Lecture #7: Simulations and Turing Machine Variants

Equivalence of Standard Turing Machines and Turing Machines with Left Failure

This document — which is not required reading — provides additional details about the proof, sketched in the preparatory material for this lecture, that the sets of recognizable and decidable languages would not be changed, if these were defined using "Turing Machines with Left Failure" instead of the (standard) Turing machines introduced in Lecture #7.

Turing Machines with Left Failure

A ***Turing machine with left failure*** is a variant of a Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

where Q, Σ, Γ, δ, q_0, q_{accept} and q_{reject} are all as before, but the transition function, δ, is applied differently: Whenever a transition

$$\delta(q, \sigma) = (r, \tau, \mathtt{L})$$

is to be applied, for $q, r \in Q$ and $\sigma, \tau \in \Gamma$, and the tape head is already at the leftmost cell then — instead of continuing without moving the location of the tape head — the computation halts and the input string is ***rejected***.

Simulating Standard Turing Machines using Turing Machines with Left Failure

Claim 4. *Let $L \subseteq \Sigma^\star$.*

(a) If L is recognizable then there exists a Turing machine, with left failure, with language L.

(b) If L is decidable then there exists a Turing machine, with left failure, that decides L.

A ***simulation*** will be used to prove this result.

Let $L \subseteq \Sigma^\star$ and suppose that L is recognizable. Then there exists a (standard) Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

such that $L = L(M)$. In order to prove the claim, let us describe a Turing machine with left failure,

$$\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\Gamma}, \widehat{\delta}, \widehat{q}_0, \widehat{q}_{\text{accept}}, \widehat{q}_{\text{reject}})$$

that simulates the given Turing machine, M.

Representing a Configuration of M

The main idea, here, will be to shift everything on the tape over by one symbol, using a special marker at the left end of the tape — so that the *simulating* machine, $\widehat{M}$ can know when the *simulated* machine is trying to move past the left end of the tape. In particular, let "#" be a symbol such that $\# \notin \Gamma$ and let

$$\widehat{\Gamma} = \Gamma \cup \{\#\}.$$

For strings $\mu, \nu \in \Gamma^\star$ and a state $q \in Q$ a configuration "$\mu q \nu$", of M, will be represented using the configuration

$$\# \mu q \nu$$

of $\widehat{M}$. All states in Q will be included in $\widehat{Q}$, so that M's configurations can be represented in this way.

Initialization

In order for $\widehat{M}$ to move from its initial configuration, for a string $\omega \in \Sigma^\star$, to its representation of M's initial configuration for this string, let

$$\widehat{Q}_{\text{Init}} = \{\widehat{q}_0\} \cup \{\widehat{q}_{1,\sigma} \mid \sigma \in \Sigma\} \cup \{\widehat{q}_2\}$$

be a set of (distinct) new states that do not belong to Q — which will be included in $\widehat{Q}$. In order to begin the definition of $\widehat{\delta}$ let us define transitions as follows.

- $\widehat{\delta}(\widehat{q}_0, \sqcup) = (q_0, \#, \mathtt{R})$.
- For every symbol $\sigma \in \Sigma$, $\widehat{\delta}(\widehat{q}_0, \sigma) = (\widehat{q}_{1,\sigma}, \#, \mathtt{R})$.
- For all symbols $\sigma, \tau \in \Sigma$, $\widehat{\delta}(\widehat{q}_{1,\sigma}, \tau) = (\widehat{q}_{1,\tau}, \sigma, \mathtt{R})$.
- For every symbol $\sigma \in \Sigma$, $\widehat{\delta}(\widehat{q}_{1,\sigma}, \sqcup) = (\widehat{q}_2, \sigma, \mathtt{L})$.
- For every symbol $\sigma \in \Sigma$, $\widehat{\delta}(\widehat{q}_2, \sigma) = (\widehat{q}_2, \sigma, \mathtt{L})$.
- $\widehat{\delta}(\widehat{q}_2, \#) = (q_0, \#, \mathtt{R})$.

Note that

$$\widehat{q}_0 \vdash \texttt{\#}\, q_0 \qquad\qquad \text{(since } \widehat{\delta}(\widehat{q}_0, \sqcup) = (q_0, \texttt{\#}, \texttt{R})\text{).}$$

Thus, when $\omega = \lambda$, then $\widehat{M}$ moves from its initial configuration for ω to its representation of M's initial configuration for ω, using a single step.

Suppose, next, that $|\omega| = 1$ — so that $\omega = \sigma$, for some symbol $\sigma \in \Sigma$. Then

$$\begin{aligned} \widehat{q}_0\, \sigma &\vdash \texttt{\#}\, \widehat{q}_{1,\sigma} && \text{(since } \widehat{\delta}(\widehat{q}_0, \sigma) = (\widehat{q}_{1,\sigma}, \texttt{\#}, \texttt{R})\text{)} \\ &\vdash \widehat{q}_2\, \texttt{\#}\sigma && \text{(since } \widehat{\delta}(\widehat{q}_{1,\sigma}, \sqcup) = (\widehat{q}_2, \sigma, \texttt{L})\text{)} \\ &\vdash \texttt{\#}\, q_0\, \sigma && \text{(since } \widehat{\delta}(\widehat{q}_2, \texttt{\#}) = (q_0, \texttt{\#}, \texttt{R})\text{).} \end{aligned}$$

Thus, when $|\omega| = 1$, $\widehat{M}$ moves from its initial configuration for ω to its representation of M's initial configuration for ω, using three steps.

Suppose, now, that $|\omega| \geq 2$. Let

$$\omega = \alpha_1 \alpha_2 \ldots \alpha_n$$

(for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma$). Then

$$\begin{aligned} \widehat{q}_0\, \omega &= \widehat{q}_0\, \alpha_1 \alpha_2 \ldots \alpha_n \\ &\vdash \texttt{\#}\, \widehat{q}_{1,\alpha_1}\, \alpha_2 \alpha_3 \ldots \alpha_n && \text{(since } \widehat{\delta}(\widehat{q}_0, \alpha_1) = (\widehat{q}_{1,\alpha_1}, \texttt{\#}, \texttt{R})\text{).} \end{aligned}$$

Exercise: Using induction on i, prove that

$$\widehat{q}_0 \omega \vdash^{\star} \texttt{\#}\alpha_1 \alpha_2 \ldots \alpha_{i-1}\, \widehat{q}_{1,\alpha_i}\, \alpha_{i+1} \alpha_{i+2} \ldots \alpha_n$$

for every integer i such that $2 \leq i \leq n-1$.

Continuing,

$$\begin{aligned} \widehat{q}_0 \omega &\vdash^{\star} \texttt{\#}\alpha_1 \alpha_2 \ldots \alpha_{n-2}\, \widehat{q}_{1,\alpha_{n-1}}\, \alpha_n && \text{(by the above result, when } i = n-1\text{)} \\ &\vdash \texttt{\#}\alpha_1 \alpha_2 \ldots \alpha_{n-1}\, \widehat{q}_{1,\alpha_n} && \text{(since } \widehat{\delta}(\widehat{q}_{1,\alpha_{n-1}}, \alpha_n) = (\widehat{q}_{1,\alpha_n}, \alpha_{n-1}, \texttt{R})\text{)} \\ &\vdash \texttt{\#}\alpha_1 \alpha_2 \ldots \alpha_{n-2}\, \widehat{q}_2\, \alpha_{n-1} \alpha_n && \text{(since } \widehat{\delta}(\widehat{q}_{1,\alpha_n}, \sqcup) = (\widehat{q}_2, \alpha_n, \texttt{L})\text{).} \end{aligned}$$

Recall that $\widehat{\delta}(\widehat{q}_2, \beta) = (\widehat{q}_2, \beta, \texttt{L})$ for every symbol $\beta \in \Sigma$.

Exercise: Using induction on i, prove that

$$\widehat{q}_0 \omega \vdash^{\star} \texttt{\#}\alpha_1 \alpha_2 \ldots \alpha_{n-i-1}\, \widehat{q}_2\, \alpha_{n-i} \alpha_{n-i+1} \ldots \alpha_n$$

for every integer i such that $1 \leq i \leq n-2$.

Continuing, once again,

$$\begin{aligned} \widehat{q}_0\omega &\vdash^\star \texttt{#}\, \widehat{q}_2\, \alpha_1\alpha_2\dots\alpha_n && \text{(by the above result, when } i = n-2\text{)}\\ &\vdash \widehat{q}_2\, \texttt{#}\alpha_1\alpha_2\dots\alpha_n && \text{(since } \widehat{\delta}(\widehat{q}_2,\alpha_1) = (\widehat{q}_2,\alpha_1,\texttt{L})\text{)}\\ &\vdash \texttt{#}\, q_0\, \alpha_1\alpha_2\dots\alpha_n && \text{(since } \widehat{\delta}(\widehat{q}_2,\texttt{#}) = (q_0,\texttt{#},\texttt{R})\text{)}\\ &= \texttt{#}\, q_0\, \omega. \end{aligned}$$

Thus $\widehat{M}$ moves from its initial configuration for ω to its representation of M's initial configuration for ω, when $|\omega| \geq 2$, as well.

Simulating a Step of M

For $q \in Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}$ and $\sigma \in \Gamma$ suppose that

$$\widehat{\delta}(q,\sigma) = \delta(q,\sigma)$$

Two situations might arise after this transition has been applied:

(a) The symbol # is *not* visible on $\widehat{M}$'s tape — so that M *did not* try to move past the left end of this tape when this step was carried out, a symbol in Γ is visible — and $\widehat{M}$ is in a configuration representing the configuration that M would reach after this step is taken (as desired).

(b) The symbol # *is* visible on $\widehat{M}$'s tape — so that M tried to move past the left end of its tape when this step was carried out.

Let us add the transition

$$\widehat{\delta}(r,\texttt{#}) = (r,\texttt{#},\texttt{R})$$

for every state $r \in Q$ to ensure that $\widehat{M}$ can go to its representation of the configuration that M would move to, in this case as well.

Cleanup

It remains only to ensure that $\widehat{M}$ moves from a representation of an accepting configuration (respectively, a rejecting configuration) to an accepting configuration (respectively, rejecting configuration) of its own. The inclusion of the transitions

$$\widehat{\delta}(q_{\text{accept}},\sigma) = (\widehat{q}_{\text{accept}},\sigma,\texttt{R}) \qquad \text{and} \qquad \widehat{\delta}(q_{\text{reject}},\sigma) = (\widehat{q}_{\text{reject}},\sigma,\texttt{R}),$$

for all $\sigma \in \Gamma$, ensures this.

Completing the Proof

Let $\omega \in \Sigma^\star$. One can now prove, by induction on t, that if t is a non-negative integer such that — when executed on input ω — M reaches a configuration

$$\mu q \nu$$

for $\mu, \nu \in \Gamma^\star$ and $q \in Q$, after taking t steps, then $\widehat{M}$ moves from configuration #$q_0\omega$ to configuration #$\mu q \nu$ after taking between t and $2t$ steps.

This can be used to prove that, for every string $\omega \in \Sigma^\star$,

- M accepts ω if and only if $\widehat{M}$ accepts ω,
- M rejects ω if and only if $\widehat{M}$ rejects ω, and
- M loops on ω if and only if $\widehat{M}$ loops on ω,

as needed to complete a proof of this claim.

Simulating Turing Machines with Left Failure Using Standard Turing Machines

Claim 5. *Let $L \subseteq \Sigma^\star$.*

(a) If there exists a Turing machine with left failure, whose language is L, then L is recognizable..

(b) If there exists a Turing machine, with left failure, that decides L, then L is decidable.

A ***simulation*** will be used to prove this result, as well.

Let $L \subseteq \Sigma^\star$, and let

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

be a Turing machine, with left failure, whose language is L. In order to prove the claim, let us describe a (standard) Turing machine,

$$\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\Gamma}, \widehat{\delta}, \widehat{q}_0, \widehat{q}_{\text{accept}}, \widehat{q}_{\text{reject}})$$

that simulates the given Turing machine with left failure, M.

Representing a Configuration of M

Once again the main idea, here, will be to shift everything on the tape over by one symbol, using a special marker at the left end of the tape — so that the *simulating* machine, $\widehat{M}$ can know when the *simulated* machine is trying to move past the left end of the tape. As above, let "#" be a symbol such that $\# \notin \Gamma$ and let

$$\widehat{\Gamma} = \Gamma \cup \{\#\}.$$

For strings $\mu, \nu \in \Gamma^\star$ and a state $q \in Q$ a configuration "$\mu q \nu$", of M, will be represented using the configuration

$$\#\mu q \nu$$

of $\widehat{M}$.

Initialization

Since $\widehat{M}$'s representation of a configuration of M is the same, for this simulation, as the one used for the simulation used to prove Claim 4, the Initialization phase that was used in the above simulation can be used, here, as well.

Simulating a Step of M

For $q \in Q \setminus \{q_{\text{accept}}, q_{\text{reject}}\}$ and $\sigma \in \Gamma$ suppose that

$$\widehat{\delta}(q, \sigma) = \delta(q, \sigma)$$

Two situations might arise after this transition has been applied:

(a) The symbol # is *not* visible on $\widehat{M}$'s tape — so that M *did not* try to move past the left end of this tape when this step was carried out, a symbol in Γ is visible — and $\widehat{M}$ is in a configuration representing the configuration that M would reach after this step is taken (as desired).

(b) The symbol # *is* visible on $\widehat{M}$'s tape — so that M tried to move past the left end of its tape when this step was carried out.

Let us add the transition

$$\widehat{\delta}(r, \#) = (q_{\text{reject}}, \#, \mathtt{R})$$

— for every state $r \in Q$ — to ensure that $\widehat{M}$ is in a configuration representing a ***rejecting*** configuration of M, whenever M would have tried to fall off the left end of its tape.

Cleanup

As with the previous simulation, it remains only to ensure that $\widehat{M}$ moves from a representation of an accepting configuration (respectively, a rejecting configuration) to an accepting configuration (respectively, rejecting configuration) of its own. The inclusion of the transitions

$$\widehat{\delta}(q_{\text{accept}}, \sigma) = (\widehat{q}_{\text{accept}}, \sigma, \mathtt{R}) \qquad \text{and} \qquad \widehat{\delta}(q_{\text{reject}}, \sigma) = (\widehat{q}_{\text{reject}}, \sigma, \mathtt{R}),$$

for all $\sigma \in \Gamma$, ensures this.

Completing the Proof

Let $\omega \in \Sigma^\star$. One can now prove, by induction on t, that if t is a non-negative integer such that — when executed on input ω — M reaches a configuration

$$\mu q \nu$$

for $\mu, \nu \in \Gamma^\star$ and $q \in Q$, after taking t steps, without trying to move past the left end of its tape, then $\widehat{M}$ moves from configuration $\#q_0\omega$ to configuration $\#\mu q \nu$ after taking t steps. If M rejects by trying to move left past the left end of its tape, after taking t steps, then $\widehat{M}$ enters a representation of a rejecting configuration of M after taking $t+1$ steps.

This can be used to prove that, for every string $\omega \in \Sigma^\star$,

- M accepts ω if and only if $\widehat{M}$ accepts ω,
- M rejects ω if and only if $\widehat{M}$ rejects ω, and
- M loops on ω if and only if $\widehat{M}$ loops on ω,

as needed to complete a proof of this claim.