

CPSC 351 — Tutorial Exercise #19

Tail Bounds

The goal of this exercise is to give you more practice in computing tail bounds for random variables.

Problems To Be Solved

1. Consider an experiment in which you roll a die (which could show any of the integers between 1 and 6), n times, for some positive integer n . Since you are interested in the sequence of values rolled, the **sample space** is the set

$$\Omega = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in \{1, 2, 3, 4, 5, 6\} \text{ for } 1 \leq i \leq n\}.$$

Here, the sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ represents the outcome where the die shows α_1 the first time it is rolled, α_2 the second time it is rolled, and so on.

Note that $|\Omega| = 6^n$. Let us suppose that this is a fair die (each possible value is showed with the same probability), and no roll of the die depends on any of the others. We will therefore use a **probability distribution** $P : \Omega \rightarrow \mathbb{R}$ such that $P(\mu) = \frac{1}{|\Omega|} = 6^{-n}$, for each outcome $\mu \in \Omega$.

Consider the number of times that “6” is rolled. This is the value of a random variable

$$X : \Omega \rightarrow \mathbb{N}$$

such that, an outcome $\mu = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega$,

$$X(\mu) = X((\alpha_1, \alpha_2, \dots, \alpha_n)) = |\{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } \alpha_i = 6\}|.$$

- (a) For $1 \leq i \leq n$, consider another random variable $X_i : \Omega \rightarrow \{0, 1\}$ such that, for $\mu = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega$,

$$X_i(\mu) = X_i((\alpha_1, \alpha_2, \dots, \alpha_n)) = \begin{cases} 1 & \text{if } \alpha_i = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Describe, as precisely as you can, how $X_1, X_2, \dots, X_n$ and X are related.

- (b) Compute $E[X_i]$ and $\text{var}(X_i)$, for each integer i such that $1 \leq i \leq n$.
- (c) Show that the random variables $X_1, X_2, \dots, X_n$ are **pairwise independent**.
- (d) Using the above information, compute $E[X]$ and $\text{var}(X)$.
- (e) Using the above information, and inequalities from lectures, find the best bounds that you can for the probability that $X < \frac{n}{12}$ as well as the probability that $X > \frac{n}{2}$ — **based on what you know, so far**.
- (f) Now show that the random variables $X_1, X_2, \dots, X_n$ are **mutually independent**.
- (g) Using this additional information, find a better bound for the probability that $X < \frac{n}{12}$ when n is large.