_n(\vec{\alpha}) = |\mathsf{Pos}_n(\vec{\alpha}|$.
- $\mathsf{SqDis}_n : \Omega_n \to \mathbb{N}$ represents that *square* of the distance from the walker to the origin — so that, by definition, $\mathsf{SqDis}_n(\vec{\alpha}) = \mathsf{Dis}_n(\vec{\alpha})^2$ for all $\vec{\alpha} \in \Omega_n$.

Note that $\mathsf{LS}_n(\vec{\alpha}) + \mathsf{RS}_n(\vec{\alpha}) = n$, for all $\vec{\alpha} \in \Omega_n$.

1. Let $P : \Omega_n \to \mathbb{R}$ be a ***probability distribution*** for the sample space Ω_n.

   (a) Consider the ***constant function*** $n : \Omega_n \to \mathbb{R}$ such that $n(\vec{\alpha})$ is equal to the *number* n for all $\vec{\omega} \in \Omega_n$.

   Show that the expected value $\mathsf{E}[n]$ of this constant function, with respect to the probability distribution P, is also equal to the *number* n.

   (b) Suppose the expected value, $\mathsf{E}[\mathsf{LS}_n]$, of the random variable LS_n with respect to this probability distribution, is a real number $\beta \in \mathbb{R}$.

   Use the above information to say as much as you can about the expected values $\mathsf{E}[\mathsf{RS}_n]$ and $\mathsf{E}[\mathsf{Pos}_n]$ of the random variables RS_n and Pos_n, respectively — with respect to the same probability distribution P.

Let us now consider the expected values of these random variables with respect to the ***uniform probability distribution***, for various choices of the positive integer n[1]. In particular, for every positive integer n, let $P_n : \Omega_n \to \mathbb{R}$ such that, for all $\vec{\alpha} \in \Omega_n$,

$$P_n(\vec{\alpha}) = \tfrac{1}{|\Omega_n|} = 2^{-n}. \tag{3}$$

[1]By doing so, we are investigating a well-known problem that is sometimes called *the Drunkard's Walk problem*.

Consider the following values — which are each defined for every positive integer n.

- *ls_n* is the expected value of the random variable LS_n with respect to the probability distribution P_n.
- *rs_n* is the expected value of the random variable RS_n with respect to the probability distribution P_n.
- *pos_n* is the expected value of the random variable Pos_n with respect to the probability distribution P_n.
- *$dist_n$* is the expected value of the random variable Dis_n with respect to the probability distribution P_n.
- *$sqdist_n$* is the expected value of the random variable SqDis_n with respect to the probability distribution P_n.

2. Let us consider these values when the number, n, is small.

   (a) Confirm that $ls_1 = rs_1 = \frac{1}{2}$, $pos_1 = 0$, and $dist_1 = sqdist_1 = 1$.

   (b) Confirm that $ls_2 = rs_2 = 1$, $pos_2 = 0$, $dist_2 = 1$, and $sqdist_2 = 2$.

   (c) Compute ls_3, rs_3, pos_3, $dist_3$ and $sqdist_3$.

   (d) Compute ls_4, rs_4, pos_4, $dist_4$, and $sqdist_4$.

   If you completed this problem correctly then you have discovered that $dist_3 = dist_4 = \frac{3}{2}$, $sqdist_3 = 3$, and $sqdist_4 = 4$.

3. Compute ls_n, rs_n and pos_n as functions of n (for all n) and explain why your answer is correct.

You should have found that $pos_n = 0$ for every positive integer n. Here is a way to prove this that is probably different from the method that you used, if you showed this: Let n be a positive integer. Consider how pos_n and pos_{n+1} are related:

$$pos_n = \sum_{\vec{\alpha} \in \Omega_n} (\text{Pos}_n(\vec{\alpha}) \times P_n(\vec{\alpha})) \tag{4}$$

and

$$pos_{n+1} = \sum_{\vec{\alpha} \in \Omega_{n+1}} (\text{Pos}_{n+1}(\vec{\alpha}) \times P_{n+1}(\vec{\alpha})). \tag{5}$$

Let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega_n$. There are two outcomes in Ω_{n+1} that "extend" the walk $\vec{\alpha}$, namely,

$$\vec{\alpha}_L = (\alpha_1, \alpha_2, \dots, \alpha_n, \mathtt{L}) \tag{6}$$

and

$$\vec{\alpha}_R = (\alpha_1, \alpha_2, \dots, \alpha_n, \mathtt{R}). \tag{7}$$

Indeed, if we set

$$\Omega_{n,L} = \{\vec{\alpha}_L \mid \vec{\alpha} \in \Omega_n\} \quad \text{and} \quad \Omega_{n,R} = \{\vec{\alpha}_R \mid \vec{\alpha} \in \Omega_n\}$$

then it is easily checked that $\Omega_{n,L} \cup \Omega_{n,R} = \Omega_{n+1}$, $\Omega_{n,L} \cap \Omega_{n,R} = \emptyset$, and for every outcome $\vec{\beta}$ in Ω_{n+1}, there is exactly one outcome $\vec{\alpha} \in \Omega_n$ such that either $\vec{\beta} = \vec{\alpha}_L$ or $\vec{\beta} = \vec{\alpha}_R$ (but not both).

It is also easily checked that

$$\mathsf{Pos}_{n+1}(\vec{\alpha}_L) = \mathsf{Pos}_n(\vec{\alpha}) - 1 \quad \text{and} \quad \mathsf{Pos}_{n+1}(\vec{\alpha}_R) = \mathsf{Pos}_n(\vec{\alpha}) + 1 \tag{8}$$

for every outcome $\vec{\alpha} \in \Omega_n$, and that

$$P_{n+1}(\vec{\alpha}_L) = P_{n+1}(\vec{\alpha}_R) = 2^{-n-1} = \frac{P_n(\vec{\alpha})}{2} \tag{9}$$

for every outcome $\vec{\alpha} \in \Omega_n$.

Now

$$\begin{aligned}
\textit{pos}_{n+1} &= \sum_{\vec{\beta} \in \Omega_{n+1}} (\mathsf{Pos}_{n+1}(\vec{\beta}) \times P_{n+1}(\vec{\beta})) \\
&= \sum_{\vec{\alpha} \in \Omega_n} (\mathsf{Pos}_{n+1}(\vec{\alpha}_L) \times P_{n+1}(\vec{\alpha}_L) + \mathsf{Pos}_{n+1}(\vec{\alpha}_R) \times P_{n+1}(\vec{\alpha}_R)) \\
&= \sum_{\vec{\alpha} \in \Omega_n} \left((\mathsf{Pos}_n(\vec{\alpha}) - 1) \times \frac{P_n(\vec{\alpha})}{2} + (\mathsf{Pos}_n(\vec{\alpha}) + 1) \times \frac{P_n(\vec{\alpha})}{2} \right) \\
&\qquad \text{(by the equations at lines (8) and (9))} \\
&= \sum_{\vec{\alpha} \in \Omega_n} \left(2 \times \mathsf{Pos}_n(\vec{\alpha}) \times \frac{P_n(\vec{\alpha})}{2} \right) \\
&= \sum_{\vec{\alpha} \in \Omega_n} (\mathsf{Pos}_n(\vec{\alpha}) \times P_n(\vec{\alpha})) \\
&= \textit{pos}_n.
\end{aligned}$$

Since $\textit{pos}_1 = 0$ and since it follows by the above that $\textit{pos}_n = \textit{pos}_{n+1}$ for every positive integer n it is now easily proved, by induction on n, that $\textit{pos}_n = 0$ for every positive integer n.

4. Modifying the above argument, as needed, show that

$$\textit{sqdist}_{n+1} = \textit{sqdist}_n + 1$$

for every positive integer n. (It follows that $\textit{sqdist}_n = n$ for every positive integer n.)

5. In order to see that assumptions about likelihoods — as reflected by the ***probability distribution*** being used — significantly affects the results being obtained, compute the expected values of the random variables Pos_n, Dis_n, and SqDis_n with respect to the following distributions — and comment on the relationship between the values you obtain (for the expected values Pos_n and SqDis_n) and what you obtained, above, when the uniform probability distribution was used.

   (a) *Dithering Walker*: Let $\gamma_{n,L}, \gamma_{n,R} \in \Omega_n$ be as follows.

   - $\gamma_{n,L} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1 = \mathtt{L}$ and

   $$\alpha_{i+1} = \begin{cases} \mathtt{R} & \text{if } \alpha_i = \mathtt{L}, \\ \mathtt{L} & \text{if } \alpha_i = \mathtt{R} \end{cases}$$

     for every integer i such that $1 \le i \le n-1$.

   - $\gamma_{n,R} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_1 = \mathtt{R}$ and

   $$\alpha_{i+1} = \begin{cases} \mathtt{R} & \text{if } \alpha_i = \mathtt{L}, \\ \mathtt{L} & \text{if } \alpha_i = \mathtt{R} \end{cases}$$

   for every integer i such that $1 \le i \le n-1$. The probability distribution to be used is the distribution $P_{D,n} : \Omega_n \to \mathbb{R}$ such that, for all $\vec{\alpha} \in \Omega_n$,

   $$P_{D,n}(\vec{\alpha}) = \begin{cases} \frac{1}{2} & \text{if } \vec{\alpha} = \gamma_{n,L}, \\ \frac{1}{2} & \text{if } \vec{\alpha} = \gamma_{n,R}, \\ 0 & \text{if } \vec{\alpha} \notin \{\gamma_{n,L}, \gamma_{n,R}\}. \end{cases}$$

   This models a "dithering" walker who starts randomly but then changes their mind, every step after that.

   (b) *Stubborn Walker:* Let $\zeta_{n,L}, \zeta_{n,R} \in \Omega_n$ be as follows:

   - $\zeta_{n,L} = (\mathtt{L}, \mathtt{L}, \ldots, \mathtt{L})$ — that is, an outcome $\vec{\alpha} \in \Omega_n$ as shown at line (2), above, where $\alpha_i = \mathtt{L}$ for every integer i such that $1 \le i \le n$.
   - $\zeta_{n,R} = (\mathtt{R}, \mathtt{R}, \ldots, \mathtt{R})$ — that is, an outcome $\vec{\alpha} \in \Omega_n$ as shown at line (2), above, where $\alpha_i = \mathtt{R}$ for every integer i such that $1 \le i \le n$.

   The probability distribution to be used is the distribution $P_{S,n} : \Omega_n \to \mathbb{R}$ such that, for all $\vec{\alpha} \in \Omega_n$,

   $$P_{S,n}(\vec{\alpha}) = \begin{cases} \frac{1}{2} & \text{if } \vec{\alpha} = \zeta_{n,L}, \\ \frac{1}{2} & \text{if } \vec{\alpha} = \zeta_{n,R}, \\ 0 & \text{if } \vec{\alpha} \notin \{\zeta_{n,L}, \zeta_{n,R}\}. \end{cases}$$

   This models a "stubborn" walker who starts randomly but stubbornly refuses the reconsider that first decision, at any point later on.

A Final Challenge Problem

6. Once again, consider the uniform probability distribution. Try to find a relationship between *dist*$_{n+1}$ and *dist*$_n$ that — somewhat — resembles the relationships between *pos*$_{n+1}$ and *pos*$_n$, and between *sqdist*$_{n+1}$ and *sqdist*$_n$, that are considered above.

   While there *is* a relationship like this that can be proved — and it can even be used to find a reasonably good "asymptotic" upper bound for *dist*$_n$ — it is more complicated than the relationships that are given above (and you should not worry about it, at all, if you are not able to find this).