

CPSC 351 — Mathematics Review

Part One: Sets, Sequences, Functions and Relations

Ideally, everything in the document should be a **review** of material that you learned about in a prerequisite course. It will be assumed that you understand and can use all of it in CPSC 351.

This document is based, heavily, on the review included in Sipser's text, *Introduction to the Theory of Computation* [3]. It should also be consistent with material in the textbook (or recommended reference) in the course in discrete mathematics that you completed as a prerequisite for this one — probably (but not necessarily) either the text of Epp [1] or Rosen [2].

Sets

Definition and Notation

A **set** is a collection of objects. Sets may contain any type of object, including numbers, symbols, or other sets.

Finite Sets. If a set has only a finite number of members, then it can be given by listing its members, separated by commas and enclosed by braces — so that, for example,

$$\{1, 3, 6, 10\}$$

is one way to describe the set of, size four, whose members are 1, 3, 6 and 10. The ordering in which elements are listed does not matter, and it also does not matter if elements are listed more than once — so that

$$\{1, 6, 10, 3\}$$

and

$$\{6, 6, 10, 3, 1, 3, 1\}$$

are also ways to describe the same set.

Thus an **empty set** — that is, a set with no members — can be represented by a pair of braces with nothing between them,

$$\{\}$$

However, the symbol \emptyset is often used to represent an empty set too.

A “set of size one” and the object contained in it are not the same thing! For example, 1 is different from $\{1\}$ — the set of size one that has the number 1 as its only member.

Infinite Sets. Several infinite sets of numbers have their own symbols. Students should be familiar with the set, \mathbb{R} , of **real numbers**, and the set, \mathbb{Z} , of **integers** (including positive whole number, negative whole numbers, and zero).

The symbol \mathbb{N} is used to denote the set of **natural numbers**, which includes all of the positive integers and none of the negative integers. Some references define \mathbb{N} so that 0 is a natural number, while others do not; in this course, 0 will generally be considered to be a natural number (but this will be clarified for students when this is needed to understand course material). The symbol \mathbb{P} might be used to represent the set of *positive* integers, so that $0 \notin \mathbb{P}$ and

$$\mathbb{N} = \mathbb{P} \cup \{0\}.$$

It is possible to describe a subset of a larger set U (where U is sometimes called a **universe**) by including a *condition* (that is, *property*) that the members of the set must satisfy. For example,

$$\{n \in \mathbb{Z} \mid n \text{ is even}\}$$

describes the set

$$\{\dots - 6, -4, -2, 0, 2, 4, 6 \dots\}$$

of integers that are even (that is, divisible by two). In general, the condition on the left (“ $n \in \mathbb{Z}$ ”, for this example) identifies the universe from which the members of set are chosen (in this example, \mathbb{Z}); “|” is used as separator; the statement to the right of the separator (“ n is even”, for this example) is the condition that the elements of the set must satisfy. In this course, this notation will almost always be used to describe *infinite* sets.

Operations on Sets

The symbol “ \in ” is used to denote **set membership**. That is, if a is an object that might belong to some set S , then the statement

$$a \in S$$

is used to state that a is a member of the set S . Thus, if $S = \{0, 3, 5\}$ then each of the statements " $0 \in S$ ", " $3 \in S$ ", and " $5 \in S$ " are true. On the other hand, the statement

$$a \notin S$$

is used to state that a is *not* a member of S — so that, for this example, the statement " $3 \notin S$ " is false and the statement " $6 \notin S$ " is true.

On the other hand, the symbol " \subseteq " is used to denote the fact that one set is a **subset** of another. That is, if S and T are sets then the statement

$$S \subseteq T$$

is used to state that every member of the set S is also a member of the set T — so that, for example,

$$\{0, 3\} \subseteq \{0, 1, 3, 6\}$$

and

$$\{0, 3\} \subseteq \mathbb{Z}$$

are true statements. Here are some related symbols that are used to state relationships between sets, and their meanings:

- " \subset " — For sets S and T , $S \subset T$ if it is true that $a \in S$ for every object a such that $a \in T$ (so that $S \subseteq T$), but there also exists at least one object b such that $b \in T$ and $b \notin S$.
- " \subsetneq " — For sets S and T , " $S \subsetneq T$ " means the same thing as " $S \subset T$ ". Sometimes " $S \subsetneq T$ " is used instead of " $S \subset T$ " because people get " \subseteq " and " \subset " mixed up.
- " \supseteq " — For sets S and T , " $S \supseteq T$ " means the same thing as " $T \subseteq S$ ".
- " \supset " — For sets S and T , " $S \supset T$ " means the same thing as " $T \subset S$ " (which means the same thing as $T \subsetneq S$ ", as noted above).
- " \supsetneq " — For sets S and T , " $S \supsetneq T$ " means the same thing as " $S \supset T$ " (or " $T \subset S$ " or " $T \subsetneq S$ ", as noted above).
- " $=$ " — For sets S and T , " $A = B$ " means that, for every object a , $a \in A$ if and only if $a \in B$.

Suppose, now, that A and B are sets.

- The **union** of A and B , $A \cup B$, is the set of objects a such that either $a \in A$ or $a \in B$, or both.
- The **intersection** of A and B , $A \cap B$, is the set of objects a such that $a \in A$ *and* $a \in B$.
- $A \setminus B$ is the set of objects a such that $a \in A$ and $a \notin B$.

Sequences

A **sequence** of objects is a list of objects in some order. There does not seem to be a standard way to represent these. For example, the sequence with “length” five, consisting of the integers 1, 3, 1 (again), 2 and 4 will generally be shown by listing the entries of the sequence, separated by commas:

$$1, 3, 1, 2, 4$$

However, a finite sequence is sometimes shown as enclosed by parenthesis,

$$(1, 3, 1, 2, 4)$$

or angle brackets,

$$\langle 1, 3, 1, 2, 4 \rangle$$

Finite sequences are sometimes called **tuples** and, if k is a positive integer, then a finite sequence with length k is sometimes called a **k -tuple**. Furthermore, 2-tuples are also called **ordered pairs**.

Cartesian Products

If A and B then the **Cartesian product** $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. More generally, if k is a positive integer and A_1, A_2, \dots, A_k are sets, then $A_1 \times A_2 \times \dots \times A_k$ is the set of k -tuples (a_1, a_2, \dots, a_k) such that $a_i \in A_i$ for every integer i such that $1 \leq i \leq k$.

Functions and Relations

If A and B are sets then a **function** from A to B is an assignment of exactly one object $b \in B$ to every object $a \in A$. Functions from A to B are sometimes called **total functions** from A to B . A **partial function** from A to B is an assignment of **at most** one value $b \in B$ to every value $a \in A$ — so that every “total function” is also a “partial function”.

The notation

$$f : A \rightarrow B$$

represents the fact that f is a partial function from A to B . For any such function f , if $a \in A$ and there exists an object $b \in B$ such that f associates b with a , then we say that f is **defined** at a and we write the value associated with a , by f , as $f(a)$. On the other hand, if there is no

value $b \in B$ associated with a by f , then we say that f is **undefined** at a (or, sometimes, say that “ $f(a)$ is undefined”).

If $f : A \rightarrow B$ then A is called the **domain** of the function f , and B is called the **range** of f .

Suppose, now, that $f : A \rightarrow B$ is a total function.

- The function f is called **injective**, or **1-1**, if it satisfies the following property: For every object $b \in B$, there exists **at most** one object $a \in A$ such that $f(a) = b$ — so that, for all objects $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
- The function f is called **surjective**, or **onto**, if f satisfies the following property: For every object $b \in B$, there exists **at least** one object $a \in A$ such that $f(a) = b$.
- A function $f : A \rightarrow B$ is called **bijective** if f is both injective and surjective — so that, for every object $b \in B$, there exists **exactly one** object $a \in A$ such that $f(a) = b$. A function $f : A \rightarrow B$, that is bijective, is called a **bijection** from A to B .

When $f : A_1 \times A_2 \times \cdots \times A_k \rightarrow B$ for some positive integer k and for sets A_1, A_2, \dots, A_k and B , we call f a **k -ary** function. This is called a “unary” function when $k = 1$ and it is called a “binary” function when $k = 2$.

A **predicate**, or **property**, is a function whose range is $\{\text{true}, \text{false}\}$.¹ A property whose domain is the set of k -tuples $A \times A \times \cdots \times A$, for some positive integer k and set A , is called a **relation** (or a **k -ary relation**, or a **k -ary relation on A**).

Infix notation is used when defining and using various familiar binary functions and relations. For the example, while the “integer addition function” would be represented as a total function $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$, based on what is given above, we generate wrote “ $a + b$ ” instead of “ $+(a, b)$ ”, as what you get when applying this function to the pair of integers a and b .

References

- [1] Susanna S. Epp. *Discrete Mathematics with Applications*. Brooks Cole, fifth edition, 2019.
- [2] Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. McGraw-Hill Education, eighth edition, 2018.
- [3] Michael Sipser. *Introduction to the Theory of Computation*. CENGAGE Learning, third edition, 2013.

¹The symbols “true” and “false” are sometimes replaced by T and F, respectively, or by 1 and 0, respectively.