Computer Science 351

Introduction to Nondeterministic Finite Automata

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Lecture #5

Goals for Today

Goals for Today:

- Introduce ***nondeterministic finite automata*** — contrasting these with "deterministic finite automata".
- Present two ways to see how a nondeterministic finite automaton processes a string.

Note: The notion of ***nondeterminism*** will be extremely important later on in this course, and in CPSC 413.

Nondeterministic Finite Automata

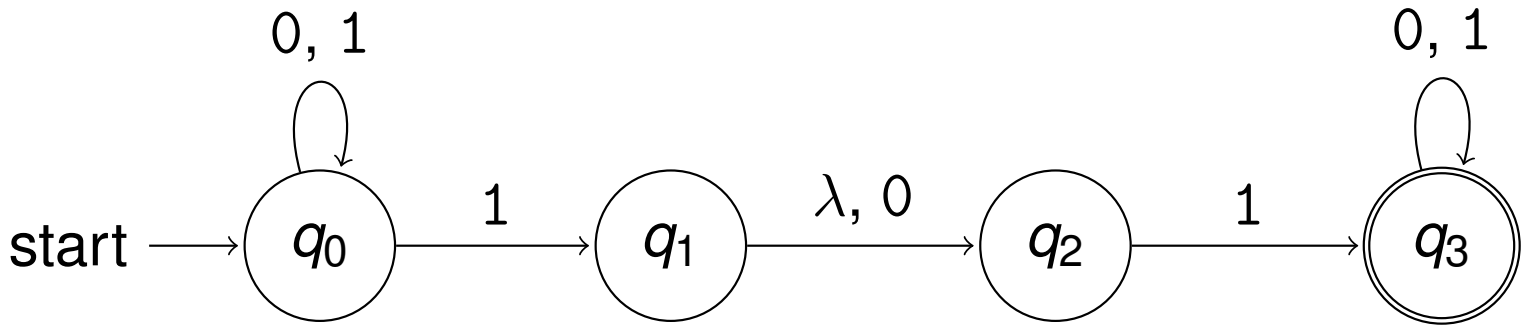
Suppose we "changed the rules" used to provide transition functions for finite automata:

- "λ-transitions" are introduced, allowing the machine to move from one state to another without processing any symbols in the input string at all, and
- the machine is allowed to move from a given state to *zero*, *one*, or *many* states when a symbol $\sigma \in \Sigma$ or λ is processed.

The resulting "finite-state machines" — now called ***nondeterministic finite automata*** — could look like the pictures shown on the next two slides.

First Example of an NFA

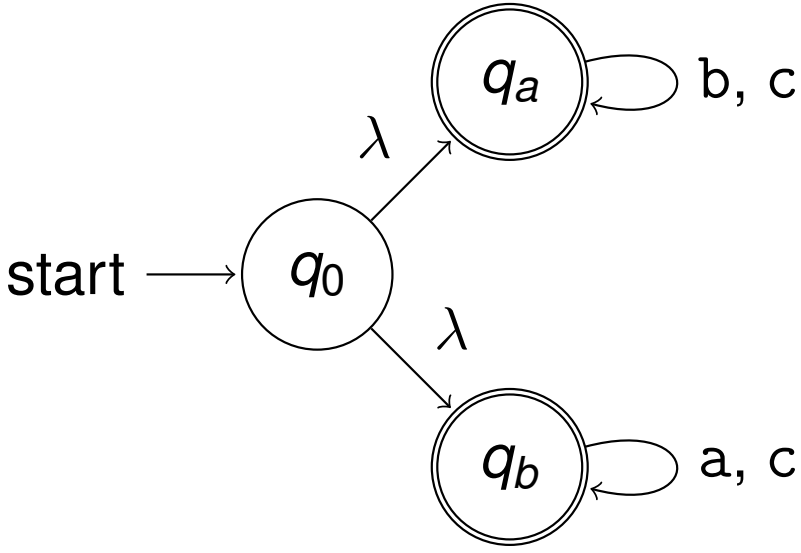
Input alphabet: $\Sigma_1 = \{0, 1\}$; NFA M_1 is as follows.



Three states — q_0, q_1 and q_2 — can be reached from the start state by processing 1.

Second Example of an NFA

Input alphabet: $\Sigma_2 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$; NFA M_2 is as follows.



- ***Three states*** — q_0, q_a and q_b — can be reached from the start state without processing any symbols at all.
- ***One state*** — q_b — can be reached from the start state by processing "a".

Processing of Strings

In a way, nondeterministic finite automata are like

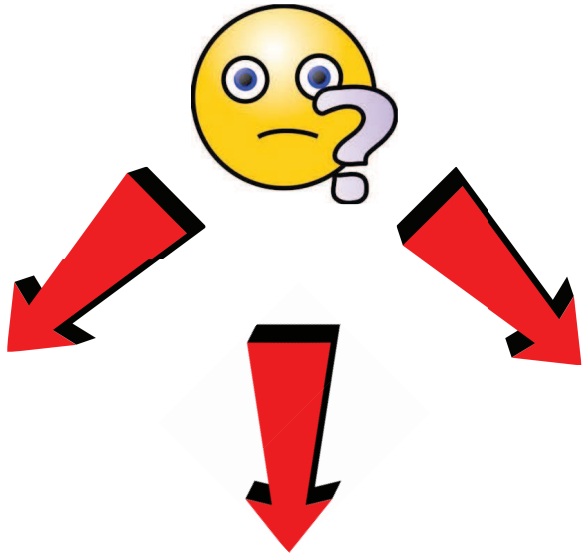
Magic!

Processing of Strings

When a ***deterministic finite automaton*** is used to process a string there is always ***exactly one*** transition that can be followed to process a symbol.

Processing of Strings

On the other hand, when a ***nondeterministic finite automaton*** is used to process a string there may be zero, one, or ***many*** transitions that you might choose.



Furthermore, if the nondeterministic finite automaton includes λ-transitions then it is possible to use one in order to change state, without processing any symbols at all!

Processing of Strings

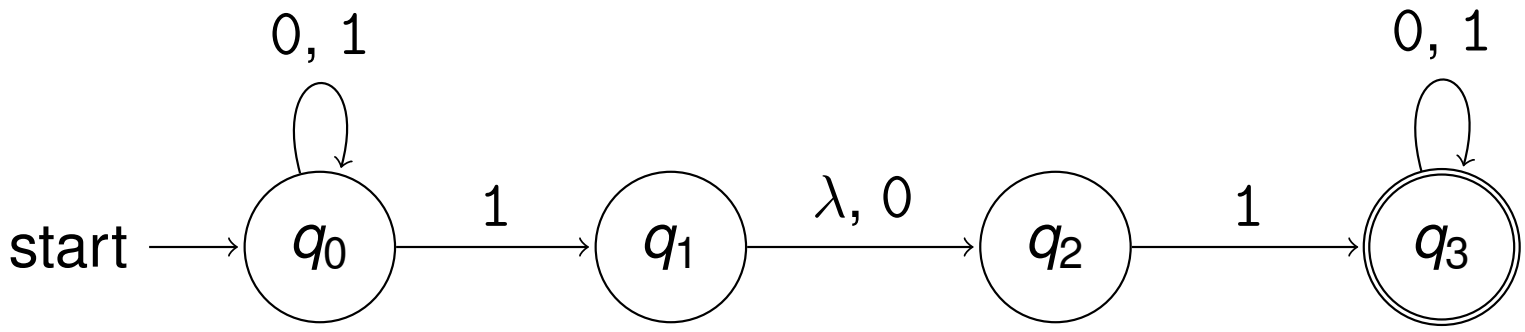
One can think about processing an input string by

guessing your way toward an accepting state....

because the string should be accepted as long as there is ***at least way*** to do this, so that an accepting state has been reached when the entire string has been processed.

Processing of Strings

Consider, again, the first NFA M_1 shown above:



When processing the string 11, one can do the following:

- Use the transition for symbol 1 from state q_0 to state q_1, in order to process the first 1 and reach state q_1;
- follow the λ-transition from q_1 to q_2 in order to reach q_2;
- follow the transition for symbol 1 from q_2 to q_3 in order to process the final 1 in the input string and reach q_3.

Since q_3 is an accepting state it follows that this NFA ***accepts*** the string 11.

Processing of Strings

One can also think about processing an input string by

keeping track of ***all*** of the states that can be reached as symbols are processed.

A process to do this is described next.

Summary of a Useful Process

To determine the set of states that are reachable by processing a string

$$\omega = \omega_1\omega_2\ldots\omega_n \in \Sigma^\star$$

1. Create a set S_λ by including all the states reachable from the start state, q_0, by following zero or more λ transitions.
2. `for` $i = 1, 2, \ldots, n$
   - Initialize $S_{\omega_1\omega_2\ldots\omega_i}$ to be $\emptyset$
   - `for` every state $q \in S_{\omega_1\omega_2\ldots\omega_{i-1}}$, add, to $S_{\omega_1\omega_2\ldots\omega_i}$, every state r that is reachable from q by following a transition (from q) for the symbol $\omega_i \in \Sigma$, and then following zero or more λ-transitions after that.
3. The set of states that are reachable from q_0 by processing the above string ω is the set $S_\omega = S_{\omega_1\omega_2\ldots\omega_n}$.

Acceptance of a String

Definition: A nondeterministic finite automaton M ***accepts*** a string $\omega \in \Sigma^\star$ if and only if the set of states that are reachable from the start state, q_0, by processing the string ω includes *one or more* accepting states $q \in F$.

Formal Definition of an NFA

Definition: Suppose S is a finite set. Then the ***power set*** of S, $\mathcal{P}(S)$, is the set of all ***subsets*** of S.

Example: If $S = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ then $\mathcal{P}(S)$ includes the following eight sets:

- $\emptyset$ (the empty set);
- $\{\mathtt{x}\}$;
- $\{\mathtt{y}\}$;
- $\{\mathtt{z}\}$;
- $\{\mathtt{x}, \mathtt{y}\}$;
- $\{\mathtt{x}, \mathtt{z}\}$;
- $\{\mathtt{y}, \mathtt{z}\}$;
- $\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$.

Formal Definition of an NFA

Definition: If Σ is an alphabet, including k symbols, then Σ_λ is a set of size $k+1$ including all the symbols in Σ as well as the empty string.

Example: if $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ then Σ_λ includes the following:

- $\mathtt{a}$;
- $\mathtt{b}$;
- $\mathtt{c}$;
- λ.

Formal Definition of an NFA

Definition: A ***nondeterministic finite automaton*** is 5-tuple

$$(Q, \Sigma, \delta, q_0, F),$$

where

1. Q is a finite (and nonempty) set of ***states***,
2. Σ is a finite (and nonempty) ***alphabet***,
3. $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the set of ***accept states***.

For $q \in Q$ and $\sigma \in \Sigma_\lambda$, $\delta(q, \sigma)$ is the *set* of states that can be reached by following a ***single*** transition for σ out of q.

Formal Definition of an NFA

The NFA M_1 can be formally modelled as $M_1 = (Q, \Sigma, \delta, q_0, F)$ where

1. $Q = \{q_0, q_1, q_2, q_3\}$;
2. $\Sigma = \Sigma_1 = \{0, 1\}$;
3. The transition function $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is shown in the table on the following slide;
4. q_0 is the start state;
5. $F = \{q_3\}$.

Formal Definition of an NFA

A table describing the transition function δ is as follows.

	0	1	λ
q_0	$\{q_0\}$	$\{q_0, q_1\}$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\{q_2\}$
q_2	$\emptyset$	$\{q_3\}$	$\emptyset$
q_3	$\{q_3\}$	$\{q_3\}$	$\emptyset$

Formal Definition of an NFA

The NFA M_2 can be formally modelled as $M_2 = (Q, \Sigma, \delta, q_0, F)$ where

1. $Q = \{q_0, q_a, q_b\}$;
2. $\Sigma = \Sigma_2 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$;
3. The transition function $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is shown in the table on the following slide;
4. q_0 is the start state;
5. $F = \{q_a, q_b\}$.

Formal Definition of an NFA

A table describing the transition function δ is as follows.

	a	b	c	λ
q_0	$\emptyset$	$\emptyset$	$\emptyset$	$\{q_a, q_b\}$
q_a	$\emptyset$	$\{q_a\}$	$\{q_a\}$	$\emptyset$
q_b	$\{q_b\}$	$\emptyset$	$\{q_b\}$	$\emptyset$

Formal Definition of an NFA

Consider a function $Cl_\lambda : Q \to \mathcal{P}(Q)$: For $q \in Q$, $Cl_\lambda(q)$ is the set of states reachable from q by following zero or more λ-transitions.

- In Example #1
  - $Cl_\lambda(q_0) = \{q_0\}$;
  - $Cl_\lambda(q_1) = \{q_1, q_2\}$;
  - $Cl_\lambda(q_2) = \{q_2\}$;
  - $Cl_\lambda(q_3) = \{q_3\}$.
- In Example #2
  - $Cl_\lambda(q_0) = \{q_0, q_a, q_b\}$;
  - $Cl_\lambda(q_a) = \{q_a\}$;
  - $Cl_\lambda(q_b) = \{q_b\}$.

$Cl_\lambda(q)$ is sometimes called the ***λ-closure*** of the state q.

Formal Definition of an NFA

It is now possible to define an ***extended transition function*** $\delta^\star : Q \times \Sigma^\star \to \mathcal{P}(Q)$: For each state $q \in Q$ and each string $\omega \in \Sigma^\star$, $\delta^\star(q, \omega)$ is the set of states that can be reached from q by processing the string ω.

This can be "formally defined" as follows:

- For every state $q \in Q$, $\delta^\star(q, \lambda) = Cl_\lambda(q)$.
- For every state $q \in Q$, every string $\omega \in \Sigma^\star$, and every symbol $\sigma \in \Sigma$,

$$\delta^\star(q, \omega \cdot \sigma) = \bigcup_{r \in \delta^\star(q,\omega)} \left(\bigcup_{s \in \delta(r,\sigma)} Cl_\lambda(s) \right).$$

Application: Evaluation of $\delta^\star$

Suppose we wish to evaluate $\delta^\star(q_0, 11)$. Setting $\omega = 1$ and $\sigma = 1$ the second part of the above definition implies that

$$\delta^\star(q_0, 11) = \bigcup_{r \in \delta^\star(q_0, 1)} \left(\bigcup_{s \in \delta(r, 1)} Cl_\lambda(s) \right). \tag{1}$$

Setting $\omega = \lambda$ and $\sigma = 1$, the second part of this definition implies that

$$\delta^\star(q_0, 1) = \bigcup_{r \in \delta^\star(q_0, \lambda)} \left(\bigcup_{s \in \delta(r, 1)} Cl_\lambda(s) \right). \tag{2}$$

Application: Evaluation of $\delta^\star$

The first part of the definition implies that

$$\delta^\star(q_0, \lambda) = Cl_\lambda(q_0) = \{q_0\}. \tag{3}$$

Applying equations (2) and (3), it now follows that

$$\begin{aligned}
\delta^\star(q_0, 1) &= \bigcup_{r \in \{q_0\}} \left(\bigcup_{s \in \delta(r,1)} Cl_\lambda(s) \right) \\
&= \bigcup_{s \in \delta(q_0,1)} Cl_\lambda(s) \\
&= Cl_\lambda(q_0) \cup Cl_\lambda(q_1) \\
&= \{q_0\} \cup \{q_1, q_2\} \\
&= \{q_0, q_1, q_2\}.
\end{aligned}$$

Application: Evaluation of $\delta^\star$

Applying this along with equation (1), it now follows that

$$\begin{aligned}
\delta^\star(q_0, 11) &= \bigcup_{r \in \{q_0, q_1, q_2\}} \left(\bigcup_{s \in \delta(r,1)} Cl_\lambda(s) \right) \\
&= \bigcup_{s \in \delta(q_0,1)} Cl_\lambda(s) \;\cup\; \bigcup_{s \in \delta(q_1,1)} Cl_\lambda(s) \;\cup\; \bigcup_{s \in \delta(q_2,1)} Cl_\lambda(s) \\
&= (Cl_\lambda(q_0) \cup Cl_\lambda(q_1)) \;\cup\; \emptyset \;\cup\; Cl_\lambda(q_3) \\
&= Cl_\lambda(q_0) \cup Cl_\lambda(q_1) \cup Cl_\lambda(q_3) \\
&= \{q_0\} \cup \{q_1, q_2\} \cup \{q_3\} \\
&= \{q_0, q_1, q_2, q_3\}.
\end{aligned}$$

Formal Definition of an NFA

Finally: For every string $\omega \in \Sigma^\star$, M ***accepts*** ω if and only if

$$\delta^\star(q_0, \omega) \cap F \neq \emptyset.$$

M ***rejects*** ω otherwise.

The ***language*** of M, $L(M)$, is the set of strings $\omega \in \Sigma^\star$ such that M accepts ω.

Which States are Reachable?

The above formal definitions — including the definition of the "extended transition function" — can be used to write a program that can be used to decide whether a given NFA M accepts a given string $\omega \in \Sigma^\star$, ***provided that*** there is an algorithm (and program) that can be used to compute the set $Cl_\lambda(q)$ for any given state $q \in Q$.

An algorithm that can be used to do this will be described in a separate document.