CPSC 351 — Tutorial Exercise #7
Even More Practice Problems

These problems will not be discussed during the tutorial, and solutions for these problems will not be made available. They are intended to give you practice working with ***multi-tape Turing machines***. Thus they give you applying material introduced in Lecture #7 that you did not use when solving problems in Tutorial Exercise #7.

Practice Problems

Consider a ***component*** of a 3-tape Turing machine, with input alphabet $\Sigma_{\text{pair}} = \{0, 1, \#\}$, tape alphabet $\Gamma = \{0, 1, \#, \dot{0}, \dot{1}, \dot{\#}, \sqcup\}$, and, and with transitions as shown in Figure 1 on page 2. In order to keep the picture simpler, the halt state is shown in the diagram as "q_H" instead of q_{halt}. Transitions out of r_0, r_1, r_2, r_3, r_4 or r_5 that are not shown will never be used (for the computation being considered) but should go to the halting state, without changing the contents of tapes or positions of tape heads.

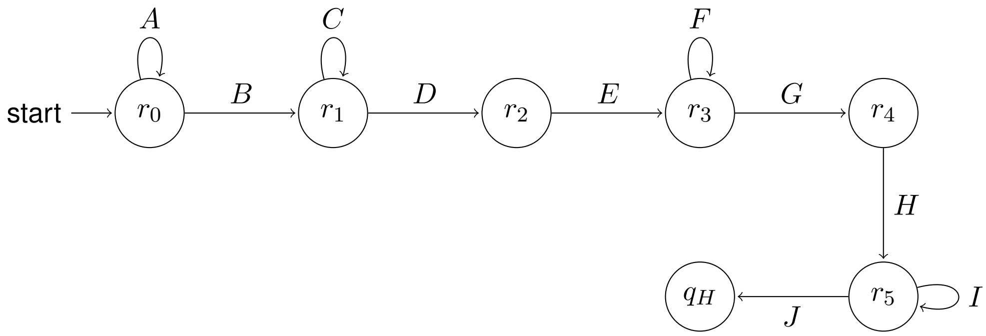
1. To begin, suppose consider an execution of this component that begins with the Turing machine in state r_0, with $\dot{0}1$ on the first tape, with the tape head pointing to the copy of "1", with $\dot{0}$ on the second tape, with the tape point pointing to the copy of "$\sqcup$" immediately to the right of the "$\dot{0}$" on the tape, and with third tape head filled copies of "$\sqcup$" and the tape head pointing to the leftmost cell — so that the Turing machine is in the configuration represented by the string

   $$\dot{0}\, r_0\, 1 \sharp \dot{0}\, r_0 \sharp r_0$$

   Show that, after a finite number of steps, the execution halts, with all tape heads filled with copies of "$\sqcup$" and with all tape heads pointing to the leftmost cells of the tapes.

2. Consider an execution of component in Figure 1 that begins with the Turing machine in a configuration represented by a string

   $$\mu_L\, r_0\, \mu_R \sharp \mu_2\, r_0 \sharp \mu_3\, r_0$$



A: For all $\sigma \in \Sigma_{\text{pair}}$, $\delta(r_0, \sigma, \sqcup, \sqcup) = (r_0, (\sigma, \mathtt{R}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$

B: $\delta(r_0, \sqcup, \sqcup, \sqcup) = (r_1, (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$

C: For all $\sigma \in \Sigma_{\text{pair}}$, $\delta(r_1, \sigma, \sqcup, \sqcup) = (r_1, (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$

D: For all $\sigma \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}, \dot{\#}\}$, $\delta(r_1, \sigma, \sqcup, \sqcup) = (r_2, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$

E: $\delta(r_2, \sqcup, \sqcup, \sqcup) = (r_3, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}))$

F: For all $\sigma \in \Sigma_{\text{pair}}$, $\delta(r_3, \sqcup, \sigma, \sqcup) = (r_3, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}))$

G: For all $\sigma \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}, \sqcup\}$, $\delta(r_3, \sqcup, \sigma, \sqcup) = (r_4, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$

H: $\delta(r_4, \sqcup, \sqcup, \sqcup) = (r_5, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{L}))$

I: For all $\sigma \in \Sigma_{\text{pair}}$, $\delta(r_5, \sqcup, \sqcup, \sigma) = (r_5, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{L}))$

J: For all $\sigma \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}, \sqcup\}$, $\delta(r_5, \sqcup, \sqcup, \sigma) = (q_{\text{halt}}, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S})))$

Figure 1: Component of a 3-Tape Turing Machine

where μ_L begins with either $\dot{\mathtt{0}}$ or $\dot{\mathtt{1}}$ and all other symbols in μ_L and μ_R are in Σ_{pair}, either μ_2 begins with one of $\dot{\mathtt{0}}$ or $\dot{\mathtt{1}}$ and all the other symbols in μ_2 are in $\{\mathtt{0}, \mathtt{1}\}$, or $\mu_2 = \lambda$, and either μ_3 begins with one of $\dot{\mathtt{0}}$ or $\dot{\mathtt{1}}$ and all the other symbols in μ_3 are in $\{\mathtt{0}, \mathtt{1}\}$, or $\mu_3 = \lambda$.

(a) ***Briefly*** explain why $\mu_L\, r_0\, \mu_R \sharp \mu_2\, r_0 \sharp \mu_3\, r_0 \vdash^\star \mu_L\, \mu_R\, r_0 \sharp \mu_2\, r_0 \sharp \mu_3\, r_0$.

(b) ***Briefly*** explain why $\mu_L\, \mu_R\, r_0 \sharp \mu_2\, r_0 \sharp \mu_3\, r_0 \vdash^\star r_2 \sharp \mu_2\, r_2 \mid \mu_3\, r_2$.

(c) ***Briefly*** explain why $r_2 \sharp \mu_2\, r_2 \sharp \mu_3\, r_2 \vdash^\star r_4 \sharp r_4 \sharp \mu_3\, r_4$.

(d) ***Briefly*** explain why $r_4 \sharp r_4 \sharp \mu_3\, r_4 \vdash^\star q_{\text{halt}} \sharp q_{\text{halt}} \sharp q_{\text{halt}}$.

It follows from this that this component can be used to continue from a configuration

$$\mu_L\, r_0\, \mu_R \sharp \mu_2\, r_0 \sharp \mu_3\, r_0$$

— with μ_L, μ_R, μ_2 and μ_3 as described here — to the end of a computation in which the empty string is returned as output.

Consider, now, another component of a Turing machine that is as shown Let us consider, now, another component of a “binary addition” Turing machine, that is as shown in Figure 2 on page 4 — where $\delta(q_0, \sqcup, \sqcup, \sqcup) = (q_{\text{halt}}, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$. In order to keep the diagram simple, transitions, to the component in Figure 1 are not shown — and these are as follows:

- $\delta(q_0, \#, \sqcup, \sqcup) = (r_0, (\dot{\#}, \mathrm{R}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.
- For all $\sigma \in \{0, 1, \sqcup\}$, $\delta(s_1, \sigma, \sqcup, \sqcup) = (r_0, (\sigma, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.
- $\delta(s_2, \sqcup, \sqcup, \sqcup) = (r_0, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.
- For $\sigma \in \{\#, \sqcup\}$, $\delta(s_3, \sigma, \sqcup, \sqcup) = (r_0, (\sigma, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.
- For all $\sigma \in \{0, 1, \#\}$, $\delta(s_4, \sigma, \sqcup) = (r_0, (\sigma, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.
- $\delta(s_5, \#, \sqcup, \sqcup) = (r_0, (\#, \mathrm{R}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$.

Some of these transitions might be followed ω either has at least two copies of “#” or does not include any copies of this symbol. Others are followed when $\omega = \mu\,\#\nu$ for $\mu \in \{0, 1\}^\star$ and $\nu \in \{0, 1, \#\}^\star$, but at least one of μ or ν begins with “0” and has length at least two.

3. Show that $q_0\,\omega \;\sharp\; q_0 \;\sharp\; q_0 \vdash^\star q_{\text{halt}} \;\sharp\; q_{\text{halt}} \;\sharp\; q_{\text{halt}}$, for $\omega \in \{0, 1, \#\}^\star$, in each of the following cases.

   (a) $\omega = \lambda$.

   (b) ω begins with “#”.

   (c) $\omega = 0$.

   (d) $\omega = 0\alpha$, for a string $\alpha \in \{0, 1, \#\}^\star$ that begins with either “0” or “1”.

   (e) ω begins with “1” but does not include any copies of “#”.

   (f) $\omega = \mu\,\#$, where μ is the unpadded binary representation of some non-negative integer.

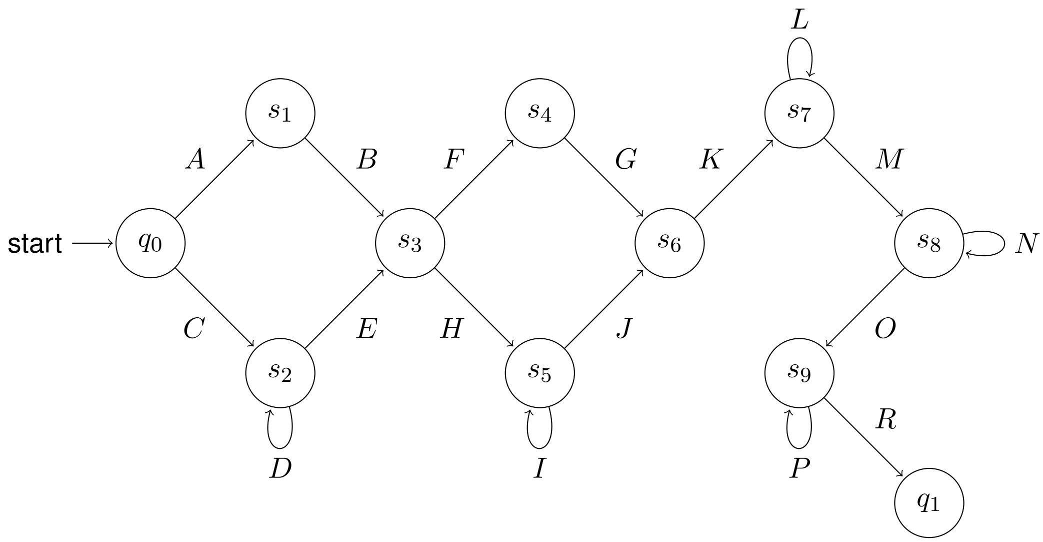
   (g) $\omega = \mu\#0\alpha$, where μ is the unpadded binary representation of some non-negative integer, and α is a non-empty string in $\{0, 1, \#\}^\star$.

   (h) ω begins with $\mu\#1$, where μ is the unpadded binary representation of some non-negative integer, but ω includes at least two copies of “#”.

   Note that if none of the above cases hold, then $\omega = \mu\,\#\,\nu$, where $\mu, \nu \in \{0, 1\}$ are the unpadded binary representations of a pair of non-negative integers.

4. Suppose, next, that $\omega = \mu\,\#\,\nu$, where μ and ν are the unpadded binary representations of a pair of non-negative integers. For a non-empty string

$$\zeta = \alpha_1 \alpha_2 \ldots \alpha_\ell$$



A: $\delta(q_0, \mathtt{0}, \sqcup, \sqcup) = (s_1, (\dot{\mathtt{0}}, \mathtt{R}), (\dot{\mathtt{0}}, \mathtt{R}), (\sqcup, \mathtt{S}))$
B: $\delta(s_1, \mathtt{\#}, \sqcup, \sqcup) = (s_3, (\mathtt{\#}, \mathtt{R}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
C: $\delta(q_0, \mathtt{1}, \sqcup, \sqcup) = (s_2, (\dot{\mathtt{1}}, \mathtt{R}), (\dot{\mathtt{1}}, \mathtt{R}), (\sqcup, \mathtt{S}))$
D For all $\sigma \in \{\mathtt{0}, \mathtt{1}\}$, $\delta(s_2, \sigma, \sqcup, \sqcup) = (s_2, (\sigma, \mathtt{R}), (\sigma, \mathtt{R}), (\sqcup, \mathtt{S}))$
E $\delta(s_2, \mathtt{\#}, \sqcup, \sqcup) = (s_3, (\mathtt{\#}, \mathtt{R}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
F: $\delta(s_3, \mathtt{0}, \sqcup, \sqcup) = (s_4, (\mathtt{0}, \mathtt{R}), (\sqcup, \mathtt{S}), (\dot{\mathtt{0}}, \mathtt{R}))$
G $\delta(s_4, \sqcup, \sqcup, \sqcup) = (s_6, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
H: $\delta(s_3, \mathtt{1}, \sqcup, \sqcup) = (s_5, (\mathtt{1}, \mathtt{R}), (\sqcup, \mathtt{S}), (\dot{\mathtt{1}}, \mathtt{R}))$
I: For all $\sigma \in \{\mathtt{0}, \mathtt{1}\}$, $\delta(s_5, \sigma, \sqcup, \sqcup) = (s_5, (\sigma, \mathtt{R}), (\sqcup, \mathtt{S}), (\sigma, \mathtt{R}))$
J: $\delta(s_5, \sqcup, \sqcup, \sqcup) = (s_6, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
K: $\delta(s_6, \sqcup, \sqcup, \sqcup) = (s_7, (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
L: For all $\sigma \in \{\mathtt{0}, \mathtt{1}, \mathtt{\#}\}$, $\delta(s_7, \sigma, \sqcup, \sqcup) = (s_7, (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}), (\sqcup, \mathtt{S}))$
M: For all $\sigma \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}\}$, $\delta(s_7, \sigma, \sqcup, \sqcup) = (s_8, (\sqcup, \mathtt{S}), (\sqcup, \mathtt{L}), (\sqcup, \mathtt{S}))$
N: For all $\sigma \in \{\mathtt{0}, \mathtt{1}\}$, $\delta(s_8, \sqcup, \sigma, \sqcup) = (s_8, (\sqcup, \mathtt{S}), (\sigma, \mathtt{L}), (\sqcup, \mathtt{S}))$
O: For all $\sigma \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}\}$, $\delta(s_8, \sqcup, \sigma, \sqcup) = (s_9, (\sqcup, \mathtt{S}), (\sigma, \mathtt{S}), (\sqcup, \mathtt{L}))$
P: For all $\alpha \in \{\dot{\mathtt{0}}, \dot{\mathtt{1}}\}$ and for all $\beta \in \{\mathtt{0}, \mathtt{1}\}$, $\delta(s_9, \sqcup, \alpha, \beta) = (s_9, (\sqcup, \mathtt{S}), (\alpha, \mathtt{S}), (\beta, \mathtt{L}))$
R: For all $\alpha, \beta \in \{\mathtt{0}, \mathtt{1}\}$, $\delta(s_9, \sqcup, \dot{\alpha}, \dot{\beta}) = (q_1, (\sqcup, \mathtt{S}), (\alpha, \mathtt{S}), (\beta, \mathtt{S}))$

Figure 2: Another Component of a 3-Tape Turing Machine

with length $\ell \geq 1$ (and with $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \{\mathtt{0}, \mathtt{1}\}$, let

$$\zeta_M = \dot{\alpha}_1 \alpha_2 \ldots \alpha_\ell$$

— so that the leftmost symbol, α, has been replaced in ζ_M by the corresponding "dotted" symbol, $\dot{\alpha}_1$, and no other symbols have been changed.

(a) Show that $q_0\,\mu\,\#\,\nu\,\sharp\,q_0\,\sharp\,q_0 \vdash^{\star} \mu_M\,\#\,s_3\,\nu\,\sharp\,\mu_M\,s_3\,\sharp\,s_3$.

(b) Show that $\mu_M\,\#\,s_3\,\nu\,\sharp\,\mu_M\,s_3\,\sharp\,s_3 \vdash^{\star} \mu_M\,\#\,\nu\,s_6\,\sharp\,\mu_M\,s_6\,\sharp\,\nu_M\,s_6$.

(c) Show that $\mu_M\,\#\,\nu\,s_6\,\sharp\,\mu_M\,s_6\,\sharp\,\nu_M\,s_6 \vdash^{\star} s_8\,\sharp\,\mu_M\,s_8\,\sharp\,\nu_M\,s_8$.

(d) Show that $s_8\,\sharp\,\mu_M\,s_8\,\sharp\,\nu_M\,s_8 \vdash^{\star} s_9\,\sharp\,s_9\,\mu_M\,\sharp\,\nu_M\,s_9$.

(e) Show that $s_9\,\sharp\,s_9\;\;\mu_9\,\sharp\,\nu_M\,s_9 \vdash^{\star} q_1\,\sharp\,q_1\,\mu\,\sharp\,q_1\,\nu$.

Note that it now follows that $q_0\,\mu\,\#\,\nu \vdash^{\star} q_1\,\sharp\,q_1\,\mu\,\sharp\,q_1\,\nu$.

5. Use the above to show that the these components correctly implement step 1 of the algorithm for the addition of binary numbers that is discussed in the presentation for Lecture #7.