CPSC 351 — Tutorial Exercise #8
Proving Undecidability

These questions are intended to give you practice in establishing ***oracle reductions*** and — primarily — ***many-one reductions*** — between languages.

1. Let Σ be an alphabet, let $L \subseteq \Sigma^\star$, and consider the language

$$L \circ L = \{\omega_1 \cdot \omega_2 \mid \omega_1, \omega_2 \in L\} \subseteq \Sigma^\star.$$

   Prove that $L \circ L \preceq_{\mathsf{O}} L$.

The following problem — which involves a many-one reduction — is of the difficulty, and length, that would be appropriate for a question on a ***test*** in CPSC 351.

2. Let $A, B \subseteq \Sigma^\star$ for $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, and let $x_{\mathsf{Yes}}, x_{\mathsf{No}} \in \Sigma^\star$, such that the following properties are satisfied.

   (i) $B = \{\mu \in \Sigma^\star \mid \mu \in A$ and μ ends with "c"$\}$.
   (ii) B is ***unrecognizable***.
   (iii) $x_{\mathsf{Yes}} \in A$ and $x_{\mathsf{No}} \notin A$.

   Give a ***many-one reduction*** to show that A is ***unrecognizable*** as well.

This final problem — which also involves a many-one reduction — is of the difficulty, and length, that would be appropriate for a question on an ***assignment*** in CPSC 351.

3. Consider the following decision problem.

   The Rejection Problem
   Instance: A Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{accept}}, q_{\mathsf{reject}})$$

   and an input string $\omega \in \Sigma^\star$ for M.
   Question: Does M ***reject*** M?

Let us use the same alphabet Σ_{TM} and encoding for Turing machines and input strings as in Lecture #8, so that the decidable language $L_{\mathsf{TM+I}} \subseteq \Sigma_{\mathsf{TM}}^{\star}$, introduced in that lecture, is the *language of instances* of this decision problem. Let $\mathsf{Reject}_{\mathsf{TM}} \subseteq L_{\mathsf{TM+I}}$ be the *language of Yes-instances* of this decision problem.

Prove that the Rejection Problem is undecidable — that is, prove that the above language, $\mathsf{Reject}_{\mathsf{TM}}$, is undecidable.

A ***hint*** for this problem is available in a separate file — but you should spend at least a little bit of time trying to solve this problem, without looking at it, before you use this hint.