Lecture #5: Nonregular Languages

What To Do Before the Lecture

1. Watch the videos for Lecture #5 —noting that they will probably be understandable if you play them at double speed. If you do not have time for this then look at the "Key Concepts" document that is found, immediately after the videos for this lecture, on the course web site, instead.

2. ***Print*** and read through the rest of this document and — if you have time — try to solve the problems! These should help you to check that you understand how to use several techniques to prove that a given language is not regular.

3. The supplemental material also include another example of the use of the Pumping Lemma to prove that a language is not regular, and (for interest only) a proof of this technical result.

Problems. To Be Solved

1. Using the Pumping Lemma, we wish to show, for $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, that the language

$$L = \{\mathtt{a}^{(n^2)} \mid n \in \mathbb{N}\} \subseteq \Sigma^\star$$

is not regular.

(a) What ***proof technique*** should be used, when the "Pumping Lemma" is to be applied to prove that a language is not regular?

(b) Why is it often a good idea to write the “Pumping Lemma” out, near the beginning of a proof where you are using it?

(c) What gets introduced (in the argument) right away? What, if anything, can be assumed about this? Why?

(d) The ***second*** thing that gets introduced, is a string s. What properties do you need to show that this string has? Why?

(e) List some reasonable choices for the “string s” that you might choose, when you are trying use the “Pumping Lemma for Regular Languages” to prove that the above language, L, is not regular.

(f) What else must be introduced, and what more do you need to do, to complete the proof?

(g) List the points that must be made (or the things that must be done) to finish — explaining *why* the various things, that must be established, are true when you use the above string s.

(h) Use the above to write a proof that the language L is not regular — that uses the above ideas, but that is organized and written in a way that make it as easy as possible for another reader to understand and believe it.

2. Consider ***closure properties*** for regular languages.

   (a) List as many closure properties for regular languages, that have been stated in this course at this point, as you can.

   (b) Describe how these can be used to prove that languages are *not* regular.

3. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. We wish to prove that the language

$$L = \{\omega \in \Sigma^\star \mid \text{the number of a's in } \omega \text{ is equal to the number of b's in } \omega\} \subseteq \Sigma^\star$$

is not a regular language.

(a) State a reasonably similar language to this one that we already know is not regular.

(b) State, as precisely as you can, how this language is related to the above language L.

(c) Using this information — and one or more closure properties (*without* using the Pumping Lemma) to prove that L is not regular.

4. Consider the following.

   Claim: For every alphabet Σ and for languages $L_1, L_2 \subseteq \Sigma^\star$, if L_1 is a regular language and L_2 is a regular language then their ***intersection***, $L_1 \cap L_2$, is also a regular language.

   (a) Use ***De Morgan's Laws*** to write $L_1 \cap L_2$ in another way (also expressing this as a function of L_1 to L_2 — but using the complements and unions of sets, instead of using set intersection.

   (b) Use this to write a proof of this claim.

(c) Use this to write a *shorter* and *simpler* proof that the language

$$L = \{\omega \in \Sigma^\star \mid \text{the number of a's in } \omega \text{ is equal to the number of b's in } \omega\} \subseteq \Sigma^\star$$

is not a regular language.