Computer Science 351

Nonregular Languages, Part Two

Instructor: Wayne Eberly

Department of Computer Science
University of Calgary

Lecture #10

Goal for Today

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- Describe how the ***closure properties for regular languages*** can *also* be used to prove that a language $L \subseteq \Sigma^\star$ is ***not*** regular.

Closure Properties

If $L_1, L_2 \subseteq \Sigma^\star$ for some alphabet Σ, and L_1 and L_2 are both regular languages, then the following languages are regular languages as well.

(a) $L_1 \cup L_2$

(b) $L_1 \circ L_2$

(c) $L_1^\star$

See the lecture slides on "Regular Operations and Regular Expressions" for information about these examples of ***closure properties***.

Closure Properties

Recall that if $L \subseteq \Sigma^\star$ then the ***complement*** of L is the language

$$L^C = \{\omega \in \Sigma^\star \mid \omega \notin L\}.$$

Theorem #1. Suppose that $L \subseteq \Sigma^\star$ for an alphabet Σ. If L is regular, then L^C is a regular language as well.

Sketch of Proof:

- There exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = L$.
- Let $\widehat{M} = (Q, \Sigma, \delta, q_0, \widehat{F})$ where

$$\widehat{F} = \{q \in Q \mid q \notin F\}.$$

- Then $L(\widehat{M}) = L^C$ so that L^C is a regular language. $\square$

Closure Properties

Since $P \Rightarrow Q$ implies that $\neg Q \Rightarrow \neg P$, the above closure properties imply the following.

(a) For all languages $L_1, L_2 \subseteq \Sigma^\star$, for any alphabet Σ, if $L_1 \cup L_2$ is ***not*** a regular language then *at least one* of L_1 or L_2 is ***not*** a regular language either.

(b) For all languages $L_1, L_2 \subseteq \Sigma^\star$, for any alphabet Σ, if $L_1 \circ L_2$ is ***not*** a regular language, then *at least one* of L_1 or L_2 is ***not*** a regular language either.

(c) For every language $L \subseteq \Sigma^\star$ over any alphabet Σ, if $L^\star$ is ***not*** a regular language then L is ***not*** a regular language either.

(d) For every language $L \subseteq \Sigma^\star$ over any alphabet Σ, if L^C is ***not*** a regular language then L is ***not*** a regular language either.

Closure Properties

Key Observation

- These results can be used to show that if one language is not a regular language then another (given) language cannot be regular, either.
- However, we need to know about some provably nonregular languages before these results can be used — so a result like the Pumping Lemma for Regular Languages is also needed.

Example

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

- Let
$$L_1 = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{Z} \text{ and } n \geq 0\} \subseteq \Sigma^\star.$$
The Pumping Lemma for Regular Languages was used, in the first lectures slides about "Nonregular Languages", to prove that L_1 is ***not*** a regular language.
- Let
$$L_2 = \{\mathtt{a}^n\mathtt{b}^m \mid n, m \in \mathbb{Z},\ n, m \geq 0, \text{ and } n \neq m\} \subseteq \Sigma^\star.$$

Question: Can we use one of the closure properties, discussed here, to prove that L_2 is *also* not a regular language?

Example

Let

$$L_3 = \{\omega \in \Sigma^\star \mid \mathtt{ba} \text{ is a substring of } \omega\} \subseteq \Sigma^\star$$

and let

$$L_4 = L_2 \cup L_3 \subseteq \Sigma^\star.$$

Lemma #2: $L_1 = L_4^C$.

How To Prove This:

- Prove that $L_1 \subseteq L_4^C$.
- Prove that $L_4^C \subseteq L_1$.

Example

Lemma #3: L_4 is ***not*** a regular language.

How To Prove This: Apply Closure Property (d) — closure under complementation.

Lemma #4: L_3 ***is*** a regular language.

One Way To Prove This: Describe a DFA $M = (Q, \Sigma, \delta, F)$ and prove that $L(M) = L_3$.

Theorem #5: L_2 is ***not*** a regular language.

How To Prove This: Apply Closure Property (a) — closure under union — along with Lemma #3 and Lemma #4.

Summary of Process

To prove that a language $L \subseteq \Sigma^\star$ (for some alphabet Σ) is not regular . . .

1. Assume — to obtain a ***contradiction*** — that L *is* a regular language.
2. If you already know — or can prove — that L^C is ***not*** a regular language then use closure property (d) — closure under complementation — to get a ***contradiction***.
3. **OR:** If you already know — or can prove — that $L^\star$ is ***not*** a regular language then use closure property (c) — closure under Kleene star — to get a ***contradiction***.

Summary of Process

4. **OR:** If you already know — or can prove — that $L \cup \widehat{L}$ is ***not*** a regular language, for some ***regular language*** $\widehat{L} \subseteq \Sigma^{\star}$ — then use closure property (a) — closure under union — to get a ***contradiction***.
5. **OR:** If you already know — or can prove — that $L \circ \widehat{L}$ is ***not*** a regular language, for some ***regular language*** $\widehat{L} \subseteq \Sigma^{\star}$ — then use closure property (b) — closure under concatenation — to get a ***contradiction***.
6. **OR:** If you already know — or can prove — that $\widehat{L} \circ L$ is ***not*** a regular language, for some ***regular language*** $\widehat{L} \subseteq \Sigma^{\star}$ — then use closure property (b) — closure under concatenation — to get a ***contradiction***.

And There's More!

- Other ***closure properties*** (involving other operations on languages) can be proved for the set of regular languages, as well.
- These "automatically" provide corresponding (new) ways to prove that languages are ***not*** regular.